27. Curvature and Structure Equations

Version 1.1

Notiztitel 30.11.2010

In this section D is again a connection on a vector bundle $\pi_E : E \to M$ over a manifold M.

The curvature operator $F = F_D$ is (cf. 26.2)

$$F(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \quad \text{for } X,Y \in \mathcal{V}(W),\ W \subset M \text{ open},$$

i.e.

$$F(X,Y) : \Gamma(W,E) \to \Gamma(W,E), \quad s \longmapsto [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]}s .$$

<u>(27.1) PROPOSITION:</u> $F(X,Y)$ is $\mathcal{E}(W)$-linear, hence

$$F(X,Y) \in \mathrm{Hom}_{\mathcal{E}(W)}\big(\Gamma(W,E), \Gamma(W,E)\big) \cong \Gamma(W, \mathrm{End}(E)) \cong \Gamma(W, E^* \otimes E)$$

[01.12.10]

<u>Proof</u>. $F(X,Y)(\lambda s + s') = \lambda F(X,Y)s + F(X,Y)s'$ is evident.

For $g \in \mathcal{E}(W)$:

$$\nabla_X \nabla_Y g s = L_X (L_Y g s + g \nabla_Y s) =$$
$$= (L_X L_Y g)s + L_Y g \, \nabla_X s + L_X g \, \nabla_Y s + g \nabla_X \nabla_Y s$$

$$[\nabla_X, \nabla_Y] gs = (L_{[X,Y]} g) s + g [\nabla_X, \nabla_Y] s$$
$$\nabla_{[X,Y]} gs = (L_{[X,Y]} g) s + g \nabla_{[X,Y]} s$$

$$\Rightarrow \quad F(X,Y) gs = g \big(F(X,Y)s\big). \qquad \square$$

(27.2) PROPOSITION: $F : \mathcal{V}(W) \times \mathcal{V}(W) \to \mathrm{End}_{\mathcal{E}(W)}(\Gamma(W,E))$ is $\mathcal{E}(W)$-bilinear and alternating. Hence, F is tensor. More precisely F is a 2form with values in $\mathrm{End}(E)$:

$$F \in \mathcal{A}^2(W, \mathrm{End}(E)) \cong \Gamma(W, \Lambda^2 TM \otimes E^* \otimes E) \cong \dots$$

Proof. $F(X,Y) = -F(Y,X)$ is evident, as well as the bilinearity with respect to $\mathbb{K}$. It remains to prove $F(gX,Y) = gF(X,Y)$ for $g \in \mathcal{E}(W)$ and $X, Y \in \mathcal{V}(W)$:

$$\nabla_{gX}\nabla_Y = g\nabla_X\nabla_Y$$
$$\nabla_Y\nabla_{gX} = \nabla_Y g\nabla_X = L_Y g\,\nabla_X + g\nabla_Y\nabla_X$$

$$\Rightarrow \quad [\nabla_{gX}, \nabla_Y] = g[\nabla_X, \nabla_Y] - L_Y g\,\nabla_X$$

$$[gX, Y] = g[X,Y] - L_Y g\, L_X$$
$$\nabla_{[gX,Y]} = g\nabla_{[X,Y]} - L_Y g\,\nabla_X$$

$$\Rightarrow \quad F(gX,Y) = g\,F(X,Y) \qquad \square$$

27.3 LOCAL FORMULAS: Let $\varphi : U \to Q \subset \mathbb{R}^n$ be a chart with coordinates q^j. Then the $\partial_j := \frac{\partial}{\partial q^j} \in \mathcal{V}(U)$ constitute

a basis of $\Gamma(U, TM) = \mathcal{W}(U)$ over $\mathcal{E}(U)$. Without loss of generality let there exists a basis $e_1, \dots, e_r \in \Gamma(U, E)$ as well, e.g. if $Q \subset \mathbb{R}^n$ is open and konvex.
The choice of ∂_j (i.e. φ) and $e_\varrho \in \Gamma(U,E)$ yields a matrix $A = (A_\varrho^\sigma)$ of oneforms

$$A_\varrho^\sigma \in \mathcal{A}^1(U),$$

s. th. $$D e_\varrho = A_\varrho^\sigma e_\sigma \; : \; A \in \mathcal{A}^1(U, \mathrm{End}(r, \mathbb{K})).$$

The A_ϱ^σ can be written as

$$A_\varrho^\sigma = \Gamma_{j\varrho}^\sigma \, dq^j \quad \text{with} \quad \Gamma_{j\varrho}^\sigma \in \mathcal{E}(U), \text{ where}$$

$$D_j e_\varrho = \Gamma_{j\varrho}^\sigma e_\sigma .$$

We know all these formulas for the connection D from 23.6 and 23.10.
Now, the curvature $F(X,Y)$ determines 2 forms $\Theta_\varrho^\sigma \in \mathcal{A}^2(U)$

$$F(X,Y)\, e_\varrho = \Theta_\varrho^\sigma (X,Y)\, e_\sigma \quad \text{and}$$

$$\Theta = (\Theta_\varrho^\sigma) \in \mathcal{A}^2(U, \mathrm{End}(r, \mathbb{K})).$$

$$\Theta_\varrho^\sigma = F_{ij\varrho}^\sigma \, dq^i \otimes dq^j \quad , \quad F_{ij\varrho}^\sigma \in \mathcal{E}(U).$$

(In Riemannian geometry $R = F$ Riemann tensor;
$F_{ij\ell}^{k} \sim R_{ij\ell}^{k}$ $i,j,k,\ell \in \{1, \dots n\}$.)

For the next local formula we define for endomorphism-valued 1-forms $A, B \in \mathcal{A}^1(U, \mathrm{End}(r, \mathbb{K})$ the two form $A \wedge B \in \mathcal{A}^2(U, \mathrm{End}(r, \mathbb{K})$ by

$$(A \wedge B)_\varrho^\tau := A_\sigma^\tau \wedge B_\varrho^\sigma \quad , \text{ i.e.}$$

$$(A \wedge B)_\varrho^\tau (X, Y) := A_\sigma^\tau (X) B_\varrho^\sigma (Y) - A_\sigma^\tau (Y) B_\varrho^\sigma (X) .$$

In general:

<u>(27.4) DEFINITION:</u> For the forms $\alpha \in \mathcal{A}^k(W, \mathrm{End}(r, \mathbb{K})$ and $\beta \in \mathcal{A}^\ell(W, \mathrm{End}(r, \mathbb{K}))$ the *exterior product* $\alpha \wedge \beta$ is defined as

$$(\alpha \wedge \beta)_\varrho^\tau := \alpha_\sigma^\tau \wedge \beta_\varrho^\sigma$$

$\alpha \wedge \beta \in \mathcal{A}^{k+\ell}(W, \mathrm{End}(r, \mathbb{K}))$.

Remark: This exterior product appears in physics texts and is particularly useful for $\mathrm{End}(r, \mathbb{K})$-valued forms which are defined locally. In a more general context this exterior product is related to the product $[\alpha, \beta]$ of forms α, β with values in a Lie algebra $\mathfrak{g}$ ($\mathrm{End}(V, \mathbb{K})$ is a Lie algebra) and then we have the general identity:

$$[\alpha, \beta] = \alpha \wedge \beta + \beta \wedge \alpha .$$

<u>(27.5) PROPOSITION</u> (Structure equation)

$1°$ $D = d + A$,

$2°$ $\Theta = dA + A \wedge A$,

$3°$ $d\Theta = \Theta \wedge A - A \wedge \Theta$.

<u>Proof</u>. $1°$ If $s = f^\varrho e_\varrho$ we have $Ds = df^\tau e_\tau + A^\tau_\varrho f^\varrho e_\tau =$ "$(d+A)f$". Or, setting $s_f(a) := \psi^{-1}(a, f^\varrho(a) \check{e}_\varrho)$: $Ds_f = s_{df} + s_{Af} = s_{(d+A)f}$.

Ad $2°$: $\nabla_X \nabla_Y e_\varrho = \nabla_X A^\sigma_\varrho(Y) e_\sigma = L_X(A^\sigma_\varrho(Y)) e_\sigma + A^\sigma_\varrho(Y) \nabla_X e_\sigma$

$$\Rightarrow [\nabla_X, \nabla_Y] e_\varrho = \left(L_X A^\sigma_\varrho(Y) - L_Y A^\sigma_\varrho(X)\right) e_\sigma +$$

$$A^\sigma_\varrho(Y) A^\tau_\sigma(X) e_\tau - A^\sigma_\varrho(X) A^\tau_\sigma(Y) e_\tau .$$

$$\nabla_{[X,Y]} e_\varrho = A_\varrho^\sigma([X,Y]) e_\sigma$$

$$\Rightarrow F(X,Y)e_\varrho = \underbrace{\left(L_X A_\varrho^\sigma(Y) - L_Y A_\varrho^\sigma(X) - A_\varrho^\sigma([X,Y]\right)}_{dA_\varrho^\sigma(X,Y)} e_\sigma + $$
$$+ \underbrace{\left(A_\sigma^\tau(X) A_\varrho^\sigma(Y) - A_\sigma^\tau(Y) A_\varrho^\sigma(X)\right)}_{(A\wedge A)_\varrho^\tau} e_\tau$$

Ad 3°: $d(\Theta_\varrho^\tau) = d(dA_\varrho^\tau + A_\sigma^\tau \wedge A_\varrho^\sigma)$
$$= dA_\sigma^\tau \wedge A_\varrho^\sigma - A_\sigma^\tau \wedge dA_\varrho^\sigma$$

And,

$$(\Theta \wedge A)_\varrho^\tau = \Theta_\sigma^\tau \wedge A_\varrho^\sigma = (dA_\sigma^\tau + A_\kappa^\tau \wedge A_\sigma^\kappa) \wedge A_\varrho^\sigma,$$

$$(A \wedge \Theta)_\varrho^\tau = A_\sigma^\tau \wedge \Theta_\varrho^\sigma = A_\sigma^\tau \wedge dA_\varrho^\sigma + A_\sigma^\tau \wedge A_\kappa^\sigma \wedge A_\varrho^\kappa.$$

Hence,

$$(\Theta \wedge A - A \wedge \Theta)_\varrho^\tau = dA_\sigma^\tau \wedge A^\sigma - A_\sigma^\tau \wedge dA_\varrho^\sigma. \qquad \square$$

<u>Remark</u>: The structure equations are sometimes formulated in a slightly different manner by making use of the coordinates q^i. We then obtain

$$A = A_j dq^j = (\Gamma_{j\varrho}^\sigma dq^j), \quad A_j = (A_{j\varrho}^\sigma), \text{ and}$$

$$\Theta = \frac{1}{2} \Theta_{jk}\, dq^j \wedge dq^k.$$

1° $D_j = \nabla_j = \partial_j + A_j$

2° $\Theta_{jk} = \partial_j A_k - \partial_k A_j + A_j \wedge A_k + A_k \wedge A_j$
$= \partial_j A_k - \partial_k A_j + [A_j, A_k]$ (or with F for Θ)

There are also local formulas for $F_{ij\varsigma}^{\sigma}$ in terms of the $\Gamma_{i\varsigma}^{\sigma}$ and its derivatives $\partial_j \Gamma_{i\varsigma}^{\sigma} = \Gamma_{i\varsigma,j}^{\sigma}$, which are similar to the Γ_{ij}^{k} of the Levi-Civita connection.

Change of Frame: For another frame $\bar{s}_\varsigma = g_\varsigma^\sigma e_\sigma$ with $g = (g_\varsigma^\sigma) \in \mathcal{E}(U, GL(r, \mathbb{K}))$ we obtain

(27.6) PROPOSITION: $\bar{\Theta} = g^{-1} \Theta g$

Proof: $\bar{\Theta} = d\bar{A} + \bar{A} \wedge \bar{A}$ with $\bar{A} = g^{-1} A g + g^{-1} dg$ (cf. 23.8), and a direct calculation yields the result.

For a global description of the curvature and structure equations we are interested in an extension of the connection $D : \mathcal{A}^0(W, E) \to \mathcal{A}^1(W, E)$ to operators

$$D = d^D : \mathcal{A}^k(W, E) \longrightarrow \mathcal{A}^{k+1}(W, E), \quad k \in \mathbb{N}.$$

(27.7) DEFINITION: The exterior covariant derivative d^D induced by a given connection D is the $\mathbb{K}$-linear map

$$D = d^D : \mathcal{A}^k(W,E) \longrightarrow \mathcal{A}^{k+1}(W,E)$$

determined by

$$d^D(\eta s) := d\eta\, s + (-1)^k \eta \wedge Ds \,, \quad \eta \in \mathcal{A}^k(W),\ s \in \mathcal{A}^0(W,E).$$

(27.8) LEMMA: d^D is well-defined. In particular, for $\beta \in \mathcal{A}^k(W,E)$, $X_i \in \mathcal{V}(W)$:

$$(d^D\beta)(X_0, X_1, \dots X_k) = \sum_{j=0}^{k} (-1)^j \nabla_{X_j} \beta(X_0, \dots \hat{X}_j, \dots X_k) +$$
$$+ \sum_{i<j} (-1)^{i+j} \beta([X_i, X_j], X_0, \dots, \hat{X}_i, \dots \hat{X}_j, \dots X_k).$$

(Proof is left to the reader.)

Notation: $D = d^D$. It is easy to deduce

(27.9) LEMMA: $D = d^D : \mathcal{A}^k(W,E) \longrightarrow \mathcal{A}^{k+1}(W,E)$ fulfills the well-known connection conditions of 23.1:

(D1) D is $\mathbb{K}$-linear

(D2) $D(f\beta) = df \wedge \beta + f\, D\beta$ for $f \in \mathcal{E}(W)$, $\beta \in \mathcal{A}^k(W,E)$.

[06.12.10]

(27.10) LEMMA: $D \circ D : \mathcal{A}^0(W,E) \to \mathcal{A}^2(W,E)$ is $\mathcal{E}(W)$-linear. It is a 2-form $D \circ D \in \mathcal{A}^2(W, \mathrm{End}(E))$. This is true also for $D \circ D : \mathcal{A}^k(W,E) \longrightarrow \mathcal{A}^{k+2}(W,E)$.

Pf. $D \circ D(fs) = d^D(df\, s + f\, Ds) = ddf\, s - df\, Ds + df\, Ds + f\, DDs$
$= f\, D \circ D(s)$.

(27.9) PROPOSITION: $F = D \circ D$.

Proof. Locally with respect to a frame $e_1, \dots e_r \in \Gamma(U,E) = \mathcal{A}^0(U,E)$ we have to show $Fe_\varrho = D \circ D\, e_\varrho$, $\varrho = 1, \dots r$. We start with $De_\varrho = A_\varrho^\sigma e_\sigma$.

$$DDe_\varrho = d^D(A_\varrho^\sigma e_\sigma) = dA_\varrho^\sigma e_\sigma - A_\varrho^\sigma \wedge De_\sigma =$$

$$= dA_\varrho^\tau e_\tau - A_\varrho^\sigma \wedge A_\sigma^\tau e_\tau = dA_\varrho^\tau + A_\sigma^\tau \wedge A_\varrho^\sigma e_\tau = (dA + A \wedge A)_\varrho^\tau e_\tau$$

$$= \Theta_\varrho^\tau e_\tau = Fe_\varrho \quad (\text{cf. } 27.5).$$

(27.10) LEMMA: Every connection D on $E \to M$ induces a natural connection $D = D^{\mathrm{End}\,E}$ on the endomorphism vector bundle $\mathcal{L}(E,E) = \mathrm{End}\,E \to M$ by

$$(D^{\mathrm{End}\,E} L)s = D(Ls) - L(Ds)$$

for $L \in \Gamma(W, \mathrm{End}\,E)$ and $s \in \Gamma(W,E)$.

Proof: Evidently, $D^{\mathrm{End}E}$ is $\mathbb{K}$-linear, i.e. (D1). Furthermore, (D1) is satisfied as well:

$$D^{\mathrm{End}E}(fL)s = D(fLs) - fL(Ds)$$
$$= df\cdot Ls + f\,D(Ls) - f\,L(Ds) = df\cdot Ls + f(D^{\mathrm{End}D}L)s.$$

What about $D^{\mathrm{End}E}(D\circ D)$? $\left(D\circ D\in \mathcal{A}^2(W,\mathrm{End}(E))\right)$. The answer is a global version of the structure equations 27.5:

<u>(27.11) PROPOSITION:</u> (Bianchi-identity):

$$D^{\mathrm{End}D}F = 0 \quad \text{or} \quad D\circ D\circ D = 0$$

Proof. It suffices to check $D^{\mathrm{End}E}F(X,Y,Z) = 0$ for vector fields $X,Y,Z\in \mathcal{V}(W)$ which commute pairwise.

$$D^{\mathrm{End}E}F(X,Y,Z) = \nabla_X^{\mathrm{End}E}F(Y,Z) + \nabla_Y^{\mathrm{End}E}F(Z,X) + \nabla_Z^{\mathrm{End}E}F(X,Y)$$

according to 27.8. Moreover:

$$\left(\nabla_X^{\mathrm{End}}F(Y,Z)\right)s = \nabla_X\left(F(Y,Z).s\right) - F(Y,Z).D_Xs \text{ etc.}$$

We obtain (replacing $F(Y,Z)$ by $[\nabla_Y,\nabla_Z]$ etc.):

$$D^{\mathrm{End}E} F(X,Y,Z) = \nabla_X [\nabla_Y, \nabla_Z] - [\nabla_Y, \nabla_Z] \nabla_X + \nabla_Y [\nabla_Z, \nabla_X] - [\nabla_Z, \nabla_X] \nabla_Y + \nabla_Z [\nabla_X, \nabla_Y] - [\nabla_X, \nabla_Y] \nabla_Z$$
$$= [\nabla_X, [\nabla_Y, \nabla_Z]] + [\nabla_Y [\nabla_Z, \nabla_X]] + [\nabla_Z, [\nabla_X, \nabla_Y]] = 0,$$

since $[\ ,\ ]$ satisfies the Jacobi-identity. $\square$

Remark: Another proof uses the local structure equations $d\Theta = \Theta \wedge A - A \wedge \Theta$ & $\Theta = dA + A \wedge A$.

Remark: The exterior covariant derivative

$$D : \mathcal{A}^k(W, E) \to \mathcal{A}^{k+1}(W, E)$$

gives a complex

$$\dots \xrightarrow{D} \mathcal{A}^k(W,E) \xrightarrow{D} \mathcal{A}^{k+1}(W,E) \xrightarrow{D} \mathcal{A}^{k+2}(E,W) \xrightarrow{D} \dots$$

if and only if $D \circ D = 0$, i.e. iff the connection D is flat (i.e. $F \equiv 0$).