25. Parallel Transport

Version 1.1

Given a vector bundle $\pi_E : E \to M$ on a manifold, e.g. the tangent bundle $TM \to M$, it is not evident how to compare vectors in E_a with those in E_b ($a, b \in M$, $a \neq b$). In particular, without additional structure we do not have a concept to call a $\xi \in E_a$ "parallel" to $\eta \in E_b$. In the trivial case $E = M \times \mathbb{K}^r$ the comparison can be made by identification of $E_a = \{a\} \times \mathbb{K}^r$ and $E_b = \{b\} \times \mathbb{K}^r$ with $\mathbb{K}^r$ and there "parallelism" is given by the translation of vectors. In the general case, a "parallelism" between vectors of E_a and E_b is at least a certain linear map $P : E_a \to E_b$. Moreover, such a parallelism should behave nicely in case of compositions $E_a \to E_b \to E_c$, in particular $E_a \to E_b \to E_a$, – and it should be smooth.

The essential discovery – more than 100 years ago – is that such a parallelism has to depend not only on the absolute positions of $a, b \in E$ but also on a curve γ connecting the two points:

There is no "absolute parallelism"! And it turns out that the best way to describe parallelism is using a connection D on E and define its induced parallelism.

Indeed, let $\gamma : [t_0, t_1] \to M$ be a piecewise smooth curve with $\gamma(t_0) = a$, $\gamma(t_1) = b$. We have seen in 23.14 that to each $\xi \in E_a$ there exists a unique horizontal lift $\beta = \beta_{\gamma,\xi}$ of γ through ξ. Therefore, the map

$$\mathbb{P}^{\gamma}_{t_0,t_1} : E_a \to E_b, \quad \mathbb{P}^{\gamma}_{t_0,t_1}(\xi) := \beta_{\gamma,\xi}(t_1),$$

is well-defined.

<u>(25.1) PROPOSITION - DEFINITION:</u> $\mathbb{P}^{\gamma}_{t_0,t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$ is the "horizontal displacement" or parallel transport along the curve γ induced by D. The maps $\mathbb{P}^{\gamma}_{t_0,t_1}$ fulfill the following properties:

- (P_1) $\mathbb{P}^{\gamma}_{t_0,t_1}$ is a $\mathbb{K}$-linear isomorphism: $\left(\mathbb{P}^{\gamma}_{t_0,t_1}\right)^{-1} = \mathbb{P}^{\gamma^-}_{t_1,t_0}$.
- (P_2) $\mathbb{P}^{\gamma}_{t,t_1} \circ \mathbb{P}^{\gamma}_{t_0,t} = \mathbb{P}^{\gamma}_{t_0,t_1}$ for all $t \in]t_0, t_1[$.
- (P_3) $\mathbb{P}^{\gamma}_{t_0,t_1}$ is independent of the parametrization of γ.
- (P_4) $\mathbb{P}_{t_0,t}$ depends differentially on t, $t \in]t_0, t_1[$.

Moreover, for sections $s \in \Gamma(W,E)$ and for tangent vectors $X = \dot\gamma(t) \in T_aM$ at the point $a = \gamma(t) \in M$:

$$(*) \quad D_X s(a) = \lim_{h \to 0} \frac{1}{h} \left(\left(P^{\gamma}_{t,t+h} \right)^{-1} \left(s \circ \gamma(t+h) \right) - s \circ \gamma(t) \right).$$

(25.2) DEFINITION: A parallel structure on a vector bundle $E \to M$ is a collection of isomorphisms $\left(P^{\gamma}_{t_0,t_1} \right)_{\gamma \ldots}$ as above with (P1) – (P2).

(25.3) PROPOSITION: Each parallel structure on a vector bundle determines a connection D by $(*)$ and it is induced by this connection (cf. 25.1).

(25.4) FACIT: We have now 4 definitions of a connection

1° as a $D : \mathcal{A}^0(M,E) \to \mathcal{A}^1(M,E)$ according to (23.1).
2° as a collection of 1-forms (A) with $*$ (cf. 23.9).
3° as a splitting C of $0 \to V_E \to TE \to \pi^*TM \to 0$.
4° as a parallel structure.

(25.5) EXAMPLES: 1° The trivial situation: $E = M \times \mathbb{F}$, $\mathbb{F} = \mathbb{K}^r$, and $D : \mathcal{A}^0(M,E) \to \mathcal{A}^1(M,E)$ a connection. We know that $D = \text{"}d + A\text{"}$ in the sense of 23.6, where $A \in \mathcal{A}^1(M, \mathrm{End}(\mathbb{F}))$: $s \in \Gamma(M,E)$ is of the form $s(a) = (a, \sigma(a))$, $\sigma \in \mathcal{E}(M,\mathbb{F})$, $s = s_\sigma$

$$Ds(a) = (a, d_a\sigma + A(\cdot)\sigma(a))$$

$$\nabla_X s(a) = (a, d_a\sigma(X) + A(X)\sigma(a))$$

$\beta : [t_0, t_1] \to E$ is a lift of $\gamma : [t_0,t_1] \to M$, if $\beta(t) = (\gamma(t), \eta(t))$, $\eta : [t_0,t_1] \to \mathbb{F}$. In 23.13 we proved

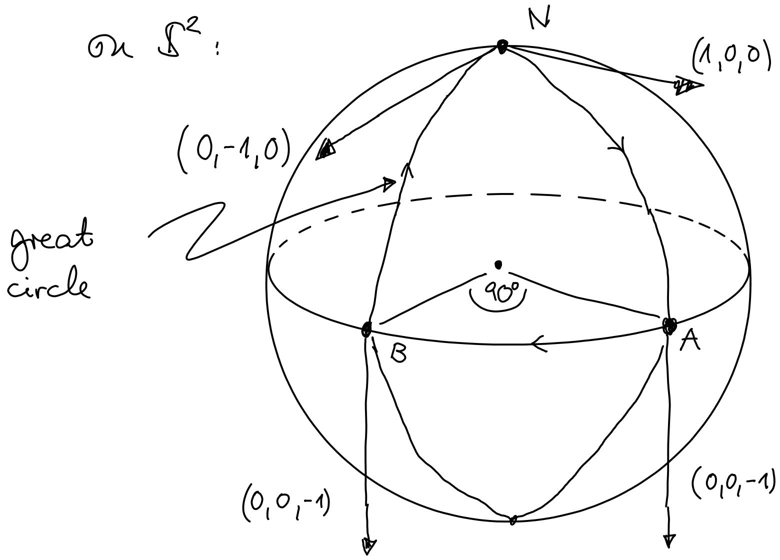
$$\beta \text{ horizontal lift} \iff \dot\eta + A(\dot\gamma)\eta = 0 .$$

According to 23.14 each $\xi \in E_a$, $a = \pi(\beta(t_0)) = \gamma(t_0)$, determines exactly one solution η, i.e. one horizontal lift $\beta = (\gamma, \eta)$ and

$$\mathbb{P}^{\gamma}_{t_0,t_1}\xi = \beta(t_1) = (\gamma(t_1), \eta(t_1)) .$$

Remark: From this description, all the properties (P1) – (P4) follow, as well as $*$.

2° The well-known case of the sphere $S^2 = M \subset \mathbb{R}^3$.
Let D on TS^2 be the Levi-Civita connection induced from the standard Riemannian metric on TM which is the restriction of the euclidian metric to the $T_a S^2 \subset \mathbb{R}^3$.
The geodesics on S^2 are the great circles of S^2. We describe the parallel transport along a geodesic triangle on S^2:



$E = TM$, $A = (1,0,0)$, $B = (0,-1,0)$

$$E_N \xrightarrow{\mathbb{P}} E_A, \qquad (1,0,0) \longmapsto (0,0,-1)$$

$$E_A \xrightarrow{\mathbb{P}} E_B, \qquad (0,0,-1) \longmapsto (0,0,-1)$$

$$E_B \xrightarrow{\mathbb{P}} E_N, \qquad (0,0,-1) \longmapsto (0,-1,0)$$

Altogether: $\mathbb{P}(1,0,0) = (0,-1,0)$
Similarly: $\mathbb{P}(0,1,0) = (1,0,0)$
We obtain a rotation $\mathbb{P}: E_N \to E_N$ of the 2-dimensional euclidian plane for the parallel transport $\mathbb{P} = \mathbb{P}^{\Delta}$ along the geodesic triangle Δ.
It is a fact of Riemannian geometry that $\mathbb{P}^{\gamma}$ is always a rotation for all closed curves γ in S^1.

Let $W(a) = \{\gamma$ piecewise smooth curve in M with $\gamma(t_0) = \gamma(t_1) = a\}$ the space of closed curves starting and ending in $a \in M$. ("Loops")

<u>(25.6) Holonomy groups:</u> Let D be a connection on the vector bundle $E \to M$. Then

$$G(a) := \{ \, \mathbb{P}^{\gamma}_{t_0,t_1} : E_a \to E_a \mid \gamma \in W(a) \}$$

is the holonomy group $\cong$ subgroup of $GL(r,\mathbb{K})$.

1° $G(a)$ is indeed a group, a subgroup of $GL(E_a)$
$G(a) \subset GL(E_a)$.

2° For $a, b \in M$ and $\alpha : [s_0,s_1] \to M$ a curve connecting a and b we have the group isomorphism

$$\Phi^{\alpha} : G(a) \to G(b), \ \mathbb{P}^{\gamma}_{t_0,t_1} \longmapsto \mathbb{P}^{\alpha}_{s_0,s_1} \circ \mathbb{P}^{\gamma}_{t_0,t_1} \circ \left(\mathbb{P}^{\alpha}_{s_0,s_1}\right)^{-1} :$$

$$\begin{tikzcd} E_a \arrow[r, "\mathbb{P}^{\gamma}"] \arrow[d, "\mathbb{P}^{\alpha}"'] & E_a \arrow[d, "\mathbb{P}^{\alpha}"] \\ E_b \arrow[r, "\Phi^{\alpha}(\mathbb{P}^{\gamma})"'] & E_b \end{tikzcd}$$

3° For a connected M, all holonomy groups are isomorphic.

4° The connection determines (for connected M) a conjugation class of subgroups in $GL(r,\mathbb{K})$. Any representative, i.e. $G'(a) := \{ H \mathbb{P} H^{-1} : \mathbb{P} \in G(a)\}$ with $H : E_a \to \mathbb{K}^r$ a linear is called <u>holonomy group</u> of D as well.

5° $G(a)$ is a closed subgroup of $GL(E_a)$ (hence $G(a)$ is a Lie group according to a general result on Lie groups).

6° The group $G_0(a) := \{ P^{\gamma}_{t_0,t_1} \mid \gamma \sim 0 \}$ „$\sim$" homotopy – called restricted holonomy group – is a normal subgroup of $G(a)$.

7° $\pi_1(M,a) \longrightarrow G(a)/G_0(a)$ is a surjective group homomorphism.

General geometric question: Which $G(a)$, $G_0(a)$ occur as holonomy groups?

We come back to holonomy groups in the context of principal fibre bundles.

And we will describe an interesting relation between holonomy and curvature, our next subject in section 26.