23. Connections on Vector Bundles

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In the following $\pi_E : E \to M$ is a vector bundle (over $\mathbb{K}$) of rank r. By $\mathcal{A}^1(M,E) := \Gamma(M, \Omega^1(TM,E))$ we denote the module (over $\mathcal{E}(M)$) of all E-valued differential 1-forms, i.e. the sections

$$s : M \longrightarrow \Omega^1(TM,E) \cong T^*M \otimes E \cong \mathcal{L}(TM,E)$$

(cf. 11.5 and 18.7). Correspondingly,

$$\mathcal{A}^k(M,E) := \Gamma(M, \Omega^k(TM,E)) \ , \ k \in \mathbb{N}, \text{ with}$$

$$\mathcal{A}^0(M,E) := \Gamma(M,E).$$

(23.1) DEFINITION: A connection on E is a map

$$D : \Gamma(W,E) \to \Gamma(W, \Omega^1(TM,E)) \quad \text{for } W \subset M \text{ open}$$
$$\left(D : \mathcal{A}^0(W,E) \to \mathcal{A}^1(W,E) \right).$$

with

$$\boxed{\begin{array}{ll} \text{(D1)} & D \text{ is } \mathbb{K}\text{-linear} \\ \text{(D2)} & D(fs) = df\,s + f\,Ds \end{array}} \text{ for } f \in \mathcal{E}(W),\ s \in \Gamma(W,E).$$

Moreover, the maps $D = D^W$ are compatible with the restrictions, i.e. for an open subset $W' \subset W$ and $s \in \Gamma(W,E)$ one requires $D^{W'}(s|_{W'}) = (D^W s)|_{W'}$.

The relation to the main concept of section 22 – the covariant derivative – is given by:

(23.2) PROPOSITION-DEFINITION: (Covariant Derivative)
Let D be a connection on the vector bundle $E \to M$. For each vector field $X \in \mathcal{V}(M)$ the associated covariant derivative of a section $s \in \Gamma(W,E)$ in the direction X is $\nabla_X s := D(s)(X)$. The maps

$$\nabla_X : \Gamma(W,E) \to \Gamma(W,E)$$

satisfy:

$$\boxed{\begin{array}{ll} (\nabla 1) & \nabla_X + \nabla_Y = \nabla_{X+Y} \\ (\nabla 2) & \nabla_{fX} = f\nabla_X \\ (\nabla 3) & \nabla_X(s+t) = \nabla_X s + \nabla_X t \\ (\nabla 4) & \nabla_X(fs) = L_X f\, s + f\nabla_X s \end{array}} \quad \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \mathcal{E}(W)\text{-linear in } X \\ \mathbb{K}\text{-linear in } s \\ \text{"derivation" in } s \end{array}$$

for all $X, Y \in \mathcal{V}(W)$, $f \in \mathcal{E}(W)$, $s, t \in \Gamma(W,E)$. Moreover, ∇ is compatible with restrictions to $W' \subset W$, W' open. Conversely, such a collection of covariant derivatives $(\nabla_X)_{X \in \mathcal{V}(W)}$ (with

$(\nabla 1)-(\nabla 4)$ defines a connection on E by $Ds(X) := \nabla_X s$.

(23.2) is in the spirit of section 22 while the equivalent definition (23.1) is oriented towards a gauge principle which we will explain later.

(23.3) EXAMPLES: 1° Let $E = TM$. Then a connection defines an (affine) covariant derivative in the sense of 22.10. And the definition 23.2 with 23.1 constitutes a straightforward generalization of 22.10 to vector bundles instead of TM.

2° On the trivial line bundle $E = M \times \mathbb{K}$ we obtain the connection

$$Df := df \qquad \left(\Gamma(M,E) \cong \mathcal{E}(M,\mathbb{K}) \right).$$

What are the other connections?

(23.4) PROPOSITION: Let D be a connection on the vector bundle $E \to M$. Then:

$$\{ D' \mid D' \text{ connection on } M \}$$
$$= \{ D + A \mid A \in \mathrm{Hom}_{\mathcal{E}(M)}\left(\mathcal{A}^0(M,E), \mathcal{A}^1(M,E)\right) \}$$

Note, that

$$\mathrm{Hom}_{\mathcal{E}(M)}(\mathcal{A}^0(M,E), \mathcal{A}^1(M,E)) = \mathrm{Hom}_{\mathcal{E}(M)}(\Gamma(M,E), \Gamma(M, T^*M \otimes E))$$

$$\cong \Gamma(M, \mathcal{L}(E, \Omega^1(M,E))) \cong \Gamma(M, E^* \otimes T^*M \otimes E) \cong \Gamma(M, T^*M \otimes \mathcal{L}(E,E)) \cong \ldots$$

<u>Proof.</u> Let D' be an arbitrary connection and set $A := D' - D$. Then $A : \mathcal{A}^0(M,E) \to \mathcal{A}^1(M,E)$ is $\mathbb{K}$-linear since D and D' are $\mathbb{K}$-linear. It remains to show that A is linear with respect to $\mathcal{E}(M)$:

$$A(fs) = D'(fs) - D(fs) = df s + f D's - df s - f Ds$$
$$= f (D' - D)(s) = f As .$$

Conversely, for $A \in \mathrm{Hom}_{\mathcal{E}(M)}(\mathcal{A}^0(M,E), \mathcal{A}^1(M,E))$ the map $D' := D + A : \mathcal{A}^0(M,E) \to \mathcal{A}^1(M,E)$ is $\mathbb{K}$-linear and satisfies (D2):

$$D'(fs) = df s + f Ds + f As = df s + f D's . \qquad \square$$

<u>REMARK:</u> The set of all connections on $E \to M$ is therefore an affine space with translation group the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathcal{E}(M)}(\mathcal{A}^0(M,E), \mathcal{A}^1(M,E)) \cong \mathcal{A}^1(M, \mathcal{L}(E,E))$.

(23.5) EXAMPLES: 1° Every connection D on the trivial line bundle $M \times \mathbb{K}$ has the form

$$D = d + \alpha, \quad \alpha \in \mathcal{A}^1(M).$$

Indeed, according to 23.3.2° and our last result 23.4 we know $D = d + A$ with $A : \mathcal{E}(M) \to \mathcal{A}^1(M) = \Omega^*(M)$, and A is determined by $\alpha := A(1) \in \mathcal{A}^1(M)$.

In local coordinates $\alpha = \alpha_j dq^j$, $\alpha_j \in \mathcal{E}(W)$, and

$$Df = \frac{\partial f}{\partial q^j} dq^j + f \alpha_j dq^j = \left((\partial_j + \alpha_j) f \right) dq^j$$

$$\nabla_j := \nabla_{\partial_j} = \partial_j + \alpha_j \qquad j = 1, 2, \dots n$$

2° Let $E = M \times \mathbb{K}^r$ the trivial vector bundle of rank r. Choose a basis $e_1, \dots e_r \in \Gamma(M, E)$ given as $a \mapsto e_\rho(a) = (a, b_\rho(a))$, $b_\rho \in \mathcal{E}(M, \mathbb{K}^r)$, $\rho = 1, 2, \dots r$. In particular, $e_1(a), \dots, e_r(a)$ is a basis of the $\mathbb{K}$-vector space $E_a = \{a\} \times \mathbb{K}^r$.
Now, to each section $s \in \Gamma(M, E)$ there corresponds unique linear combination

$$s = s^\rho e_\rho, \quad s^\rho \in \mathcal{E}(M).$$

With
$$Ds := ds^\varrho e_\varrho \quad (= ds^\varrho \otimes e_\varrho)$$
a connection $D : \mathcal{A}^0(M,E) \to \mathcal{A}^1(M,E)$ is defined. Evidently, D is $\mathbb{K}$-linear (D1). Moreover,
$$D(fs) = d(fs^\varrho)e_\varrho = df\, s^\varrho e_\varrho + f\, ds^\varrho e_\varrho = df\, s + f\, Ds,$$
hence (D2). All the other connections on E have the form
$$D + A, \quad A \in \Gamma(M, \mathcal{L}(E, \Omega^1(M,E))).$$

<u>(23.6) LOCAL FORMULAS:</u> Let D be a connection on the $\mathbb{K}$-vector bundle E of rank $r \in \mathbb{N}$. Let $\mathcal{U} \subset M$ be an open subset such that $E|_{\mathcal{U}}$ is trivial. Then there is a basis $e_1, \dots e_r \in \Gamma(\mathcal{U}, E)$ of $\Gamma(\mathcal{U}, E)$ (over $\mathcal{E}(\mathcal{U})$) also called *frame* of E over $\mathcal{U}$. Now, each $De_\varrho \in \Gamma(\mathcal{U}, \Omega^1(TM, E))$ has the form
$$De_\varrho = A_\varrho^\sigma e_\sigma \; (= A_\varrho^\sigma \otimes e_\sigma) : \; a \longmapsto (X \longmapsto A_\varrho^\sigma(X) e_\sigma(a)),$$
with uniquely defined 1-forms $A_\varrho^\sigma \in \mathcal{A}^1(\mathcal{U})$. In particular,
$$\nabla_X e_\varrho = De_\varrho(X) = A_\varrho^\sigma(X) e_\sigma.$$

Let us denote the matrix (A_ϱ^σ) of one forms on $\mathcal{U}$ by A. Then $A \in \Gamma(\mathcal{U}, T^*M \otimes \mathcal{L}(E,E)) = \mathcal{A}^1(\mathcal{U}, \mathcal{L}(E,E))$. Or, with $\mathrm{End}(E) := \mathcal{L}(E,E)$: $A \in \mathcal{A}^1(\mathcal{U}, \mathrm{End}(E))$. We obtain for a general local section $s = s^\varrho e_\varrho \in \Gamma(\mathcal{U}, E)$:

$$
\begin{aligned}
D(s) &= D(s^\varrho e_\varrho) = \\
&= ds^\varrho e_\varrho + s^\varrho De_\varrho \\
&= ds^\sigma e_\sigma + s^\varrho A_\varrho^\sigma e_\sigma \\
&= (ds^\sigma + A_\varrho^\sigma s^\varrho) e_\sigma \quad , \text{ i.e}
\end{aligned}
$$

$$
\boxed{Ds = (ds + As)^\sigma e_\sigma},
$$

where $As = A_\varrho s^\varrho$, $(As)^\sigma = A_\varrho^\sigma s^\varrho$ $(= s^\varrho A_\varrho^\sigma)$. □

(23.7) DEFINITION: The 1-form $A \in \mathcal{A}^1(\mathcal{U}, \mathrm{End}(E))$ is called the (local) connection 1-form associated with a frame (s_ϱ) over $\mathcal{U}$.

(23.8) PROPOSITION: (Change of Frame) Any other frame $(\bar{s}_\varrho)$ of E over $\mathcal{U}$ is given by a unique smooth $g: \mathcal{U} \to GL(r, \mathbb{K})$, the change of frame:

$$\bar{e}_\rho = g\, e_\rho \quad \text{or} \quad \bar{e}_\rho = g_\rho^\sigma e_\sigma \,, \quad g = (g_\rho^\sigma)\,.$$

The connection one forms $\bar{A}$, A transform as follows:

$$* \quad \boxed{\begin{aligned} \bar{A} &= g^{-1} A g + g^{-1} dg \,, \quad \text{or} \\ \bar{A}_\sigma^\rho &= (g^{-1})_\mu^\rho \, A_\tau^\mu \, g_\sigma^\tau + (g^{-1})_\tau^\rho \, d g_\sigma^\tau \,. \end{aligned}}$$

<u>Proof</u>. We have

$$D\bar{e}_\sigma = \bar{A}_\sigma^\rho \bar{e}_\rho = \bar{A}_\sigma^\rho g_\rho^\tau e_\tau = g_\rho^\tau \bar{A}_\sigma^\rho e_\tau \,, \quad \text{and also}$$

$$D\bar{e}_\sigma = D(g_\sigma^\rho e_\rho) = dg_\sigma^\rho e_\rho + g_\sigma^\rho De_\rho = dg_\sigma^\tau e_\tau + g_\sigma^\rho A_\rho^\tau e_\tau = (A_\rho^\tau g_\sigma^\rho + dg_\sigma^\tau) e_\tau \,.$$

Thus

$$g_\mu^\tau \bar{A}_\sigma^\mu = A_\mu^\tau g_\sigma^\mu + dg_\sigma^\tau \quad \text{and multplying with } (g^{-1})_\tau^\rho$$

immediately gives the above formulas $*$. $\square$

<u>(23.9) REMARK:</u> The above change of frame is <u>the</u> example of a gauge transformation. And a collection of $\mathrm{End}(E)$-valued 1-forms (A) associated to the various frames have the interpretation of local potentials if they transform according to $*$ under an arbitrary

change of frame. We also say that the collection (A) is gauge invariant. Only gauge invariant 1-forms might be physical potentials.

(23.10) LOCAL FORMULAS II: In order to obtain local formulas in terms of coordinates let $\varphi = (q^1, \dots q^n) : U \to Q \subset \mathbb{R}^n$ be a chart. The forms A^σ_ϱ have the form

$$A^\sigma_\varrho = \Gamma^\sigma_{j\varrho}\, dq^j$$

with $\Gamma^\kappa_{j\mu} \in \mathcal{E}(U)$. The $\Gamma_{j\mu}$ are the connections coefficients or the Christoffel symbols of D. Altogether:

$$\boxed{\begin{aligned} De_\varrho &= \Gamma^\sigma_{j\varrho}\, dq^j e_\sigma &&= \Gamma^\sigma_{j\varrho}\, dq^j \otimes e_\sigma \\ De_\varrho(X) &= X^j \Gamma^\sigma_{j\varrho}\, e_\sigma &&= \nabla_X e_\varrho \\ \nabla_j e_\varrho &= De_\varrho\left(\frac{\partial}{\partial q^j}\right) &&= \Gamma^\sigma_{j\varrho}\, e_\sigma \end{aligned}} \quad \text{if } X = X^i \partial_i$$

and for general $X = X^j \partial_j$ and $s = s^\varrho e_\varrho$:

$$\begin{aligned} \nabla_X s = \nabla_X s^\varrho e_\varrho &= ds^\varrho(X)\, e_\varrho + s^\varrho \nabla_X e_\varrho \\ &= \partial_j s^\varrho X^j e_\varrho + s^\varrho X^j \Gamma^\sigma_{j\varrho} e_\sigma = \left(\partial_j s^\kappa + \Gamma^\kappa_{j\mu} s^\mu\right) X^j e_\kappa . \end{aligned}$$

Hence

$$\boxed{\nabla_X s = \left(\partial_j s + \Gamma_{j\rho}\, s^\rho\right)^\sigma X^j e_\sigma .}$$

$\square$

(23.11) PROPOSITION: Assume M has countable topology. Then every vector bundle $E \to M$ has a connection.

Sketch of a proof. We know that in the situation of an open subset $U \subset M$ such that $E|_U$ there exists a connection according to 23.5.2°. These connections can be glued together by a smooth partition of unity to yield a connection on M.

$\square$

(23.12) DEFINITION: Let D be a connection on $\pi: E \to M$.

1° A section $s \in \Gamma(W, E)$ is called horizontal (or parallel) if $Ds = 0$.

Let $\gamma: [t_0, t_1] \to M$ (piecewise smooth) curve.

2° $\beta: [t_0, t_1] \to M$ is called a lift of γ if $\pi \circ \beta = \gamma$ and if β is piecewise smooth.

3° A lift is called horizontal (or parallel), if

$$\nabla_{\dot{\gamma}} \beta = 0.$$

Evidently, if $s \in \Gamma(W,E)$ is a horizontal section and $\gamma : [t_0, t_1] \to W$ is a curve, then $\beta := s \circ \gamma$ is a horizontal lift of γ.

<u>(23.13) PROPOSITION:</u> Let D be a connection on $E \to M$, $\varphi : E|_U \to U \times \mathbb{K}^r$ a local trivialization and $\gamma : [t_0, t_1] \to U$ a curve in U. Any lift β of γ has the form $\beta(t) = \varphi^{-1}(\gamma(t), \eta(t))$ with η a curve in $\mathbb{K}^r$. The lift β is horizontal if and only if

$$\dot{\eta} + A(\dot{\gamma})\eta = 0,$$

with A as in 23.6 (local connection form).

<u>Proof.</u> We use the sections $e_\varrho(a) := \varphi^{-1}(a, \hat{e}_\varrho)$ $(\hat{e}_\varrho = (\delta_\varrho^\sigma))$, and have $De_\varrho = A_\varrho^\sigma e_\sigma$, $A = (A_\varrho^\sigma)$, $\beta = \eta^\varrho e_\varrho$. Similar calculations as in 23.6 show that

$$D\beta = D(\eta^\varrho e_\varrho) = (\dot{\eta}^\sigma + A_\varrho^\sigma \eta^\varrho) e_\sigma .$$

Hence, $\nabla_{\dot{\gamma}} \beta = 0 \iff \dot{\eta}^\sigma + A_\varrho^\sigma(\dot{\gamma})\eta^\varrho = 0, \ \sigma = 1, \dots r.$

$\square$

(23.14) COROLLARY: Let D be a connection on $E \to M$ and let $\gamma : [t_0, t_1] \to M$ be a curve through a point $a \in M$, $a = \gamma(t_*)$, $t_* \in \,]t_0, t_1[$. To every $s_* \in E_a$ there exists a unique horizontal lift β of γ with $\beta(t_*) = s_*$.

Proof. $\dot{\eta} + A(\dot{\gamma})\eta = 0$ with $\eta(t_*) \in \mathbb{K}^r$ given by $\varphi(s_*) = (a, \eta(t_*))$ is an initial value problem with a unique (local) solution.

$\square$