22. Semi-Riemannian Geometry

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The objective of this section is to introduce geometric concepts on the special vector bundle TM over M. These geometric concepts will be generalized to arbitrary vector bundles in the next sections. In particular, we shall come to the notion of the Levi-Civita connection of a metric on the tangent bundle $TM \to M$ leading to the concept of a general connection on a vector bundle $E \to M$ in the next section.

(22.1) **DEFINITION**: A semi-Riemannian mfd is a mfd M together with a tensor $g \in \mathcal{T}_2^0(M)$* with

1° $g : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{E}(M)$ is symmetric, i.e. $g(X,Y) = g(Y,X)$ for all $X, Y \in \mathcal{V}(M)$.

2° g is non-degenerate, i.e for $X \in \mathcal{V}(M)$: $g(X,Y) = 0$ for all $Y \in \mathcal{V}(M) \Rightarrow X = 0$.

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* By definition, $B : \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{E}(M)$ is linear over $\mathcal{E}(M)$ in each argument, cf. section 10.

g is called the metric or metric tensor („Maßtensor" in german) of the semi-Riemannian manifold (M,g).

Already the case of a manifold Q given as an open subset $O \subset \mathbb{R}^n$ is quite interesting. We treat this case as the case of a chart

$$\varphi = (q^1, \dots q^n) : \mathcal{U} \to Q \subset \mathbb{R}^n$$

of a general manifold M with $\mathcal{U} \subset M$ open.
In these local coordinates $(q^1, \dots, q^n)$ the restriction $g|_{\mathcal{U}}$ of the metric tensor is determined by the coefficients

$$g_{ij} = g\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) \in \mathcal{E}(\mathcal{U}) :$$

$$g|_{\mathcal{U}} = g_{ij}\, dq^i \otimes dq^j \quad (\text{cf. section } 10).$$

The matix $(g_{ij}(a)) \in \mathbb{R}^{n \times n}$ is symmetric and invertible.

In classical mechanics such a matrix may given as

„mass" matrix (g_{ij}). In many cases such a mass matrix g will be positive definite, i.e.

$$g(X,X) > 0 \text{ for } X \neq 0.$$

A pos. def. semi-Riem. metric is a Riemannian metric.

It is a Lorentz-metric if to each point $a \in M$ there exists a chart with $(g_{ij}(a)) = \mathrm{diag}(-1,+1,..+1)$

or $= \mathrm{diag}(+1,-1,..-1)$

(22.2) Calculus of Variations: Let $L : TM \to \mathbb{R}$ be a smooth function, called Lagrangian. (M,L) is then a Lagrangian system. One wants (originally), to find (piecewise smooth) curves $\gamma : [t_0,t_1] \to M$ from $a = \gamma(t_0)$ to $b = \gamma(t_1)$ such that

$$\int_{t_0}^{t_1} L(\gamma(t))dt \equiv \text{minimal}$$

$$\text{or} \equiv \text{maximal}$$

A necessary condition for this to hold is

$$\boxed{\frac{d}{d\varepsilon} \int_{t_0}^{t_1} L(\Gamma(t,\varepsilon))\, dt \Big|_{\varepsilon=0} = 0}$$

where $\Gamma(t,\varepsilon)$ is a variation of $\gamma(t)$, i.e. for example a continuous map

$$\Gamma : [t_0, t_1] \times]-r_0, r_0[\to M \quad , \; r_0 > 0,$$

being smooth in ε and piecewise smooth in t with $\gamma(t) = \Gamma(t,0)$ and $\Gamma(t_0,\varepsilon) = a$, $\Gamma(t_1,\varepsilon) = b$, for all $\varepsilon \in]-r_0, r_0[$.

By definition, γ is a motion of the Lagrange system (M,L) if

$$\frac{d}{d\varepsilon} \int L(\dot{\Gamma}(t,\varepsilon))\, dt \Big|_{\varepsilon=0} = 0 \quad .$$

The curve is then a stationary point in the space of curves from a to b.

(22.3) PROPOSITION: γ is a motion of (M,L) $\iff$ for all charts $\varphi : U \to Q$ the Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\frac{\partial \hat{L}}{\partial v} = \frac{\partial \hat{L}}{\partial q}} \quad (\text{in } Q\times\mathbb{R}^n = TQ)$$

are satisfied.
Here, $\hat{L}(q,v) := L(T_q\psi(v))$, $(q,v)\in Q\times\mathbb{R}^n$, where $\psi = \varphi^{-1} : Q \to U$, $T_q\psi : T_qQ \to T_{\psi(q)}M$.

<u>Proof</u>: We restrict to the case of

$$\Gamma(t,\varepsilon) = \psi\left(\varphi\circ\gamma(t) + \varepsilon h(t)\right)$$

with $h : [t_0,t_1] \to \mathbb{R}^n$ smooth satisfying $h(t_0) = h(t_1) = 0$:
Let $q(t) = \varphi\circ\gamma(t)$, then $\dot\Gamma(t,\varepsilon) = T_q\psi(\dot q + \varepsilon\dot h)$ and

$$\hat{L}(q+\varepsilon h, \dot q + \varepsilon \dot h) = L(T_q\psi(\dot q + \varepsilon\dot h)) = L(\dot\Gamma(t,\varepsilon)).$$

The condition $0 = \frac{d}{d\varepsilon}\int_{t_0}^{t_1} L(\dot\Gamma(t,\varepsilon))\,dt\Big|_{\varepsilon=0}$ implies

$$0 = \frac{d}{d\varepsilon}\int_{t_0}^{t_1}\hat{L}(q+\varepsilon h, \dot q + \varepsilon\dot h)\,dt\Big|_{\varepsilon=0} = \int_{t_0}^{t}\left(\frac{\partial \hat{L}}{\partial q^\mu}h^\mu + \frac{\partial\hat{L}}{\partial v^\mu}\dot h^\mu\right)dt$$

Because of

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^\mu}h^\mu\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial v^\mu}\right)h^\mu + \frac{\partial L}{\partial v^\mu}\dot h^\mu$$

we obtain by partial integration and the

$$\Rightarrow 0 = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q^\mu}h^\mu - \frac{d}{dt}\left(\frac{\partial L}{\partial v^\mu}\right)h^\mu\right)dt = \int_{t_0}^{t_1}\left(\frac{\partial L}{\partial q^\mu}h^\mu - \frac{d}{dt}\left(\frac{\partial L}{\partial v^\mu}\right)\right)h^\mu\,dt$$

"boundary condition" $h^\mu(t_0) = h^\mu(t_1) = 0$ that

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial \hat{L}}{\partial q^\mu} h^\mu - \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial v^\mu}\right) h^\mu \right) dt = \int_{t_0}^{t} \left(\frac{\partial \hat{L}}{\partial q^\mu} - \frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial v^\mu}\right)\right) h^\mu dt .$$

Since this has to hold for all such h we conclude

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^\mu}\right) = \frac{\partial L}{\partial q^\mu} , \quad \mu = 1, \dots n \quad . \qquad \square$$

(22.4) PROPOSITION: Let (M, g) be a semi-Riemannian mfd with $L(X) = \frac{1}{2} g(X, X)$, $X \in TM$, as its Lagrangian. Then:

γ is a motion of (M, L) <=> In local coordinates $\boxed{\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0 .}$

where

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\mu} \left(g_{i\mu,j} + g_{j\mu,i} - g_{ij,\mu} \right) , \quad g_{k\ell,\nu} = \frac{\partial}{\partial q^\nu} g_{k\ell} ,$$

with $(g^{k\mu})$ being the inverse matrix of (g_{ij}).

Γ^k_{ij} are the Christoffel symbols.

Proof: We have to check that

$$\ddot{\gamma}^k + \Gamma^k_{ij}\dot{\gamma}^i\dot{\gamma}^j = 0$$

are the Euler-Lagrange equations for the local Lagrangian

$$\hat{L}(q,v) = \frac{1}{2} g_{ij}(q)\, v^i v^j :$$

$$\frac{\partial \hat{L}}{\partial v^\mu} = g_{\mu j} v^j \quad , \quad \frac{\partial \hat{L}}{\partial q^\mu} = \frac{1}{2} g_{ij,\mu}(q)\, v^i v^j$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial v^\mu}(\gamma,\dot{\gamma})\right) = \frac{d}{dt}\left(g_{\mu j}(\gamma)\,\dot{\gamma}^j\right) = g_{\mu j,\nu}(\gamma)\,\dot{\gamma}^\nu\dot{\gamma}^j + g_{\mu j}(\gamma)\,\ddot{\gamma}^j$$

$\Rightarrow$

$$g^{k\mu}\left(\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial v^\mu}(\gamma,\dot{\gamma})\right) - \frac{\partial L}{\partial q^\mu}\right) =$$

$$g^{k\mu} g_{\mu j,\nu}\,\dot{\gamma}^\nu\dot{\gamma}^j + g^{k\mu} g_{\mu j}\,\ddot{\gamma}^j - \frac{1}{2} g^{k\mu} g_{ij,\mu}(q)\,\dot{\gamma}^i\dot{\gamma}^j = 0$$

$$\Rightarrow \quad \ddot{\gamma}^j + \frac{1}{2} g^{k\mu}\left\{ g_{j\mu,i} + g_{i\mu,j} - g_{ij,\mu}\right\}\dot{\gamma}^i\dot{\gamma}^j = 0 \qquad \square$$

(22.5) Corollary: At every point $a \in M$ and for all directions $X \in T_aM$ there exists a unique motion γ of (M,L) with $\gamma(0) = a$ and $\dot{\gamma}(a) = X$.

This is an immediate consequence of the

Picard-Lindelöf theorem on the existence and uniqueness of initial value problems with smooth coefficients.

(22.6) DEFINITION: Let (M, g) be a Riemannian mfd.

$1°$ $\gamma : [t_0, t_1] \to M$ piecewise smooth is naturally parametrized if

$$I_{t_0}^t \gamma := \int_{t_0}^t \sqrt{g(\dot\gamma, \dot\gamma)}\, dt = t - t_0 \quad \text{for } t \in [t_0, t_1]$$

$2°$ γ is a geodesic if γ nat. param. and if γ is a motion of (M, L^*), $L^*(X) = \sqrt[2]{g(X,X)}$ (arclength).

(22.7) Lemma. $1°$ γ nat. param. $\iff g(\dot\gamma, \dot\gamma) = 1$.

$2°$ If γ is regular, i.e. smooth and $\dot\gamma(t) \neq 0$ always, then there is a param. $\varphi : [s_0, s_1] \to [t_0, t_1]$ s.th $\tilde\gamma := \gamma \circ \varphi$ is nat. param.

Proof: $1°$ ✓ $2°$ $\beta(t) := \int_{t_0}^t \sqrt{g(\dot\gamma, \dot\gamma)}\, dt$. Then $\dot\beta(t) = \sqrt{g(\dot\gamma(t), \dot\gamma(t))} > 0$. Hence $\beta : [t_0, t_1] \to [0, B]$, $B = \beta(t_1)$, is invertible. Let $\varphi := \beta^{-1} : [0, B] \to [t_0, t_1]$ & $\tilde\gamma := \gamma \circ \varphi$.
We obtain $\dot{\tilde\gamma} = \dot\gamma(\varphi)\dot\varphi = \dot\gamma(\varphi)\frac{1}{\dot\beta}$, $g(\dot{\tilde\gamma}, \dot{\tilde\gamma}) = g(\dot\gamma, \dot\gamma)\frac{1}{\dot\beta^2} = 1$

(22.8) PROPOSITION: Let $\gamma : [t_0, t_1] \to M$ be a naturally parametrized curve in a Riemannian manifold (M, g). Then

$$\gamma \text{ geodesic} \iff \gamma \text{ motion of } (M, \tfrac{1}{2} g)$$
$$\iff \ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0$$

Proof: $L = \frac{1}{2} g$ and $L^* = \sqrt{g} = \sqrt{2L}$.

$$\frac{\partial L^*}{\partial v} = \frac{1}{2} \cdot \frac{1}{L^*} 2 \frac{\partial L}{\partial v} \text{ in general and } \frac{\partial L^*}{\partial v} = \frac{\partial L}{\partial v} \quad \text{for } L^* = 1$$

$$\frac{\partial L^*}{\partial q} = \frac{1}{2} \frac{1}{L^*} 2 \frac{\partial L}{\partial q} \quad \text{" "} \quad \frac{\partial L^*}{\partial q} = \frac{\partial L}{\partial v} \quad \text{"}$$

$\square$

Remark: 1° Note that the motions of $L = \sqrt{g}$ and of $\frac{1}{2} L^2$ are essentially the same!

2° Christoffel symbols are not tensors!

But they define a connection (a covariant derivative)!

Let $\gamma(t)$ be a curve and $Y(t)$ a vector field along $\gamma(t)$, i.e. $Y(t) \in T_{\gamma(t)} M$. Set

$$\nabla_{\dot{\gamma}(t)} Y(t) := \left(L_{\dot{\gamma}} Y^k(t) + \Gamma^k_{ij} \dot{\gamma}^i Y^j \right) \frac{\partial}{\partial q^k} .$$

This defines a map

$$\nabla_{\dot{\gamma}(t)} : \mathfrak{X}(W) \to \mathfrak{X}(W) \quad \text{the } \underline{\text{covariant derivative}}.$$

(22.10) DEFINITION: An (affine) covariant derivative on a manifold M is a map

$$\nabla : \mathcal{V}(W) \times \mathcal{V}(W) \longrightarrow \mathcal{V}(W) \quad \text{for all } W \subset M \text{ open}$$

such that

1° $X \longmapsto \nabla_X Y := \nabla(X,Y)$ is $\mathcal{E}(W)$-linear for fixed $Y \in \mathcal{V}(W)$: $\quad \nabla_{fX+X'} Y = f\nabla_X Y + \nabla_{X'} Y$

2° $Y \longmapsto \nabla_X Y$ is $\mathbb{R}$-linear and

$$\boxed{\nabla_X (fY) = (L_X f) Y + f\,\nabla_X Y}$$

Moreover, ∇ is compatible to restrictions $W' \subset W$.

(22.11) Examples: 1° ∇ with the Γ^k_{ij} as above related to (M,g): $\quad \nabla_{X(t)} Y(t) = \left(L_X Y^k + \Gamma^k_{ij} X^i Y^k\right) \frac{\partial}{\partial q^k}$

• in X $\mathcal{E}(W)$-lin, • in Y a derivation.

$\nabla = \nabla^g$ is called the Levi-Civita covariant derivative (resp. connection). Note, that the Levi-Civita connection $\nabla = \nabla^g$ is completely determined by

$$\boxed{\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k ,}$$

and it can be defined by this formula, once the

Christoffel symbols Γ_{ij}^{k} are given locally.

The Levi-Civita derivative respects the metric g in the following sense: For all vector fields $X, Y, Z \in \mathcal{V}(W)$, $W \subset M$ open, we have

$$L_X\, g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

This has to checked on $\partial_i, \partial_j, \partial_k$ only:

$$g(\nabla_{\partial_k}\partial_i, \partial_j) + g(\partial_i, \nabla_{\partial_k}\partial_j) =$$

$$= g(\Gamma_{ki}^{\ell}\partial_\ell, \partial_j) + g(\partial_i, \Gamma_{kj}^{\ell}\partial_\ell) = \Gamma_{ki}^{\ell} g_{\ell j} + \Gamma_{kj}^{\ell} g_{i\ell}.$$

$$\Gamma_{ki}^{\ell} g_{\ell j} = \frac{1}{2} g^{\ell\mu} g_{\ell j}\left\{g_{k\mu,i} + g_{i\mu,k} - g_{ki,\mu}\right\} = \frac{1}{2}\left(g_{kj,i} + g_{ij,k} - g_{ki,j}\right)$$

(since $g^{\ell\mu} g_{\ell j} = \delta_j^{\mu}$). Therefore,

$$\Gamma_{ki}^{\ell} g_{\ell j} + \Gamma_{kj}^{\ell} g_{i\ell} = g_{ij,k} = \partial_k g_{ij} = L_{\frac{\partial}{\partial q^k}}\, g(\partial_i, \partial_j). \qquad \square$$

2° Let M parallelizable, i.e. TM trivial. There are $B_1, \dots B_n \in \mathcal{V}(M)$, such that $(B_1, \dots B_n)$ is a Basis of $\mathcal{V}(W)$: Each $Y \in \mathcal{V}(W)$ is of the form $Y = Y^j B_j$ with uniquely defined $Y^j \in \mathcal{E}(W)$.

$$\nabla_X Y := (L_X Y^k) B_k|_W$$

This defines a covariant derivative! It depends on the choice of the basis $B_1, \dots B_n$ of $\mathcal{V}(M)$. We see

$$\nabla_X B_k = 0 \quad \forall X \in \mathcal{V}(M):$$

B_k is parallel!

(22.12) DEFINITION: 1° Given a covariant derivative ∇ on TM a section $Y \in \mathcal{V}(W) = \Gamma(W, TM)$ is called parallel $:\Longleftrightarrow \nabla_X Y = 0 \ \forall X \in \mathcal{V}(W)$.

2° A vector field along a curve $\gamma : [t_0, t_1] \to M$ is a (piecewise) smooth $Y : [t_0, t_1] \to TM$ with $\gamma = \tau \circ Y$.
$Y(t)$ parallel along $\gamma(t)$ $:\Longleftrightarrow \nabla_{\dot\gamma} Y = 0$.

3° A curve γ in M is called autoparallel with respect to a given covariant derivative if and only if

$$\nabla_{\dot\gamma} \dot\gamma = 0,$$

i.e. if $\dot\gamma$ is parallel along γ.

Remarks: 1° If Y is parallel, $Y \in \mathcal{V}(W)$. Then for every curve

$\gamma : [t_0, t_1] \to W \subset M$ the „lift" $Y(t) := Y(\gamma(t))$ is parallel along γ.

2° If $\gamma : [t_0, t_1] \to W$ is autoparallel it satisfies the equation

$$\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = 0,$$

where Γ^k_{ij} is defined by $\nabla_{\partial_i} \partial_j := \Gamma^k_{ij} \partial_j$. Consequently, in the case of the Levi-Civita connection of a semi-Riemannian manifolg (M, g) the autoparallel curves are the motions of $(M, \frac{1}{2} g)$, cf. 22.4.

<u>(22.13) DEFINITION:</u> Let ∇ be a cov. derivative. The torsion of ∇ is

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y], \quad \text{for } X, Y \in \mathcal{V}(M).$$

T is a tensor $\in \mathcal{T}^0_2(M, TM) = \mathrm{Hom}_{\mathcal{E}(W)}(\mathcal{V}(M), \mathcal{V}(M); \mathcal{V}(M))$

<u>(22.14) EXAMPLES:</u> 1° Γ^k_{ij} given by $\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$ (locally)

$T = 0 \iff \Gamma^k_{ij} = \Gamma^k_{ji} \quad (\iff T(\partial_i, \partial_j) = 0)$

2° Levi-Civita connection ∇^g: $\Gamma^k_{ij} = \Gamma^k_{ji} \implies T = 0$.

3° $B_1, .. B_n$: $T = 0 \iff T(B_i, B_j) = 0 \iff [B_i, B_j] = 0$ (e.g. $B_j = \frac{\partial}{\partial q^j}$ on $M \subset \mathbb{R}^n$ open). But if $[B_j, B_k] \neq 0$: $T \neq 0$.

<u>(22.15) FUNDAMENTAL THEOREM</u> of semi-Riemannian geometry:
For a semi-Riemannian mfd (M,g) there exists exactly one torsionfree connection ∇ with

$$\boxed{L_X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)}$$ for all $X,Y,Z \in \mathcal{V}(M)$.

<u>Proof</u>: $A(X,Y,Z) := L_X g(Y,Z) + L_Y g(X,Z) - L_Z g(X,Y)$

Assume ∇ is such a connection. Then

$$\begin{aligned} A(X,Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y X, Z) + g(X, \nabla_Y Z) \\ &\qquad - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) = \\ &= 2g(\nabla_X Y, Z) + g(\nabla_Y X - \nabla_X Y, Z) + g(\nabla_Y Z - \nabla_Z Y, X) \\ &\qquad + g(\nabla_X Z - \nabla_Z X, Y) = \\ &= 2g(\nabla_X Y, Z) + g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X) \end{aligned}$$

$$\Rightarrow \quad g(\nabla_X Y, Z) = \frac{1}{2}\Big(A(X,Y,Z) + g([Y,X], Z) + g([X,Z], Y) + g([Y,Z], X) \Big)$$

This implies the uniqueness of ∇ (or $\nabla_X Y$), since g is non-degenerated.

At the same time the formula shows the existence since it may be used as a definition. $\square$

<u>Final remark</u>: According to 22.11.1° the covariant derivative in the theorem is the Levi-Civita derivative.