21. Classifying Map

Version 1.1

Notiztitel

Let $G_{r,m}(\mathbb{K})$ be the Grassmannian manifold of all r-dimensional linear subspaces $Y \subset \mathbb{K}^m$ (for $r, m \in \mathbb{N}$, $r \le m$), and let

$$\pi : U_{r,m} \to G_{r,m}(\mathbb{K})$$

denote the tautological bundle over $G_{r,m}(\mathbb{K})$: The fibre of $U_{r,m}$ over a "point" $Y \in G_{r,m}(\mathbb{K})$ is Y.

(21.1) THEOREM: Let E be a vector bundle over M of rank r. There always exists a smooth map

$$g : M \to G_{r,m}(\mathbb{K})$$

such that E is (isomorphic to) the pullback $g^* U_{r,m}$.

Proof: According to the subsequent Lemma (21.4) there exists $f \in \mathcal{E}(B, \mathbb{K}^m)$ for a suitable m s.th. $f_a := f|E_a$ is a monomorphism for each $a \in M$. f induces a map $g : M \to G_{r,m}(\mathbb{K})$, $g(a) := f(E_a)$. g is smooth.

Let us define a vector bundle homomorphism (over g)

$$h : E \to \mathcal{U}_{r,m}$$

by

$$h(v) := f(v) \in f(E_a) = g(a) = (\mathcal{U}_{r,m})_{g(a)}, \quad v \in E_a .$$

Then $h_a := h|_{E_a} : E_a \to (\mathcal{U}_{r,m})_{g(a)}$ is injective and linear, hence an isomorphism of vector spaces and h is smooth with $\pi \circ h = g \circ \pi_E$:

$$\begin{array}{ccc} E & \xrightarrow{h} & \mathcal{U}_{r,m} \\ \pi_E \downarrow & & \downarrow \pi \\ B & \xrightarrow[g]{} & G_{r,m}(\mathbb{K}) \end{array}$$

Hence, by 16.5 E is isomorphic to the pullback $g^*\mathcal{U}_{r,m}$. $\square$

<u>Remark:</u> The proof is not quite complete, the smoothness of the maps g and h is not checked. In order to do this we have to explain the smooth structure of $G_{r,m}$ and $\mathcal{U}_{r,m}$. This should be done somewhere, or left as an exercise.

The map $g : E \to \mathcal{U}_{r,m}$ with $g^*\mathcal{U}_{r,m} \cong E$ is a classifying map.

(21.2) <u>PROPOSITION:</u> A vector bundle E over M which is a subbundle of a vector bundle H over M always has a complement F, i.e. a subbundle F of H such that $E \oplus F = H$.

This can be shown with the aid of a hermitian (or euclidean) metric on the vector bundle H (cf. section 26), but also with the basic facts about subbundles and quotient bundles of vector bundles which are described below.

(21.3) <u>PROPOSITION:</u> Every vector bundle E is the subbundle of a trivial vector bundle of a suitable rank, i.e. there is a monomorphism $\alpha : E \longrightarrow M \times \mathbb{K}^m$ (in (vect_B), i.e. α is smooth, $\pi_E = pr_1 \circ \alpha$ and α is linear in the fibres as well as injective).

<u>Proof:</u> According to ~~proposition~~ 19.3 the $\mathcal{E}(M)$-module $\Gamma(M,E)$ is finitely generated, by $s_1, s_2, \ldots, s_m \in \Gamma(M,E)$ say. Then $\beta : M \times \mathbb{K}^m \to E$, $\beta(a,\lambda) := \lambda^j s_j$, defines

a smooth map, $\pi_E \circ \beta = pr_1$ and linear in the fibres. Hence β is a vector bundle homomorphism. Moreover, β is surjective (and hence called an *epimorphism*), since the s_j generate $\Gamma(M,E)$ by 19.2.2°. It follows, that there exists a monomorphism $\alpha : E \to M \times \mathbb{K}^m$ with $\beta \circ \alpha = id_E$. This can be shown by again introducing a hermitian (or euclidean) metric on $M \times \mathbb{K}^m = H$, or by a general discussion of subbundles and quotient bundles (see below).

(21.4) COROLLARY: Let $E \to M$ be a vector bundle. Then there exists $f \in \mathcal{E}(M, \mathbb{K}^m)$ for some $m \in \mathbb{N}$, such that the restrictions $f|_{E_a} : E_a \to \mathbb{K}^m$ are all linear and injective.

Proof: We use the monomorphism $\alpha : E \to M \times \mathbb{K}^m$ (cf. 21.3) and set $f := pr_2 \circ \beta$. Or we use an isomorphism $\varphi : E \oplus F \to M \times \mathbb{K}^m$ (cf. 21.2 and 21.3) and set $f := pr_2 \circ \varphi|_E$.

We finally describe some basic results about subbundles and quotient bundles.

(21.5) DEFINITION: A (vector) subbundle F of a vector bundle E over M is a submanifold $F \subset E$ such that $\pi_E|_F : F \to M$ defines the structure of a vector bundle, or slightly more general, a subbundle is an injective morphism $\varphi : F \to E$ (i.e. a monomorphism in (vect_M)).

Remark: If $\varphi : F \to E$ is a monomorphism, then $\varphi(F) \subset E$ is a subbundle in E.

(21.6) Fact: More generally, for a general morphism $\varphi : F \to E$ (in (vect_M)), $\varphi(F) \subset E$ is a subbundle if all $\varphi_a(F_a) = \mathrm{im}\,\varphi_a \subset E_a$, $a \in M$, have the same dimension. This condition is equivalent to $a \longmapsto \mathrm{rk}\,\varphi_a$ being constant. $\varphi(F)$ is denoted by $\mathrm{im}\,\varphi$.

(21.7) EXAMPLE: The morphism $M \times \mathbb{K} \to M \times \mathbb{K}$, $(r,y) \longmapsto (r, ry)$, over $M = \mathbb{R}$ does not have constant rank.

(21.8) <u>Conclusion</u>: 1° Let $\varphi : H \to E$ be an epimorphism of vector bundles. Then $\operatorname{Ker}\varphi \subset H$ is a subbundle.

2° Let $F \subset H$ be a subbundle in H, then the quotient $H/_F$ is a vector bundle over M with $\operatorname{rk} H/_F = \operatorname{rk} H - \operatorname{rk} F$. The fibres of $H/_F$ are $(H/_F)_a = H_a/_{F_a}$.

(21.9) <u>PROPOSITION:</u> Let $\varphi : H \to E$ be an epimorphism of vector bundles over E then there is an isomorphism $\beta : H \to F \oplus E$, $\beta(v) := \alpha(v) \oplus \varphi(v)$, $v \in H$. Here $F = H/_{\operatorname{Ker}\varphi}$.

<u>Proof</u>: It is easy to see that β is smooth and fibrewise a linear isomorphism.

<u>REMARK:</u> The last statement can be reformulated in the following way: For every <u>exact sequence</u>

$$0 \to E \to F \to G \to 0$$

of vector bundles we have $F \cong E \oplus G$.