16. Transition Functions

Let $\xi = (T, \pi, B, F)$ be a (locally trivial) fibration. There exists an open covering $(U_j)_{j\in I}$ of the base manifold B such that there are local trivializations

$$\psi_j : \pi^{-1}(U_j) \longrightarrow U_j \times F \qquad \text{for each } j \in I.$$

For every $a \in U_j$ the restriction to $T_a = F_a = \pi^{-1}(a)$

$$\psi_{j,a} = pr_2 \circ \psi_j \big|_{T_a} : T_a \longrightarrow F$$

is a diffeomorphism (cf. 15.2.1°) and one can recover ψ_j by

$$\psi_j(t) = (\pi(t), \psi_{j,\pi(t)}(t)) \quad , \ t \in U_j .$$

For $a \in U_i \cap U_j$ the "transition"

$$\psi_{ij}(a) := \psi_{i,a} \circ \psi_{j,a}^{-1} : F \longrightarrow F$$

turns out to be a diffeomorphism, and the collection of all $\psi_{ij}(a)$ determine the fibration, as we will see in the sequel. Moreover,

$$\psi_i \circ \psi_j^{-1}(a,y) = (a, \psi_{ij}(a).y) \quad , (a,y) \in (U_i \cap U_j) \times F.$$

Let $\mathrm{Diff}(F)$ be the group of all diffeomorphisms of F.
The ψ_{ij} are mappings

$$\psi_{ij} : U_{ij} \longrightarrow \mathrm{Diff}(M)$$

which are called the transition functions (of ξ with respect to the covering (U_j)).

(16.1) Proposition: For ξ and (U_i) as above we have:

1° $(a,y) \longmapsto \psi_{ij}(a).y = \psi_{ij}(a)(y)$ is a smooth map $U_i \cap U_j \times F \longrightarrow F$.

2° The ψ_{ij} satisfy the cocycle condition:

$$\boxed{\begin{aligned} &\psi_{ii} = id_F \\ &\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = id_F \quad \text{on } U_i \cap U_j \cap U_k \neq \emptyset. \end{aligned}}$$

Conversely,

(16.2) Proposition: Let B and F be manifolds and let $(U_j)_{j\in I}$ an open cover of B. Let $\psi_{ij} : U_i \cap U_j \to \mathrm{Diff}(F)$ be a collection of maps satisfying 1° and 2° of (16.1).

Then there exists a fibration $\xi = (T, \pi, B, F)$ with ψ_{ij} as transition functions, which is unique up to isomorphism.

<u>Proof</u>: $S := \bigcup \{ U_i \times F \times \{i\} : i \in I \}$ (disjoint union). On S we define the equivalence relation

$$(a, y, i) \sim (b, y', j)$$
$$:\Longleftrightarrow a = b \in U_i \cap U_j \text{ and } y = \psi_{ij}(a)\, y'.$$

This is in fact an equivalence relation because of 2°. For example, symmetry: From $(a, y, i) \sim (b, y', j)$ we obtain $a = b$ & $y = \psi_{ij}(a).y'$. By 2° $\psi_{ji} \circ \psi_{ij} \circ \psi_{jj} = id_F$ and $\psi_{jj} = id_F$. Hence $\psi_{ji}^{-1} = \psi_{ij}$. Hence $y' = \psi_{ji}(a).y$, which implies $(b, y', j) \sim (a, y, i)$.

Now let $T := S/\sim$ with $\pi : T \to B$, $[(a, y, i)] \mapsto a$, as the projection. On $\pi^{-1}(U_i) = \{ [(a, y, i)] : (a, y) \in U_i \times F \}$ we obtain the bijective map

$$\psi_i : \pi^{-1}(U_i) \to U_i \times F, \; [(a, y, i)] \mapsto (a, y),$$

satisfying $pr_1 \circ \psi_i = \pi|_{\pi^{-1}(U_i)}$.

On $U_i \cap U_j \neq \emptyset$ the map

$$\psi_i \circ \psi_j^{-1} : (U_i \cap U_j) \times F \longrightarrow (U_i \cap U_j) \times F$$

has the form $(a, y) \longmapsto (a, \psi_{ij}(a).y)$. Hence it is smooth as well as $\psi_j \circ \psi_i^{-1}$, consequently $\psi_i \circ \psi_j^{-1}$ is a diffeomorphism.

The set $T = S/_\sim$ will be equipped with the quotient topology. We get for $V \subset \pi^{-1}(U_j)$:

$$V \subset T \text{ open } \Longleftrightarrow \psi_i(V) \subset U_i \times F \text{ open}$$

and for a general $V \subset T$:

$$V \subset T \text{ open } \Longleftrightarrow \forall i \in I : \psi_i(V \cap U_i) \subset U_i \times F \text{ open}.$$

As a result, T is a Hausdorff space (and metrizable B and F are metrizable, connected if B and F are connected etc.). The ψ_i define a smooth structure on T (by bundle charts) such that $\xi = (T, \pi, B, F)$ is a fibration with (ψ_{ij}) as transition functions. Note, that T is the quotient of S in the category (mfd). If there is another fibration $\xi' = (T', \pi', B, F)$ with the same transition functions one obtains an isomorphism $h: T \to T'$ with $\pi = \pi' \circ h$ by the next proposition. □

(16.3) Proposition: Let $\xi = (T, \pi, B, F)$ and $\xi' = (T', \pi', B, F)$ be fibrations over B with identical fibre F and let $(U_j)_{j \in I}$ an open covering of B such that there exists local trivializations ψ_j, ψ_j' of ξ, ξ' over U_j, $j \in I$:

$$\psi_j : \pi^{-1}(U_j) \xrightarrow{\sim} U_j \times F,$$
$$\psi_j' : \pi'^{-1}(U_j) \xrightarrow{\sim} U_j \times F.$$

Under these circumstances a fibre preserving diffeomorphism $h: T \to T'$ with $\pi = \pi' \circ h$ (i.e. an isomorphism in (fib_B^F)) exists if and only if there is a family (h_j) of maps

$$h_j : U_j \to \mathrm{Diff}(F)$$

with

$$\boxed{\begin{array}{l} 1^\circ \quad U_j \times F \longrightarrow F, \ (a,y) \longmapsto h_j(a).y, \text{ is smooth and} \\ 2^\circ \quad \psi_{ij}' = h_i \circ \psi_{ij} \circ h_j^{-1}, \quad \text{for all } i,j \in I \end{array}}$$

where ψ_{ij}, ψ'_{ij} are the transition functions $U_i \cap U_j \to F$.

Proof: Given (h_j) with 1° and 2° set

$$H_j : \bar{\pi}^{-1}(U_j) \to (\bar{\pi}')^{-1}(U_j) \;,\; \psi_j^{-1}(a,y) \mapsto (\psi'_j)^{-1}(a, h_j(a).y),$$

$(a,y) \in U_j \times F$. H_j is smooth because of 1° and on $U_{ij} := U_i \cap U_j$ the condition 2° implies

$$H_j|_{\bar{\pi}^{-1}(U_{ij})} = H_i|_{\bar{\pi}^{-1}(U_{ij})}$$

If $\psi_j^{-1}(a,y) = t = \psi_i^{-1}(a,\bar{y})$, we have $\bar{y} = \psi_{ij}(a).y$. By definition

$$H_j(t) = H_j(\psi_j^{-1}(a,y)) = (\psi'_j)^{-1}(a, h_j(a).y), \text{ hence}$$

$$\begin{aligned}\psi'_i(H_j(t)) &= \psi'_i \circ \psi_j'^{-1}(a, h_j(a).y) = (a, \psi'_{ij}(a)\, h_j(a).y) \\ &\underset{2°}{=} (a, h_i(a)\, \psi_{ij}(a).y) = (a, h_i(a).\bar{y}) \\ &= \psi'_i(H_i(\psi_i^{-1}(a,\bar{y}))) = \psi'_i(H_i(t)).\end{aligned}$$

It follows $H_j(t) = H_i(t)$, $t \in \bar{\pi}^{-1}(U_{ij})$

As a result, the $(H_j)_{j\in I}$ define a smooth H by $H|_{\bar{\pi}^{-1}(U_j)} := H_j$, $j \in I$, which satisfies all required properties.

Conversely let $h : T \to T'$ be an isomorphism in (fib_B^F). Define $h_j : U_j \to \mathrm{Diff}(F)$ by

$$\left(a,\ h_j(a).y\right) = \psi'_j \circ h\left(\psi_j^{-1}(a,y)\right) \quad \text{for} \quad (a,y) \in U_j \times F.$$

Then $h_j(a) \in \mathrm{Diff}(F)$ and $h_j(a).y = \mathrm{pr}_2\left(\psi_j \circ h(\psi_j^{-1}(a,y))\right)$, hence $(a,y) \longmapsto h_j(a).y$ is smooth (i.e. 1°).

Moreover, the commutative diagram

$$\begin{tikzcd}
 & (a,y) \arrow[r, mapsto] & (a,h_j(a).y) & \\
(a,y) \arrow[r, phantom, "\in"] \arrow[dd, mapsto, "\psi_i\circ\psi_j^{-1}"'] & U_{ij}\times F \arrow[u, phantom, sloped, "\in"] \arrow[r] & U_{ij}\times F \arrow[u, phantom, sloped, "\in"] \arrow[r, phantom, "\ni"] & (a,\bar{y}) \arrow[dd, mapsto, "\psi'_i\circ\psi'^{-1}_j"] \\
 & \pi^{-1}(U_{ij}) \arrow[u, "\psi_j"'] \arrow[r, "h"] \arrow[d, "\psi_i"] & (\pi')^{-1}(U_{ij}) \arrow[u, "\psi'_j"] \arrow[d, "\psi'_i"] & \\
(a,\psi_{ij}(a).y) \arrow[r, phantom, "\in"] & U_{ij}\times F \arrow[r] \arrow[d, phantom, sloped, "\ni"] & U_{ij}\times F \arrow[d, phantom, sloped, "\ni"] \arrow[r, phantom, "\ni"] & (a,\psi'_{ij}(a).\bar{y}) \\
 & (a,\bar{y}) \arrow[r, mapsto] & (a,h_i(a).\bar{y}) &
\end{tikzcd}$$

shows $h_i(a).\psi_{ij}(a).y = \psi'_{ij}(a).h_j(a)y$, i.e. 2°. $\square$

if there is a

<u>(16.4) Remarks:</u> 1° Cohomological interpretation:
0 - cochains are (g_j), $g_j : U_j \to \mathrm{Diff}(F)$ with 1°
1 - cochains are (g_{ij}), $g_{ij} : U_{ij} \to \mathrm{Diff}(F)$ with 1°
1 - cochains are cocycles if $g_{ii} = 1$ and $g_{ij} g_{jk} g_{ki} = 1$ $(= id_F)$ on $U_{ijk} = U_i \cap U_j \cap U_k$. Two cocycles g_{ij} and g'_{ij} are cohomologically equivalent (or cohomologeous) if and only if there is a 0-cochain (h_j) with $g'_{ij} = h_i \circ g_{ij} \circ h_j^{-1}$.
The (1-) cohomology space with respect to $\mathfrak{U} = (U_j)$ is

$$H^1(\mathfrak{U}, \mathrm{Diff}(F)) := \{1\text{-cochains}\}/\sim .$$

As a consequence of 16.2 and 16.3 the space $H^1(\mathfrak{U}, \mathrm{Diff}(F))$ is the set of isomorphism classes in (fib_B^F) with local trivializations over U_j, $j \in I$.
Note, that in the case that the fibre F is diffeomorphic to $\mathbb{R}^k$ there exists a cover (U_j) of the manifold such that any fibration ξ has a local trivializations over each U_j, $j \in I$, cf. 16.10. With respect to such a covering $\mathfrak{U} = (U_j)$ the set of isomorphism classes in (fib_B^F) is $H^1(\mathfrak{U}, \mathrm{Diff}(F)) = H^1(B, \mathrm{Diff}(F))$.

2° The cohomological description of fibrations persists when we study

i) vector bundles of rank r

ii) principal fibre bundles with Lie group G as fibre or

iii) fibre bundles with structure group G.

In case i) $\mathrm{Diff}(F)$ will be replaced by $GL(\mathbb{R}, r)$ or $GL(\mathbb{C}, r)$. In case ii) $\mathrm{Diff}(F)$ is replaced by G itself (acting as inner automorphisms) and in case iii) $\mathrm{Diff}(F)$ is again replaced by G (acting as diffeomorphisms on F).

Application (of 16.2):

<u>(16.5) Proposition - Definition</u>: Let $\xi = (T, \pi, B, F)$ be a fibration and let $g : B' \to B$ be a smooth map. Then there is a fibration $\xi' = (T', \pi', B', F)$ with a morphism $h : \xi' \to \xi$ over g whose restrictions $h_{b'} : T'_{b'} \to T_b$, $b = g(b')$, are diffeomorphisms. (Note that the smooth map $h : T' \to T$ satisfies $\pi \circ h = g \circ \pi'$.) Moreover, ξ' is unique up to isomorphism in $(fib^F_{B'})$.

<u>Proof</u>: Existence. Let (U_j) be an open cover of B

and $\psi_j : T_{U_j} \to U_j \times F$ local trivializations. (ψ_{ij}) are the corresponding transition functions of ξ with respect to (U_j): $\psi_{ij} : U_{ij} \to \mathrm{Diff}(F)$ satisfying 1° and 2° of 16.1. Now, $U_j' := \bar{g}^{-1}(U_j)$ yields an open cover of B' and the $\psi_{ij}' := \psi_{ij} \circ g : U_{ij}' \to \mathrm{Diff}(F)$ satisfy 16.1.1° & 2°. Let $\xi' = (T', \pi', B', F)$ denote the corresponding fibration according to 16.2. with transition functions ψ_{ij}' with respect to the cover (U_j') and bundle charts $\psi_j' : T'_{U_j'} \to U_j' \times F$ with $\psi_i' \circ \psi_j'^{-1}(b', y) = (b', \psi_{ij}'(b').y)$ for $b' \in U_{ij}'$, $y \in F$.
Let $H_j : T'_{U_j'} \to T_{U_j}$ given by $H_j(\psi_j'^{-1}(b', y)) := \psi_j^{-1}(g(b'), y)$, $b' \in U_j'$, $y \in F$. Then H_j is smooth, $\pi \circ H_j = g \circ \pi'$ on U_j', and the restrictions of H_j to the fibres $T'_{b'}$ are diffeomorphisms.
It suffices to prove $H_j|_{T'_{U_{ij}'}} = H_i|_{T'_{U_{ij}'}}$ in order to glue the H_j to a morphism H with $H_j = H|_{T'_{U_j'}}$:
Let $t' = \psi_j'^{-1}(b', y) = \psi_i'^{-1}(b', \bar{y})$. From

$$(b', \bar{y}) = \psi_i' \circ \psi_i'^{-1}(b', \bar{y}) = \psi_i' \circ \psi_j'^{-1}(b', y) = (b', \psi_{ij}'(b').y)$$

we obtain $\bar{y} = \psi_{ij}'(b').y$, hence

$$H_i(t') = \psi_i^{-1}(g(b'), \bar{y}) = \psi_i^{-1}(g(b'), \psi_{ij}(b)\, y) = \psi_j^{-1}(g(b'), y)$$

hence $H_i(t') = H_j(t')$

Uniqueness. If $\bar{h} : \bar{\xi} \to \xi$ is another such morphism with $\bar{h} : \bar{T} \to T$ fibre preserving ($\pi \circ \bar{h} = g \circ \bar{\pi}$) and all $\bar{h}_{b'} : \bar{T}_{b'} \to T'_{g(b')}$, $b' \in B'$, diffeomorphisms, then $f_a := h_a^{-1} \circ \bar{h}_a$ gives a fibre preserving diffeomorphism $f : \bar{T} \to T'$ with $f_{B'} = id_{B'}$, i.e. an isomorphism $f : \bar{\xi} \to \xi'$ in $(fib^F_{B'})$.

$\square$

(16.6) <u>Remark</u>: The fibration ξ' induced by ξ and $g : B' \to B$ as in 16.5 is called the <u>pullback</u> of ξ by g, and it is denoted by $g^*\xi$. Under a different point of view it is called <u>base change</u>. In any case, it can be constructed in a different way as the submanifold

$$T' := \{ (a', t) \in B' \times T \mid g(a') = \pi(t) \} \subset B' \times T$$

of $B' \times T$ with $\pi'(a', t) := a'$ and $h(a', t) := t$. Evidently, $\pi \circ h(a', t) = g \circ \pi'(a', t)$.

In the following, B (and T) are supposed to have a countable* topology.

(16.7) Dimension Lemma: Let $(\mathcal{U}_j)_{j\in I}$ be an open cover of an n-dimensional manifold B. Then $(\mathcal{U}_j)$ has a refinement** of the form $(W_{k\nu})_{(k,\nu)\in H\times\mathbb{N}}$, $W_{k\nu}\subset B$ open, such that $W_{k\mu}\cap W_{k\nu}=\emptyset$ for $k\in H, \mu\neq\nu$, with H finite.

Source: HUREWICZ – WALLMANN. M has Lebesgue dimension n.

(16.8) Corollary: 1° Every mfd B has an atlas with finitely many charts. 2° Every fibration (T,π,B,F) has an atlas with finitely many bundle charts.

(16.9) Proposition: Every mfd B has an open cover $(\mathcal{U}_j)$ such that all intersections $\mathcal{U}_i, \mathcal{U}_{ij}, \mathcal{U}_{ijk}, \mathcal{U}_i\cap\mathcal{U}_j\cap\mathcal{U}_k\cap\mathcal{U}_\ell, \dots$ are empty or contractable.***

(16.10) Corollary: There exists an open cover $(\mathcal{U}_j)$ of B such that all fibrations ξ over B with typical fibre F diffeomorphic to $\mathbb{K}^r$ are trivial over $\mathcal{U}_j$ for all $j\in I$.

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* B has countable topology $\Leftrightarrow$ B has a countable base of topology, i.e. $(D_n)_{n\in\mathbb{N}}$ s.th.: $\mathcal{U}\subset B$ open $\Leftrightarrow$ $\exists A\subset\mathbb{N}: \mathcal{U}=\bigcup\{D_n \mid n\in A\}$

** $(V_\lambda)_{\lambda\in L}$ refinement of $(\mathcal{U}_j)_{j\in I}$ $\Leftrightarrow$ $\exists\tau: L\to I\ \forall\lambda\in L: V_\lambda\subset\mathcal{U}_{\tau(\lambda)}$

*** A topol. space Y is contractible $\Leftrightarrow$ $\exists H: Y\times[0,1]\to Y$ continuous s.th. $H(y,0)=y$, $y\in Y$, and $H(y,1)=y_1$ constant.