15. Fibrations

Version 1.1

We want to study a rather general structure which we call fibration (Faserung). In other texts these structure are called locally trivial fibrations or fibre bundles. Note that in the topological context simply a surjective continuous map is called already a fibre bundle, or bundle.

(15.1) Definition: A *fibration* with *typical fibre* F is a quadruple $\xi = (T, \pi, B, F)$ with:

1° T, B, F are manifolds and $\pi \in \mathcal{E}(T,B)$

2° For each $a \in B$ there exists an open neighbourhood $U \subset B$ of a and a diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$ with $pr_1 \circ \psi = \pi|_{\pi^{-1}(U)}$:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\psi", "\sim"'] \arrow[d, "\pi"'] & U \times F \arrow[dl, "pr_1"] \\ U & \end{tikzcd}$$

T is called the *total space*, B the *base space*, or base, F the *typical fibre* and π the *projection* of ξ. Instead of ξ the whole fibration is sometimes denoted by T. Notation:

$$\begin{tikzcd} T \arrow[d, "\pi"'] & F \\ B & \end{tikzcd}$$

ψ is called a local trivialization.

(15.2) Remarks. Let $\xi = (T, \pi, B, F)$ be a fibration.

1° For $a \in B$ and ψ as in 16.1.2° the map
$$\psi_a : F_a := \pi^{-1}(a) \longrightarrow F \ , \ t \mapsto pr_2 \circ \psi(t)$$
(i.e. $\psi_a = pr_2 \circ \psi|F_a$) is a diffeomorphism. By the way, F_a is a submanifold.

2° Let $\psi : \pi^{-1}(U) \to U \times F$ as in 16.1.2° an $\varphi : U \to V$ a chart of the maximal atlas of B. Then
$$\hat{\varphi} : \pi^{-1}(U) \longrightarrow V \times F \ , \quad \hat{\varphi} = (\varphi \times id_F) \circ \psi ,$$
is called a bundle chart (although ξ is (not yet) a bundle in our sense, and although $\hat{\varphi}$ is not a true chart)

(15.3) Definitions: Let $\xi = (T, \pi, B, F)$ and $\xi' = (T', \pi', B', F')$ be fibrations. A map $h : T \to T'$ is called a morphism (of fibrations) if $h \in \mathcal{E}(T, T')$ and if there exists $h_B \in \mathcal{E}(B, B')$ such that $\pi' \circ h = h_B \circ \pi$:
$$\begin{array}{ccc} T & \xrightarrow{h} & T' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[h_B]{} & B' \end{array}$$

In this way we obtain the category (fib) of fibrations and for a fixed manifold B the full subcategory (fib_B) of fibrations over B.
A section of ξ over an open subset $W \subset B$ is a smooth map $s: W \to \xi$ with $\pi \circ s = id_W$. The set of sections over W is denoted by $\Gamma(W, \xi)$.

(15.4) Examples:

1° When E and B are manifolds the product $B \times F$ with $\pi = pr_1 : B \times F \to B$ defines a fibration.
We have $\Gamma(W, B\times F) \cong \mathcal{E}(W, F)$.
A fibration $\xi = (T, \pi, B, F)$ is called "trivial", if there exists a global trivialization, i.e. a diffeomorphism

$$\varphi : T \to B \times F \quad \text{with} \quad \pi = pr_1 \circ \varphi .$$

(equivilantly: an isomorphism $h: T \to B \times F$ with $h_B = id_B$).
In fact, a fibration ξ which is isomorphic to a trivial fibration is itself trivial.

2° Let ξ be a fibration and $W \subset B$ open. Then

$\xi_W = \xi_{|W} := (\pi^{-1}(W), \pi|_W, W, F)$ is a fibration, the restriction.

3° $(TM, \tau, M, \mathbb{R}^n)$ is a fibration (cf. 6.1) and the same is true for $(T^*M, \tau, M, \mathbb{R}^n)$ (cf. 9.3.1). (There is more structure, since the fibres a vector spaces.) In general, TM is not trivial. Example: $M = \mathbb{S}^2 \subset \mathbb{R}^3$, $\mathbb{S}^2$ the twosphere.

4° TM and T^*M are isomorphic.

5° $\pi : \mathbb{R} \to \mathbb{S}^1$, $\pi(t) := \exp(2\pi i t)$, is a fibration with typical fibre $\mathbb{Z}$ ($\mathbb{Z}$ as a discrete space)

Local trivializations:

Let $a \in \mathbb{S}^1$, $a = \exp(2\pi i t_0)$ for some $t_0 \in \mathbb{R}$.

$U := \{\exp(2\pi i t) \mid |t - t_0| < \frac{1}{2}\}$

$\pi^{-1}(U) :=]t_0 - \frac{1}{2}, t_0 + \frac{1}{2}[+ \mathbb{Z} = \mathbb{R} \setminus \{t_0 + \frac{1}{2} + \mathbb{Z}\}$

$$\begin{tikzcd} \psi : \pi^{-1}(U) \arrow[r] \arrow[d, "\pi"'] & U \times \mathbb{Z} \arrow[dl, "pr_1"] \\ U & \end{tikzcd} \qquad t + k \longmapsto (\pi(t), k)$$

There is no global section since there is no global Log! Hence, the fibration is not trivial.

6° Similarly, $\pi : \mathbb{C} \to \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, $\pi(z) = e^{2\pi i z}$.

Fibre $\mathbb{Z}$, not trivial. The sections are the branches of the logarithm.

7° $z^k: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$, $z \longmapsto z^k$, is a fibration with fibre isomorphic to $\{1, \dots k\} \cong \mathbb{Z}/_{k\mathbb{Z}}$. And the same is true for $z^k: \mathbb{S}^1 \to \mathbb{S}^1$. Both these fibrations are not trivial since there does not exists a global k^{th} root. z^k has no global section.

8° $\pi: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$, $x \longmapsto \frac{x}{\|x\|}$, is a fibration with fibre isomorphic to $\mathbb{R}_+ =]0, \infty[\cong \mathbb{R}$. The fibration turns out to be trivial. A natural global section is $x \longmapsto x$.

9° $\varkappa: \mathbb{K}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}_n(\mathbb{K})$ is a non-trivial fibration with fibre type $\mathbb{K}$.

10° A covering map $\pi: T \to B$ is a fibration with discrete fibre isomorphic to a quotient of the fundamental group $\pi_1(B)$. In the case of a universal covering the fibre type is $\pi_1(B)$.