12. Differential Forms

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Differential forms on a manifold M are special tensors and the have locally the form

$$\eta = \eta_{i_1 \dots i_k} dq^{i_1} \wedge \dots \wedge dq^{i_k} \quad .$$

We explain this again in a general form for E-modules. In particular, we want to describe and prove some important formulas for differential forms in this general frame.

As before let V be a module over a commutative $\mathbb{K}$-algebra E with 1.

<u>(12.1) Definition:</u> An s-form (a form of degree s) $s \in \mathbb{N}$, is an alternating $\binom{0}{s}$-tensor η. That is

$$\eta \in T^0_s V = \mathrm{Hom}_E(V^s, E), \ \eta : V^s \to E \text{ multilinear}$$

$$\eta(X_1, \dots X_i, X_{i+1}, \dots X_s) = -\eta(X_1, \dots X_{i+1}, X_i, \dots X_s)$$

for all $X_j \in V$ & $i = 1, \dots, n-1$.

Equivalently: $\eta(X_1, \dots X_s) = (\operatorname{sign}\sigma)\,\eta(X_{\sigma(1)}, \dots X_{\sigma(s)})$
for the permutations $\sigma : \{1, \dots s\} \to \{1, \dots s\}$.

Let $A^s V \subset T^0_s V$ denote the E-module of s-forms.
We have $A^0 V \cong E$, $A^1 V = V^*$, $A^2 V = \{\omega : V \times V \to E \mid$
ω bilinear and alternating$\}$.

Analogously we define the symmetric
tensors $S^k V \subset T^0_k V$.

<u>(12.2) Fact</u>: $T^0_2 V = A^2 V \oplus S^2 V$, but
$A^k V \oplus S^k V \neq T^0_k V$ for $k > 2$, in general.

<u>(12.3) Def</u> (Exterior Product - step by step):
1° $\alpha, \beta \in A^1 V$: $\alpha \wedge \beta (X, Y) := \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$
for $X, Y \in V$.
2° $\alpha^1, \dots, \alpha^k \in A^1 V$:

$$\alpha^1 \wedge \dots \wedge \alpha^k (X_1, \dots X_k) := \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \; \alpha^1(X_{\sigma(1)}) \dots \alpha^k(X_{\sigma(k)})$$

with S_k the group of permutations of $\{1, \dots k\}$.

3° $\eta \in A^k V$, $\omega \in A^l V$

$$\eta \wedge \omega (X_1, \dots X_k, X_{k+1}, \dots X_{k+l}) :=$$

$$:= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\, \eta(X_{\sigma(1)}, \dots X_{\sigma(k)})\, \omega(X_{\sigma(k+1)}, \dots X_{\sigma(k+l)}).$$

4° The exterior algebra (Grassmann algebra)

$$AV := \bigoplus_{k \in \mathbb{N}} A^k V$$

is an algebra over E, as well.

Analogously, we yield the symmetric algebra

$$SV := \bigoplus_{k \in \mathbb{N}} S^k V,$$

which we don't need in differential geometry.

(12.4) Fact: Let V have the basis $(\partial_1, \dots \partial_n)$ with dual basis $d^1, \dots d^n$ of V^*. Then

$$\left(d^{\mu_1} \wedge d^{\mu_2} \wedge \dots \wedge d^{\mu_k}\right)_{\mu_1 < \dots < \mu_k}$$

is a basis of $A^k V$.

As a consequence, $A^k V$ is free of finite rank if V

is free of finite rank. Moreover, if V is free of rank n, then $A^k V = \{0\}$ for $k > n$.

(12.5) <u>Definition-Proposition</u>: Let V be a reflexive E-module as before with the additional property that there exists an E-module isomorphism

$$L : V \longrightarrow \mathrm{Der}_{\mathbb{K}}(E) \quad , \quad X \longmapsto L_X \quad (\text{cf. } 7.6).$$

Then we define the exterior derivative $d : A^k V \to A^{k+1} V$ by

$$\boxed{\begin{aligned}(d\eta)(X_0, X_1, \dots, X_k) := & \sum_{j=0}^{k} (-1)\, L_{X_j}\left(\eta(X_0, \dots, \hat{X}_j, \dots X_k)\right) + \\ & + \sum_{i<j} (-1)^{i+j}\, \eta\left([X_i, X_j], X_0, \dots, \hat{X}_i \dots \hat{X}_j \dots X_k\right)\end{aligned}}$$

for $\eta \in A^k V$, $X_0, X_1, \dots X_n \in V$.

In this context $[X, Y]$ is defined as before (cf. 7.6): $L_X L_Y - L_Y L_X$ turns out to be a derivation. Hence, it is of the form $L_Z = L_X L_Y - L_Y L_X$ and we define $[X, Y] := Z$.
V with the bracket $[\,,\,]$ is a Lie algebra over $\mathbb{K}$.

It is not obvious, that $d\eta$ is multilinear! Let us present the main arguments:

Proof: First, it is not difficult to show that

$$d\eta : V^{k+1} \to E$$

is additive and linear over $\mathbb{K}$ in each argument. Since we have $L_{X+Y} = L_X + L_Y$ and $[X+X', Y] = [X,Y] + [X', Y]$. It remains to prove

$$d\eta(X_0, \dots, fX_j, \dots X_k) = f\, d\eta(X_0, \dots, X_j, \dots X_k)$$

for all $X_i \in V$ and $f \in E$, and it is enough to check the case $j=0$. Since L is an isomorphism of E-modules we have $L_{fX} = fL_X$ and from this we conclude

$$[fX, Y] = f[X,Y] - (L_Y f)X :$$

Indeed

$$\begin{aligned} L_{fX} L_Y - L_Y L_{fX} &= fL_X L_Y - L_Y f L_X \\ &= fL_X L_Y - fL_Y L_X - (L_Y f)L_X, \end{aligned}$$

since $L_Y f L_X(g) = L_Y f\, L_X g + f L_Y L_X g$, $g \in E$.

Now,

$$
\begin{aligned}
&d\eta(fX_0, X_1, \dots X_k) \\
&= L_{fX_0}(\eta(X_1 \dots X_k)) + \sum_{j=1}^{k} (-1)^j L_{X_j} \eta(fX_0, \dots \hat{X}_j, \dots X_k) \\
&+ \sum_{0<j} (-1)^j \eta([fX_0, X_j], X_1, \dots \hat{X}_j, \dots X_k) + \sum_{0<i<j} (-1)^{i+j} \eta([X_i, X_j], fX_0, \dots \hat{X}_i, \dots \hat{X}_j, \dots X_k) \\
&= f L_{X_0}(\eta(X_1, \dots X_k) + f \sum_{j=1}^{k} (-1)^j L_{X_j} \eta(X_0, \dots, \hat{X}_j, \dots X_k) + \\
&\qquad + \sum_{j=1}^{k} (-1)^j (L_{X_j} f) \, \eta(X_0, \dots, \hat{X}_j, \dots X_k) \\
&+ \sum_{0<j} (-1)^j \eta([fX_0, X_j], X_1, \dots \hat{X}_j, \dots X_k) + \sum_{0<i<j} (-1)^{i+j} \eta([X_i, X_j], fX_0, \dots \hat{X}_i, \dots \hat{X}_j, \dots X_k) \\
&= f\left(\sum_{j=0}^{k} (-1)^j L_{X_j} (\eta(X_0, \dots, \hat{X}_j, \dots X_k)\right) + \sum_{j=1}^{k} (-1)^j (L_{X_j} f) \, \eta(X_0, \dots, \hat{X}_j, \dots X_k) \\
&+ \sum_{0<j} (-1)^j \eta(f[X_0, X_j] - (L_{X_0} f) Y_j, X_1, \dots \hat{X}_j, \dots X_k) + \\
&+ f\left(\sum_{0<i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \dots \hat{X}_i, \dots \hat{X}_j, \dots X_k)\right) \\
&= f\left(\sum_{j=0}^{k} (-1)^j L_{X_j} (\eta(X_0, \dots, \hat{X}_j, \dots X_k)\right) + \sum_{j=1}^{k} (-1)^j (L_{X_j} f) \, \eta(X_0, \dots, \hat{X}_j, \dots X_k) + \\
&+ f\left(\sum_{0<j} (-1)^j \eta([X_0, X_j], \dots \hat{X}_j, \dots X_k) - \sum_{j=1}^{n} (-1)^j (L_{X_j} f) \, \eta(X_0, \dots, \hat{X}_j, \dots X_k\right) + \\
&+ f\left(\sum_{0<i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \dots \hat{X}_i, \dots \hat{X}_j, \dots X_k)\right) \\
&= f \, d\eta(X_0, \dots, X_k). \qquad \square
\end{aligned}
$$

We obtain a series of useful formulas:

(12.6) Proposition: The exterior derivative $d : AV \to AV$ satisfies

1° $df(X) = L_X f$ for all $X \in V$, $f \in E$.

2° $d(f\,dg) = df \wedge dg$ for all $f, g \in E$ & $d(dg) = 0$

3° $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ for $\alpha \in A^k V$, $\beta \in A^s V$.

4° $d(f\,dg_1 \wedge dg_2 \wedge \ldots \wedge dg_k) = df \wedge dg_1 \wedge \ldots \wedge dg_k$ for $f, g_j \in E$.

$$\boxed{5° \quad d \circ d = 0}$$

(12.7) Proposition. Let V be free with basis $\partial_1, \ldots \partial_n$ and dual basis $d^j \in V^*$. Each $\eta \in A^k V$ has the unique form

$$\eta = \sum_{\mu_1 < \ldots < \mu_k} \eta_{\mu_1 \ldots \mu_k} d^{\mu_1} \wedge \ldots \wedge d^{\mu_k} \quad \text{with } \eta_{\mu_1 \ldots \mu_k} \in E$$

and we have

$$d\eta = \sum_{\mu_1 < \ldots < \mu_k} d\eta_{\mu_1 \ldots \mu_k} \wedge d^{\mu_1} \wedge \ldots \wedge d^{\mu_k} \qquad (10.11.4°)$$

with $$d\eta_{\mu_1 \ldots \mu_k} = \sum_{j=1}^{k} \partial_j \eta_{\mu_1 \ldots \mu_k} d^j .$$

One has transformation rules as in 11.2.

(12.8) Examples:

1° $A = A_1 d^1 + A_2 d^2 + A_3 d^3 \qquad A_j \in E$

$dA = dA_1 \wedge d^1 + dA_2 \wedge d^2 + dA_3 \wedge d^3$

Now, assume that d^1, d^2, d^3 is the dual basis of the basis $\partial_1, \partial_2, \partial_3$ of V (as in the case of $\mathfrak{W}(U) = V$ with coordinates q^1, q^2, q^3 and $\partial_j = \frac{\partial}{\partial q^j}$, $d^\mu = dq^\mu$. Then

$$dA = \partial_j A_1\, d^j \wedge d^1 + \partial_j A_2\, d^j \wedge d^2 + \partial_j A_3\, d^j \wedge d^3 = \partial_j A_\mu\, d^j \wedge d^\mu$$

$$= (\partial_1 A_2 - \partial_2 A_1)\, d^1 \wedge d^2 + (\partial_2 A_3 - \partial_3 A_2)\, d^2 \wedge d^3 + (\partial_1 A_3 - \partial_2 A_3)\, d^1 \wedge d^3$$

2° $F \in \Lambda^2 V$ with V free of rank 3:

$$F = F_{12}\, d^1 \wedge d^2 + F_{23}\, d^2 \wedge d^3 + F_{13}\, d^1 \wedge d^3 \qquad , F_{ij} \in E$$

$$= \frac{1}{2} \sum_{i,j=1}^{3} F_{ij}\, d^i \wedge d^j \qquad \text{with } F_{ji} := -F_{ij} \ , \ 1 \leq i < j \leq 3.$$

$$dF = (\partial_1 F_{23} - \partial_2 F_{13} + \partial_3 F_{12})\, d^1 \wedge d^2 \wedge d^3$$

In particular, with $F = X^1 d^2 \wedge d^3 + X^2 d^3 \wedge d^1 + X^3 d^1 \wedge d^2$:

$$dF = (\partial_1 X^1 + \partial_2 X^2 + \partial_3 X^3)\, d^1 \wedge d^2 \wedge d^3 = \text{„}\operatorname{div} X\text{“}\, d^1 \wedge d^2 \wedge d^3$$

(12.9) Definition: $X \in V$

1° $i_X : A^k V \to A^{k-1} V$, $i_X \eta(X_1, \dots, X_{k-1}) := \eta(X, X_1, \dots X_{k-1})$

2° $L_X : A^k V \to A^k V$, $L_X := i_X d + d i_X$

<u>(12.10) Proposition:</u> i_X is E-linear and L_X is $\mathbb{K}$-linear satisfying

1° $L_X \circ d = d \circ L_X$

2° $L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$

At the end of this section we come to the definition of a differential form as a section in the corresponding bundle. Let M be a manifold of dimension n and let $k \in \mathbb{N}$, $k \leq n$. The set

$$\Omega^k M := \bigcup \{ A^k T_a M \mid a \in M \}$$

can be endowed with a natural smooth structure such that the projection $\pi : \Omega^k M \to M$, $\pi(A^k T_a M) = \{a\}$, is smooth and the bundle charts (as in 11.1) are diffeomorphisms. $\Omega^k M = \Omega^k$ is a vector bundle, the <u>bundle of k-forms</u>.

<u>(12.11) Alternative definitions:</u> A differential form of degree k on an open subset $W \subset M$ of a manifold M is

1° a section $\eta : W \to \Omega^k M$ in the bundle of k-forms, or

2° a map $\eta \in A^k \mathcal{D}(W) =: \mathcal{A}^k(W)$, or

3° a family of coefficient $(\eta^{\mathcal{U}}_{i_1 \dots i_k})_{\mathcal{U} \in I}$ for each chart $\varphi : \mathcal{U} \to Q \subset \mathbb{R}^n$ such that

$$\eta^{\bar{\mathcal{U}}}_{i_1 \dots i_k} = \frac{\partial q^{j_1}}{\partial \bar{q}^{i_1}} \cdots \frac{\partial q^{j_k}}{\partial \bar{q}^{i_k}} \eta^{\mathcal{U}}_{j_1 \dots j_k} \qquad (i_1 < \dots < i_k \,;\, j_1 < \dots < j_k)$$

(Here, I is any atlas on M defining the smooth structure.)

In particular: The $\mathcal{E}(W)$-module $\mathcal{A}^k(W) := A^k \mathfrak{D}(W)$ is in a natural way isomorphic to the $\mathcal{E}(W)$-module $\Gamma(W, \Omega^k M)$ of sections over W in $\Omega^k M$.

There is a vector-valued version of the theory of k-forms. Instead of $A^k V$ we obtain $A^k(V, Z)$ and in the case of manifolds we have

$$\mathcal{A}^k(W, F) := A^k(\mathfrak{D}(W), \Gamma(W, F))$$

the differential k-forms with values in a vector bundle over F