10. Multilinear Algebra

Version 1.2

Tensors at a point of a manifold M behave in many respects similar to tensor fields on M. In particular the standard algebraic and differential manipulations with both classes of objects are analogous. This analogy can be expressed in a simple way by multilinear algebra.

In the following E is a commutative algebra over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with 1. In our context of analysis on smooth manifolds, E is the field $\mathbb{R}$ or $\mathbb{C}$ or E is the $\mathbb{K}$-algebra $\mathcal{E}(M,\mathbb{K})$ of smooth functions (scalars) on M with values in $\mathbb{K}$.

V is an E-module which is supposed to be <u>reflexive</u>, i.e. the natural map

$$V \longrightarrow V^{**}, \ v \longmapsto (\mu \longmapsto \mu(v)), \quad v \in V, \mu \in V^*,$$

is an isomorphism of E-modules.

Let Z denote another E-module which is in many cases the Ring E itself.

(10.1) Examples: 1° A finite dimensional vector space V over $\mathbb{K}$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) is an E-module with $E = \mathbb{K}$. V is reflexive, since $V \to V^{**}$, $v \mapsto (\mu \mapsto \mu(v))$, is an isomorphism of vector spaces as is well-known. Moreover, V is finitely generated over $E = \mathbb{K}$ by any basis $e_1, \dots e_n$ of V, for exple.

2° For $M \subset \mathbb{R}^n$ open the set $\mathcal{V}(M)$ of vector fields is an $\mathcal{E}(M)$-module. $V = \mathcal{V}(M)$ is finitely generated over $E = \mathcal{E}(M)$, we even have a basis, e.g. $\partial_j := \frac{\partial}{\partial q^j}$, $j = 1, \dots n$, for the (euclidean) coordinates q^j (cf. 7.2): Every vector field $X \in V$ has a representation

$$X = \sum_{j=1}^{n} X^j \partial_j = X^j \partial_j$$

with uniquely defined $X^j \in E = \mathcal{E}(M)$. Altogether, V is a free E-module of rank n. Let $dq^j \in V^*$ be given by

$$dq^j(X) := X^j, \quad j = 1, 2, \dots n, \text{ where } X = X^j \partial_j.$$

Then $dq^1, \dots, dq^n$ constitutes a basis of the dual E-module $V^* = \{\mu : V \to E \mid \mu \text{ is } E \text{ linear}\}$. Now, $\theta_j : V^* \to E$, $\mu = \mu_j dq^j \mapsto \mu_j$, defines a basis of V^{**} and $v \mapsto (\mu \mapsto \mu(v))$ yields an isomorphism of

E-modules $V \mapsto V^{**}$ sending ∂_j to θ_j!
What we have shown is nothing else than the fact that the dual V^* of a free E-module of rank n is again a free E-module of rank n and that V is reflexive.

3° For a general smooth manifold M the set $\mathcal{V}(M)$ of vector fields on M is again an $\mathcal{E}(M)$-module (cf. 7.1). $\mathcal{V}(M)$ is reflexive* and $\mathcal{V}(M)$ is finitely generated if M has countable topology.

4° In case of $V = \mathcal{V}(M)$ we know – in addition – to 2° and 3°: $\mathcal{V}(M) \cong \mathrm{Der}_{\mathbb{K}}(\mathcal{E}(M))$ (cf. 7.5): $E = \mathcal{E}(M)$ is a $\mathbb{K}$-algebra and $V \cong \mathrm{Der}_{\mathbb{K}}(E)$.

5° $\mathbb{Z}/k\mathbb{Z}$ is not reflexive as a $\mathbb{Z}$-module.

---

* In order to prove this statement one first shows that the E-linear forms $V^* = \mathrm{Hom}_E(V, E)$ for $V = \mathcal{V}(M)$ and $E = \mathcal{E}(M)$ are the differential one forms: $\mathcal{V}(M)^* = \mathcal{V}^*(M)$ (cf. 9.3.3°). Now, any $\theta \in V^{**} = \mathrm{Hom}_E(V^*, E)$ induces for charts $(q^1, \dots q^n) : U \to Q \subset \mathbb{R}^n$ the corresponding components $\theta^i := \theta(dq^i)$ which define a vector field X by $X|_U = \theta^i \frac{\partial}{\partial q^i}$ such that $\theta(\mu) = \mu(X)$ for all $\mu \in \mathcal{V}(M)$. Hence $\theta = \hat{X}$, i.e. $X \mapsto \hat{X}$ is an isomorphism.

(10.2) Definition: A *tensor* t *of type* $\binom{r}{s}$ (for $r, s \in \mathbb{N}$) is a multilinear map

$$t : (V^*)^r \times V^s \longrightarrow E,$$

i.e. t has to be E-linear in each of its $r+s$ arguments. t is called also an $\binom{r}{s}$-tensor, and a Z-valued tensor of type $\binom{r}{s}$ is correspondingly an $(r+s)$-linear map

$$t : (V^*)^r \times V \longrightarrow Z.$$

Notation:

$$T^r_s(V, Z) := \{ t \mid t \text{ is } \tbinom{r}{s}\text{-tensor with values in } Z \}$$

$$T^r_s V := T^r_s(V, E)$$

In particular,

- $\binom{0}{0}$-tensors are the elements of E: $T^0_0 V = E$
- $\binom{0}{1}$-tensors are the *1-forms*: $T^0_1 V = \mathrm{Hom}_E(V, E) = V^*$
- $\binom{1}{0}$-tensors are the linear forms on V^*, hence essentially the elements of V $\quad T^1_0 V = V^{**} \simeq V$

(10.3) Examples: 1° Let V be an n-dimensional vector space over $\mathbb{K}$. $T^1_1 V$, $T^2_0 V$, $T^0_2 V$ have the dimension n^2. Applied to $V = T_a M$ we obtain the tensors at a point $a \in M$: $T^r_s T_a M =: T^r_{s,a} M$

2° Applied to $V = \mathcal{V}(M)$ we obtain the tensor fields $t \in T^r_s \mathcal{V}(M) =: \mathcal{T}^r_s(M)$ of type $\binom{r}{s}$ on M.

(10.4) Fact - Definition: $T^r_s V$ is an E-module for all $r, s \in \mathbb{N}$ and the direct sum

$$TV := \bigoplus_{r,s \in \mathbb{N}} T^r_s V$$

becomes an algebra over E, the (mixed) tensor algebra, through the tensor product $\otimes : TV \times TV \to TV$ induced by $t \otimes t'$ for $t \in T^r_s V$, $t' \in T^{r'}_{s'} V$, where

$$t \otimes t'(\eta^1, \dots, \eta^r, \eta^{r+1}, \dots \eta^{r+r'}, X_1, \dots X_s, X_{s+1}, \dots X_{s+s'}) :=$$
$$:= t(\eta^1, \dots \eta^r, X_1, \dots X_s)\, t'(\eta^{1+r}, \dots, \eta^{r+r'}, X_{1+s}, \dots X_{s+s'}) .$$

The tensor product $\otimes$ satisfies the following rules (among others):

$$(f+g)t = ft + gt \qquad (TV \text{ is an } E\text{-module}),$$
$$f(t\otimes t') = (ft)\otimes t' = t\otimes ft',$$
$$(t+t')\otimes t'' = t\otimes t'' + t'\otimes t''.$$

(10.5) Lemma: If V has a basis $\partial_1, \dots \partial_n$ (over E) with the dual basis $d^1, \dots d^n \in V^*$ defined by $d^j(\partial_k) = \delta^j_k$ (Kronecker!) (e.g. $V = \mathcal{D}(U)$ with $\partial_j = \frac{\partial}{\partial q^j}$ & $d^k = dq^k$ from local coordinates), then each $t \in T^r_s V$ has the form

$$t = t^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s}\, \partial_{\mu_1} \otimes \dots \otimes \partial_{\mu_r} \otimes d^{\nu_1} \otimes \dots \otimes d^{\nu_s}$$

where

$$t^{\mu_1 \dots \mu_r}_{\nu_1 \dots \nu_s} = t(d^{\mu_1}, \dots, d^{\mu_r}, \partial_{\nu_1}, \dots \partial_{\nu_s}) \in E$$

Similarly, the vector-valued case.

See Section 11 for the transformation rules induced by a change of basis.

Tensor product of modules:

With the concept of the tensor product of E-modules or of $\mathbb{K}$-vector spaces the tensors and spaces of tensors can be described in a slightly different way. We don't need this alternative description in the course, but we give a brief introduction for the sake of completeness.

For E-modules V, V' and Z the space of mappings

$$\beta : V \times V' \to Z$$

which are linear over E in each entry are denoted by $\mathrm{Bil}_E(V, V'; Z)$ or $\mathrm{Hom}_E(V, V'; Z)$.

<u>(10.6) Definition:</u> A tensor product of V and V' is an E-module U together with a $\tau \in \mathrm{Bil}_E(V, V'; U)$ such that

1° Every $\beta \in \mathrm{Bil}_E(V, V'; Z)$ has a unique factorization $\beta = \hat{\beta} \circ \tau$ with $\hat{\beta} \in \mathrm{Hom}_E(U, Z)$, i.e

$$\begin{tikzcd} V \times V' \arrow[r, "\tau"] \arrow[dr, "\beta"'] & U \arrow[d, "\hat{\beta}"] \\ & Z \end{tikzcd}$$

is commutative.

2° Whenever (φ', U') is another pair with 1° there exists a unique morphism $\theta \in \mathrm{Hom}_E(U', U)$ with $\varphi' = \varphi \circ \theta$. i.e. the diagram is commutative.

$$\begin{tikzcd} V \times V' \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & V \otimes V' \arrow[d, "\theta = \hat{\varphi}'"] \\ & W \end{tikzcd}$$

(10.7) Proposition: The tensor product of V, V' exists and it is unique up to isomorphism. It is denoted by $\otimes : V \times V' \to V \otimes V'$.

Proof. Let F be the free E-module over $V \times V'$ as a set, i.e. the E-module of finite E-linear combinations

$$h^{jk}(v_j, v'_k) \quad , \; h^{jk} \in E, \; v_j \in V, \; v'_k \in V'.$$

Let $N \subset F$ denote the submodule generated by all

$$\begin{array}{ll} (f v_1 + g v_2, v') - f(v_1, v') - g(v_2, v') & f, g \in E, \; v_i, v \in V \\ (v, f v'_1 + g v'_2) - f(v, v'_1) - g(v, v'_2) & v'_i, v' \in V \end{array}$$

Then the quotient E-module F/N with the projection

$$\varphi : V \times V' \to F/N \quad , \; (v, v') \longmapsto [(v, v')] =: v \otimes v',$$

satisfies 1° & 2°.

We conclude:

$$T_2^0(V,Z) = Bil_E(V,V;Z) \cong Hom_E(V\otimes V,Z)$$
$$T_1^1(V,Z) = Bil_E(V^*,V;Z) \cong Hom_E(V^*\otimes V,Z)$$
$$T_0^2(V,Z) = Bil_E(V^*,V^*;Z) \cong Hom_E(V^*\otimes V^*,Z)$$

Similarly, there is the notion of p-linear map

$$\beta : V_1 \times \ldots \times V_p \to Z$$

and its E-module $Hom_E(V_1,\ldots V_p;Z)$ of all p-linear maps. And there exists (uniquely up to isomorphisms) the tensor product

$$\otimes : V_1 \times \ldots \times V_p \to V_1 \otimes V_2 \otimes \ldots \otimes V_p$$

such that

$$\begin{aligned} Hom_E(V_1\otimes V_2\otimes\ldots\otimes V_p, Z) &\longrightarrow Hom_E(V_1,\ldots V_p;Z) \\ \mu &\longmapsto \mu\circ\otimes \end{aligned}$$

is an isomorphism and $(\otimes, V_1\otimes\ldots\otimes V_p)$ is universal, i.e. minimal.

We conclude

$$\mathrm{Hom}_E(\underbrace{V^*\otimes\ldots\otimes V^*}_{r}\otimes\underbrace{V\otimes\ldots\otimes V}_{s}, Z) \cong \mathrm{Hom}_E(V^{*r}, V^s; Z) = T^r_s(V,Z)$$

$$\mu \longmapsto \mu\circ\otimes$$

In the special case of a finite dimensional vector space V over $\mathbb{K}$ or in the case of $V = \mathcal{D}(M)$, we have in addition:

$$\mathrm{Hom}_E(V,Z) \cong V^*\otimes Z$$

since $\mathrm{Hom}_E(V,Z)$ is generated by the set of maps $X \longmapsto \mu(X)z = \mu\otimes z(X)$, where $\mu\in V^*$, $z\in Z$.

Consequently,

$$\mathrm{Hom}_E(V^*, Z) \cong V\otimes Z \text{ and}$$

$$(V\otimes V)^* \cong V\otimes V\ ;\ (V^*\otimes V)^* \cong V\otimes V^*;\ \text{etc.}$$

(10.8) <u>Fact</u>: In that cases

$$\boxed{T^r_s(V,Z) \cong V^{\otimes r}\otimes V^{*\otimes s}\otimes Z,}$$

where $V^{\otimes r} := V\otimes V\otimes\ldots\otimes V$ r times.