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Tutorial in mathematical gauge theory
Exercise 1

The projective space $\mathbb{K}P^n$

We wish to define the projective space $\mathbb{K}P^n$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For that purpose let $a, b \in \mathbb{K}^{n+1} \setminus \{0\}$ and define an equivalence relation $\sim$ by

$$a \sim b :\Leftrightarrow \exists \lambda \in \mathbb{K} : \lambda a = b\,.$$

We denote an equivalence class in $\mathbb{K}^{n+1} \setminus \{0\}$ as

$$\pi(b) := (b^0 : b^1 : \ldots : b^n) := \{a \in \mathbb{K}^{n+1} | a \sim b\}\,.$$

Furthermore, let $\mathbb{K}P^n$ denote the set of all equivalence classes.

1. Show that $U_i := \{((b^0, \ldots : b^n) | b^i \neq 0)\} \subset \mathbb{K}P^n$ is open in the quotient topology of $\mathbb{K}P^n$.
2. Deduce that

$$\phi_i : U_i \longrightarrow \mathbb{K}^n\,, \quad (b^0 : \ldots : b^n) \longmapsto \frac{1}{b^i}(b^0, \ldots, b^{i-1}, b^{i+1}, \ldots, b^n)$$

   defines a chart (U_i, ϕ_i) in $\mathbb{K}P^n$.
3. Show that $\mathfrak{A} := \{(U_i, \phi_i) | i = 0, \ldots, n\}$ defines an atlas on $\mathbb{K}P^n$.
4. Verify that π is differentiable.
5. Proove that $\mathbb{K}P^n$ exists as a differentiable (smooth) manifold.
   Hint: Use the definition given in the lecture. Then define for all open $U_j \subset \mathbb{K}P^n$ a set $W_j := (b^0, b^1, \ldots, b^n) \in \mathbb{K}^{n+1} | b^j = 1\}$ and $\sigma_j : U_j \to W_j$ by

$$\sigma_j((b^0 : \ldots : b^n)) := \frac{1}{b^j}(b^0, \ldots, b^{j-1}, b^j, b^{j+1}, \ldots, b^n)$$

   Use them to show that $f : \mathbb{K}P^n \to N$ for some differentiable manifold N is differentiable if and only if $f \circ \pi$ is differentiable.