The Unitary Group in Its Strong Topology

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• $\mathcal{H}$ denotes a complex Hilbert space.

• A unitary operator on $\mathcal{H}$ is a complex linear mapping $T : \mathcal{H} \to \mathcal{H}$ which is surjective and satisfies $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f, g \in \mathcal{H}$.

• Unitary operators are bounded and injective, hence their inverses exist and are unitary as well.

• $\text{U}(\mathcal{H})$ is the group of all unitary operators on $\mathcal{H}$, the unitary group.
The unitary group appears in various different applications in mathematics and physics.

- Operator Theory, Mathematical Physics

- Representation Theory (of Groups, Algebras, ...)

- Geometry and Topology (action on spaces and bundles), e.g. K-Theory
In its **norm topology** $U(\mathcal{H})$ is a real Banach Lie group with models in the Banach space of self-adjoint bounded operators on $\mathcal{H}$.

The **strong topology** is the topology of pointwise convergence on $U(\mathcal{H})$. It is generated by the seminorms

$$p_\varphi(T) = \|T(\varphi)\|, \ T \in U(\mathcal{H}), \ \varphi \in \mathcal{H}.$$ 

The **strong topology** $\tau_s$ is weaker than the **compact open topology** $\tau_c$, which is in turn weaker than the **norm topology** $\tau_n$:

$$\tau_s \subset \tau_c \subset \tau_n$$
The norm topology, however, is too strong for many purposes. For example, for a compact group $G$, the left regular representation

$$\lambda : G \rightarrow \mathbb{U}(L^2(G)),$$ with $\lambda_g \varphi(h) := \varphi(g^{-1}h), \varphi \in L^2(G)$

is in general not continuous with respect to the norm topology.
For this reason, the continuity condition in case of representations $\pi : G \to U(\mathcal{H})$ for a topological group $G$ is the continuity of the 'action' 

$$G \times \mathcal{H} \to \mathcal{H}, \ (g, \varphi) \mapsto \pi(g)(\varphi).$$

This joint continuity turns out to be equivalent to the continuity of $\pi : G \to U(\mathcal{H})$ in the strong topology.

In this sense, the strong topology is the 'right' topology.
In research articles and books one often finds the statement: "‘The unitary group is not a topological group with respect to the strong topology.’"

(Bargmann, Simms, Atiyah)

As a consequence, several proofs related to the strong topology become complicated or unnatural constructions are carried through to compensate the missed property.

However,

the unitary group is a topological group!
**Fact 1:** $\mathcal{U}(\mathcal{H})$ in its strong topology is a topological group.

**Proof:** For the continuity of $(S, S') \mapsto S \circ S'$ at $(T, T') \in \mathcal{U}(\mathcal{H}) \times \mathcal{U}(\mathcal{H})$ one has to show:

$\forall \varepsilon > 0$ and $\forall \varphi \in \mathcal{H}$ there are neighbourhoods $\mathcal{V}$ of $T$ and $\mathcal{V}'$ of $T'$ such that

$\forall (S, S') \in \mathcal{V} \times \mathcal{V}' : S \circ S' \in B_\varphi(T \circ T', \varepsilon),$

where $B_\psi(A, r) = \{ U \in \mathcal{U}(\mathcal{H}) : \| U(\psi) - A(\psi) \| < r \}$ is the semiball given by $\psi$.

With the choices $\psi = T' \varphi$, $\mathcal{V} = B_\psi(T, \varepsilon/2)$ and $B_\varphi(T', \varepsilon/2)$ this condition is satisfied:

$$\| S \circ S'(\varphi) - T \circ T'(\varphi) \| = \| S \circ S'(\varphi) - T \circ S'(\varphi) + T \circ S'(\varphi) - T \circ T'(\varphi) \|$$

$$\leq \| S \circ S'(\varphi) - T \circ S'(\varphi) \| + \| T \circ S'(\varphi) - T \circ T'(\varphi) \|$$

$$\leq \| S(\varphi) - T(\varphi) \| + \| S'(\varphi) - T'(\varphi) \|$$

$$= p_\psi(S - T) + p_\varphi(S' - T') < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$
In order to show the continuity of $S \mapsto S^{-1}$ at $T$ let $\varphi \in \mathcal{H}$ and $\varepsilon > 0$.

With $\psi := T^{-1}\varphi$ we conclude for every $S \in B_\psi(T, \varepsilon)$:

$$\|S^{-1}\varphi - T^{-1}\varphi\| = \|S^{-1} \circ T\psi - \psi\| = \|S \circ S^{-1} \circ T\psi - S\psi\| = \|T\psi - S\psi\| < \varepsilon,$$

hence $S^{-1}(B_\psi(T, \varepsilon)) \subset B_\varphi(T, \varepsilon)$. 

□
Another statement in the mathematical literature is: "\( \tau_s \neq \tau_c \) on \( U(H) \)."

However:

**Fact 2:** *The strong topology \( \tau_s \) on \( U(H) \) coincides with the compact open topology \( \tau_c \).*

This can easily be shown by using the fact that \( U(H) \) is equicontinuous.
The next result is also easy to show.

**Fact 3:** *The unitary group $U(\mathcal{H})$ in its strong topology is metrizable in case of a separable Hilbert space $\mathcal{H}$.***
Fact 4: The unitary group $U(\mathcal{H})$ in its strong topology is contractible in case of $\dim \mathcal{H} = \infty$.

This is a deep result due to Kuiper which is proven along the lines of the corresponding proof for the norm topology: The unitary group $U(\mathcal{H})$ and its complexification $GL(\mathcal{H})$ are contractible in the norm topology $\tau_n$ if $\mathcal{H}$ is infinite dimensional.
**Question:** Does there exist a manifold structure on the unitary group $U(\mathcal{H})$ with its strong topology $\tau_s = \tau_c$?

**Observations:**

- Every s.a. operator $A$ on $\mathcal{H}$ (bounded or unbounded) generates a one-parameter group $t \mapsto \exp itA$ of unitary operators.

- The question has to be made precise since the collection of all s.a. operators is not a vector space.

- Several considerations in physics seem to assume formally that the unitary group has such a Lie group structure.

Thank You very much!