Chapter 11
Mathematical Aspects of the Verlinde Formula

The Verlinde formula describes the dimensions of spaces of conformal blocks (cf. Sect. 9.3) of certain rational conformal field theories (cf. [Ver88]). With respect to a suitable mathematical interpretation, the Verlinde formula gives the dimensions of spaces of generalized theta functions (cf. Sect. 11.1). These dimensions and their polynomial behavior (cf. Theorem 11.6) are of special interest in mathematics. Prior to the appearance of the Verlinde formula, these dimensions were known for very specific cases only, e.g., for the classical theta functions (cf. Theorem 11.5).

The Verlinde formula has been presented by E. Verlinde in [Ver88] as a result of physics. Such a result is, of course, not a mathematical result, it will be considered as a conjecture in mathematics. However, the physical insights leading to the statement of the formula and its justification can be of great help in proving it. Several mathematicians have worked on the problem of proving the Verlinde formula, starting with [TUY89] and coming to a certain end with [Fal94]. These proofs are all quite difficult to understand. For a recent review on general theta functions we refer to the article [Fal08*] of Faltings.

In this last chapter of the present notes we want to explain the Verlinde formula in the context of stable holomorphic bundles on a Riemann surface, that is as a result in function theory or in algebraic geometry. Furthermore, we will sketch a strategy for a proof of the Verlinde formula which uses a kind of fusion for compact Riemann surfaces with marked points. This strategy is inspired by the physical concept of the fusion of fields in conformal field theory as explained in the preceding chapter. We do not explain the interesting transformation from conformal field theory to algebraic geometry. Instead we refer to [TUY89], [Uen95], [BF01*], [Tyu03*].

11.1 The Moduli Space of Representations and Theta Functions

In the following, S is always an oriented and connected compact surface of genus $g = g(S) \in \mathbb{N}_0$ without boundary. The *moduli space of representations* for the group G is

$$\mathscr{M}^G := \mathrm{Hom}(\pi_1(S), G) \big/ G \, .$$

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The equivalence relation indicated by "$/G$" is the conjugation

$$g \sim g' \Longleftrightarrow \exists h \in G : g = hgh^{-1}.$$

Theorem 11.1. *$\mathscr{M}^G$ has a number of quite different interpretations. In the case of $G = \mathrm{SU}(r)$ these interpretations can be formulated in form of the following one-to-one correspondences (denoted by "$\cong$"):*

1. *$\mathscr{M}^{\mathrm{SU}(r)} = \mathrm{Hom}(\pi_1(S), \mathrm{SU}(r)) / \mathrm{SU}(r)$.*

   Topological *interpretation: the set $\mathscr{M}^{\mathrm{SU}(r)}$ is a topological invariant, which carries an amount of information which interpolates between the fundamental group $\pi_1(S)$ and its abelian part*

   $$H_1(S) = \pi_1(S) / [\pi_1(S), \pi_1(S)] ,$$

   the first homology group of S.

2. *$\mathscr{M}^{\mathrm{SU}(r)} \cong$ set of equivalence classes of flat $\mathrm{SU}(r)$-bundles.*

   Geometric *interpretation: there are two related (and eventually equivalent) interpretations of "flat" $\mathrm{SU}(r)$-bundles; "flat" in the sense of a flat vector bundle with constant transition functions and "flat" in the sense of a vector bundle with a flat connection (corresponding to $\mathrm{SU}(r)$ in both cases). Two such bundles are called* equivalent *if they are isomorphic as flat bundles.*

3. *$\mathscr{M}^{\mathrm{SU}(r)} \cong \check{H}^1(S, \mathrm{SU}(r)) \cong H^1(\pi_1(S), \mathrm{SU}(r))$.*

   Cohomological *interpretation: $\check{H}^1(S, \mathrm{SU}(r))$ denotes the first Čech cohomology set with values in $\mathrm{SU}(r)$ (this is not a group in the non-abelian case) and $H^1(\pi_1(S), \mathrm{SU}(r))$ denotes the group cohomology of $\pi_1(S)$ with values in $\mathrm{SU}(r)$.*

4. *$\mathscr{M}^{\mathrm{SU}(r)} \cong \mathscr{A}_0 / \mathscr{G}$.*

   Interpretation as a phase space*: $\mathscr{A}$ is the space of differentiable connections on the trivial bundle $S \times \mathrm{SU}(r) \to S$, $\mathscr{A}_0 \subset \mathscr{A}$ is the subspace of flat connections and $\mathscr{G}$ is the corresponding gauge group of bundle automorphisms, that is*

   $$\mathscr{G} \cong \mathscr{C}^\infty(S, \mathrm{SU}(r)).$$

   $\mathscr{A}_0 / \mathscr{G}$ appears as the phase space of a three-dimensional Chern–Simons theory with an internal symmetry group $\mathrm{SU}(r)$ with respect to a suitable gauge (cf. [Wit89]*).*

5. *$\mathscr{M}^{\mathrm{SU}(r)} \cong$ moduli space of semi-stable holomorphic vector bundles E on S of rank r with* $\det E = \mathscr{O}_S$.

   Complex analytical *interpretation: here, one has to introduce a complex structure J on the surface S such that S equipped with J is a Riemann surface S_J. The vector bundles in the above moduli space are holomorphic with respect to this complex structure and the sheaf $\mathscr{O}_S$ is the structure sheaf on S_J. To emphasize the dependence on the complex structure J on S, we denote this moduli space by $\mathscr{M}_J^{\mathrm{SU}(r)}$.*

To prove the above bijections "$\cong$" in the cases 2., 3., and 4. is an elementary exercise for understanding the respective concepts. Case 5. is a classical theorem of Narasimhan and Seshadri [NS65] and is much more involved.

In each of these cases, "$\cong$" is just a bijection of sets. However, the different interpretations yield a number of different structures on $\mathscr{M}^{\mathrm{SU}(r)}$. In 1., for instance, $\mathscr{M}^{\mathrm{SU}(r)}$ obtains the structure of a subvariety of $\mathrm{SU}(r)^{2g}/\mathrm{SU}(r)$ (because of the fact that $\pi_1(S)$ is a group of $2g$ generators and one relation, cf. (11.4) below), in 4. the set $\mathscr{M}^{\mathrm{SU}(r)}$ obtains the structure of a symplectic manifold and in 5., according to [NS65], the structure of a Kähler manifold outside the singular points of $\mathscr{M}^{\mathrm{SU}(r)}$.

Among others, there are three important generalizations of Theorem 11.1:

- to other Lie groups G instead of $\mathrm{SU}(r)$,
- to higher-dimensional compact manifolds M instead of S and, in particular, to Kähler manifolds in connection with 5.
- to $S \setminus \{P_1, \ldots, P_m\}$ instead of S with points $P_1, \ldots, P_m \in S$ (cf. Sect. 11.3) and a suitable fixing of the vector bundle structure near the points $P_1, \ldots, P_m \in S$.

To begin with, we do not discuss these more general aspects, but rather concentrate on $\mathscr{M}^{\mathrm{SU}(r)}$. The above-mentioned structures induce the following properties on $\mathscr{M}^{\mathrm{SU}(r)}$:

- $\mathscr{M}^{\mathrm{SU}(r)}$ has a natural symplectic structure, which is induced by the following 2-form ω on the affine space

$$\mathscr{A} \cong \mathscr{A}^1(S, \mathfrak{su}(r))$$

of connections:

$$\omega(\alpha, \beta) = c \int_S \mathrm{tr}(\alpha \wedge \beta) \tag{11.1}$$

for $\alpha, \beta \in \mathscr{A}^1(S, \mathfrak{su}(r))$ with a suitable constant $c \in \mathbb{R} \setminus \{0\}$.

Here,

$$\mathrm{tr} : \mathfrak{su}(r) \to \mathbb{R}$$

is the trace of the complex $r \times r$-matrices with respect to the natural representation. In what sense this defines a symplectic structure on $\mathscr{A}$ and on $\mathscr{A}_0/\mathscr{G}$ will be explained in more detail in the following.

In fact, for a connection $A \in \mathscr{A}$ the tangent space $T_A\mathscr{A}$ of the affine space $\mathscr{A}$ can be identified with the vector space $\mathscr{A}^1(S, \mathfrak{su}(r))$ of $\mathfrak{su}(r)$-valued differentiable 1-forms. Hence, a 2-form on $\mathscr{A}$ is given by a family $(\omega_A)_{A \in \mathscr{A}}$ of bilinear mappings ω_A on $\mathscr{A}^1(S, \mathfrak{su}(r)) \times \mathscr{A}^1(S, \mathfrak{su}(r))$ depending differentiably on $A \in \mathscr{A}$. Now, the map

$$\omega : \mathscr{A}^1(S, \mathfrak{su}(r)) \times \mathscr{A}^1(S, \mathfrak{su}(r)) \to \mathbb{C}$$

defined by (11.1) is independent of $A \in \mathscr{A}$ with respect to the natural trivialization of the cotangent bundle

$$T^*\mathscr{A} = \mathscr{A} \times \mathscr{A}^1(S, \mathfrak{su}(r))^*.$$

Consequently, ω with (11.1) is a closed 2-form. It is nondegenerate since $\omega(\alpha,\beta) = 0$ for all α implies $\beta = 0$. Hence, it is a symplectic form on $\mathscr{A}$ defining the symplectic structure. Moreover, it can be shown that the pushforward of $\omega|_{\mathscr{A}_0}$ with respect to the projection $\mathscr{A}_0 \to \mathscr{A}_0/\mathscr{G}$ gives a symplectic form $\omega_{\mathscr{M}}$ on the regular part of $\mathscr{A}_0/\mathscr{G}$. Indeed, $\mathscr{A}_0/\mathscr{G}$ is obtained by a general Marsden–Weinstein reduction of $(\mathscr{A},\omega)$ with respect to the action of the gauge group $\mathscr{G}$ where the curvature map turns out to be a moment map.

This symplectic form $\omega_{\mathscr{M}}$ is also induced by Chern–Simons theory (cf. [Wit89]). $\mathscr{A}_0/\mathscr{G}$ with this symplectic structure is the phase space of the classical fields.

- Moreover, on $\mathscr{M}^{\mathrm{SU}(r)}$ there exists a natural line bundle $\mathscr{L}$ (the determinant bundle) – which is uniquely determined up to isomorphism – together with a connection ∇ on $\mathscr{L}$ whose curvature is $2\pi i\omega$. With a fixed complex structure J on S, for instance, the line bundle $\mathscr{L}$ has the following description:

$$\Theta := \left\{ [E] \in \mathscr{M}_J^{\mathrm{SU}(r)} : \dim_{\mathbb{C}} H^0(S,E) \geq 1 \right\}$$

is a Cartier divisor (the "theta divisor") on $\mathscr{M}_J^{\mathrm{SU}(r)}$, for which the sheaf

$$\begin{aligned} \mathscr{L} = \mathscr{L}_\Theta = \mathscr{O}(\Theta) &= \text{sheaf of meromorphic functions } f \text{ on } \mathscr{M}_J^{\mathrm{SU}(r)} \\ &\quad \text{with } (\mathrm{f}) + \Theta \geq 0 \end{aligned}$$

is a locally free sheaf of rank 1. Hence, $\mathscr{L}$ is a complex line bundle, which automatically is holomorphic with respect to the complex structure on the moduli space induced by J. ($H^0(S,E)$ is the vector space of holomorphic sections on the compact Riemann surface $S = S_J$ with values in the holomorphic vector bundle E and $[E]$ denotes the equivalence class represented by E.)

Definition 11.2. The space of holomorphic sections in $\mathscr{L}^k$, that is

$$H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)},\ \mathscr{L}^k\right),$$

is the space of *generalized theta functions* of level $k \in \mathbb{N}$.

Here, $\mathscr{L}^k$ is the k-fold tensor product of $\mathscr{L}$: $\mathscr{L}^k = \mathscr{L} \otimes \ldots \otimes \mathscr{L}$ (k-fold). Since $\mathscr{M}_J^{\mathrm{SU}(r)}$ is compact, $H^0(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k)$ is a finite-dimensional vector space over $\mathbb{C}$.

In the context of geometric quantization, the space

$$H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)},\ \mathscr{L}\right)$$

can be interpreted as the quantized state space for the phase space $(\mathscr{M}^{\mathrm{SU}(r)},\omega)$, prequantum bundle $\mathscr{L}$ and holomorphic polarization J. A similar result holds for $H^0(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k)$. To explain this we include a short digression on geometric quantization (cf. [Woo80] for a comprehensive introduction):

Geometric Quantization. Geometric quantization of a classical mechanical system proceeds as follows. The classical mechanical system is supposed to be represented by a symplectic manifold (M,ω). For quantizing (M,ω) one needs two additional geometric data, a prequantum bundle and a polarization. A *prequantum bundle* is a complex line bundle $L \to M$ on M together with a connection ∇ whose curvature is $2\pi i\omega$. A *polarization* F on M is a linear subbundle F of (that is a distribution on) the complexified tangent bundle $TM^{\mathbb{C}}$ fulfilling some compatibility conditions. An example is the bundle F spanned by all "y-directions" in $M = \mathbb{R}^2$ with coordinates (x,y) or on $M = \mathbb{C}^n$ the complex subspace of $TM^{\mathbb{C}}$ spanned by the directions $\frac{\partial}{\partial z_j}, j = 1,\ldots,n$. This last example is the holomorphic polarization which has a natural generalization to arbitrary complex manifolds M. Now the (uncorrected, see (11.3)) state space of geometric quantization is

$$Z := \{s \in \Gamma(M,L) : s \text{ is covariantly constant on } F\}.$$

Here, $\Gamma(M,L)$ denotes the $\mathscr{C}^\infty$-sections on M of the line bundle L and the covariance condition means that $\nabla_X s = 0$ for all local vector fields $X : U \to F \subset TM^{\mathbb{C}}$ with values in F. In case of the holomorphic polarization the state space Z is simply the space $H^0(M,L)$ of holomorphic sections in L.

Back to our moduli space $\mathscr{M}_J^{\mathrm{SU}(r)}$ with symplectic form $\omega_{\mathscr{M}}$, the holomorphic line bundle $\mathscr{L} \to \mathscr{M}_J^{\mathrm{SU}(r)}$, and holomorphic polarization one gets the following: for every $k \in \mathbb{N}$, $\mathscr{L}^k$ is a prequantum bundle of $(\mathscr{M}_J^{\mathrm{SU}(r)}, k\omega_{\mathscr{M}})$. Consequently, $H^0(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k)$ is the (uncorrected) state space of geometric quantization.

In order to have a proper quantum theory constructed by geometric quantization it is necessary to develop the theory in such a way that the state space Z obtains an inner product. By an appropriate choice of the prequantum bundle and the polarization one has to try to represent those observables one is interested in as self-adjoint operators on the completion of Z (see [Woo80]). We are not interested in these matters and only want to point out that the space of generalized theta functions has an interpretation as the state space of a geometric quantization scheme: The space

$$H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k\right)$$

is the (uncorrected) quantized state space of the phase space

$$\left(\mathscr{M}_J^{\mathrm{SU}(r)}, k\omega\right),$$

for the prequantum bundle $\mathscr{L}^k$ and for the holomorphic polarization on $\mathscr{M}_J^{\mathrm{SU}(r)}$.

Before continuing the investigation of the spaces of generalized theta functions we want to mention an interesting connection of geometric quantization with representation theory of compact Lie groups which we will use later for the description of parabolic bundles. In fact, to a large extent, the ideas of geometric quantization developed by Kirillov, Kostant, and Souriau have their origin in representation theory.

Let G be a compact, semi-simple Lie group with Lie algebra $\mathfrak{g}$ and fix an invariant nondegenerate bilinear form $<,>$ on $\mathfrak{g}$ by which we identify $\mathfrak{g}$ and the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. For simplicity we assume G to be a matrix group. Then G acts on $\mathfrak{g}$ by the *adjoint action*

$$\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g},\ X \to gXg^{-1},$$

$g \in G$, and on $\mathfrak{g}^*$ by the *coadjoint action*

$$\mathrm{Ad}_g^* : \mathfrak{g}^* \to \mathfrak{g}^*, \xi \to \xi \circ \mathrm{Ad}_g,$$

$g \in G$. The orbits $\mathscr{O} = G\xi = \{\mathrm{Ad}_g^*(\xi) : g \in G\}$ of the coadjoint action are called *coadjoint orbit*s. They carry a natural symplectic structure given as follows. For $A \in \mathfrak{g}$ let $X_A : \mathscr{O} \to T\mathscr{O}$ be the Jacobi field, $X_A(\xi) = \frac{d}{dt}(\mathrm{Ad}^*_{e^{tA}}\xi)\,|_{t=0}$. Then by

$$\omega_\xi(X_A, X_B) := \xi([A,B])$$

for $\xi \in \mathscr{O}$, $A,B \in \mathfrak{g}$, we define a 2-form which is nondegenerate and closed, hence a symplectic form.

The coadjoint orbits have another description using the isotropy group $G_\xi = \{g \in G : \mathrm{Ad}_g^*\xi = \xi\}$, namely

$$\mathscr{O} \cong G/G_\xi \cong G^{\mathbb{C}}/B,$$

where $G^{\mathbb{C}}$ is the complexification of G and $B \subset G^{\mathbb{C}}$ is a suitable Borel subgroup. In this manner $\mathscr{O} \cong G^{\mathbb{C}}/B$ is endowed with a complex structure induced from the complex homogeneous (flag) manifold $G^{\mathbb{C}}/B$. ω turns out to be a Kähler form with respect to this complex structure, such that $(\mathscr{O}, \omega)$ is eventually a Kähler manifold. Assume now that we find a holomorphic prequantum bundle on $\mathscr{O}$. Then G acts in a natural way on the state space $H^0(\mathscr{O}, \mathscr{L})$. Based on the Borel–Weil–Bott theorem we have the following result.

Theorem 11.3 (Kirillov [Kir76]). *Geometric quantization of each coadjoint orbit of maximal dimension endowed with a prequantum bundle yields an irreducible unitary representation of G. Every irreducible unitary representation of G appears exactly once amongst these (if one takes account of equivalence classes of prequantum bundles $\mathscr{L} \to \mathscr{O}$ only).*

To come back to our moduli spaces and spaces of holomorphic sections in line bundles we note that a close connection of the spaces of generalized theta functions with conformal field theory is established by the fact that $H^0(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k)$ is isomorphic to the space of conformal blocks of a suitable conformal field theory with gauge symmetry (cf. Sect. 9.3). This is proven in [KNR94] for the more general case of a compact simple Lie group G.

At the end of this section we want to discuss the example $G = \mathrm{U}(1)$ which does not completely fit into the scheme of the groups $\mathrm{SU}(r)$ or groups with a simple complexification. However, it has the advantage of being relatively elementary, and it explains why the elements of $H^0(\mathscr{M}_J^{\mathrm{SU}(1)}, \mathscr{L}^k)$ are called generalized theta functions:

Example 11.4. (e.g. in [Bot91*]) Let G be the abelian group $\mathrm{U}(1)$ and let J be a complex structure on the surface S. Then $\mathscr{M}_J^{\mathrm{U}(1)}$ is isomorphic (as a set) to

1. the moduli space of holomorphic line bundles on the Riemann surface $S = S_J$ of degree 0.
2. the set of equivalence classes of holomorphic vector bundle structures on the trivial C^∞ vector bundle $S_J \times \mathbb{C} \to S_J$.
3. $\mathrm{Hom}(\pi_1(S), \mathrm{U}(1)) \cong \check{H}^1(S, \mathrm{U}(1)) \cong H^1(S_J, \mathscr{O}) \big/ H^1(S, \mathbb{Z})$,
   which is a complex g-dimensional torus where $\mathscr{O}$ is the sheaf of germs of holomorphic functions in S_J.
4. $\mathbb{C}^g \big/ \Gamma \cong$ Jacobi variety of S_J.

Let $\mathscr{L} \to \mathscr{M}_J^{\mathrm{U}(1)}$ be the theta bundle, given by the theta divisor on the Jacobi variety. Then

- $H^0(\mathscr{M}_J^{\mathrm{U}(1)}, \mathscr{L}) \cong \mathbb{C}$ is the space of classical theta functions and
- $H^0(\mathscr{M}_J^{\mathrm{U}(1)}, \mathscr{L}^k)$ is the space of classical theta functions of level k.

Theorem 11.5. $\dim_{\mathbb{C}} H^0(\mathscr{M}_J^{\mathrm{U}(1)}, \mathscr{L}^k) = k^g$ *(independently of the complex structure).*

The Verlinde formula is a generalization of this dimension formula to other Lie groups G instead of $\mathrm{U}(1)$. Here we will only treat the case of the Lie groups $G = \mathrm{SU}(r)$.

11.2 The Verlinde Formula

Theorem 11.6 (Verlinde Formula). *Let*

$$z_k^{\mathrm{SU}(r)}(g) := \dim_{\mathbb{C}} H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k\right).$$

Then

$$\begin{aligned} z_k^{\mathrm{SU}(2)}(g) &= \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin^2 \frac{j\pi}{k+2}\right)^{1-g} \quad \textit{and} \\ z_k^{\mathrm{SU}(r)}(g) &= \left(\frac{r}{k+r}\right)^{g} \sum_{\substack{S\subset\{1,\ldots,k+r\}\\ |S|=r}} \prod_{\substack{s\in S, t\notin S\\ 1\le t\le k+r}} \left|2\sin\pi\frac{s-t}{r+k}\right|^{g-1} \end{aligned} \tag{11.2}$$

for $r \geq 2$.

The theorem (cf. [Ver88], [TUY89], [Fal94], [Sze95], [Bea96], [Bea95], [BT93], [MS89], [NR93], [Ram94], [Sor95]) has a generalization to compact Lie groups for which the complexification is a simple Lie group $G^{\mathbb{C}}$ of one of the types A, B, C, D, or G ([BT93], [Fal94]).

Among other aspects the Verlinde formula is remarkable because

- the expression on the right of the equation actually defines a natural number,
- it is polynomial in k, and
- the dimension does not depend on the complex structure J.

Even the transformation of the second formula into the first for $r = 2$ requires some calculation. Concerning the independence of J: physical insights related to rational conformal field theory imply that the space of conformal blocks does not depend on the complex structure J on S. This makes the independence of the dimension formula of the structure J plausible. However, a mathematical proof is still necessary.

From a physical point of view, the Verlinde formula is a consequence of the fusion rules for the operator product expansion of the primary fields (cf. Sect. 9.3). We will discuss the fusion mathematically in the next section. Using the fusion rules formulated in that section, the Verlinde formula will be reduced to a combinatorical problem, which is treated in Sect. 11.4.

There is a shift $k \to k + r$ in the Verlinde formula which also occurs in other formulas on quantum theory and representation theory. This shift has to do with the quantization of the systems in question and it is often related to a central charge or an anomaly (cf. [BT93]). In the following we will express the shift within geometric quantization or rather metaplectic quantization. This is based on the fact that $H^0(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k)$ can be obtained as the state space of geometric quantization. Indeed, the shift has an explanation as to arise from an incomplete quantization procedure. Instead of the ordinary geometric quantization one should rather take the metaplectic correction.

Metaplectic Quantization. In many known cases of geometric quantization, the actual calculations give rise to results which do not agree with the usual quantum mechanical models. For instance, the dimensions of eigenspaces turn out to be wrong or shifted. This holds, in particular, for the Kepler problem (hydrogen atom) and the harmonic oscillator. Because of this defect of the geometric quantization occurring already in elementary examples one should consider the metaplectic correction which in fact yields the right answer in many elementary classical systems, in particular, in the two examples mentioned above. To explain the procedure of metaplectic correction we restrict to the case of a Kähler manifold (M, ω) with Kähler form ω as a symplectic manifold. In this situation a *metaplectic structure* on M is given by a spin structure on M which in turn is given by a square root $K^{\frac{1}{2}}$ of the canonical bundle K on M. (K is the holomorphic line bundle $\det T^*M$ of holomorphic n-forms, when n is the complex dimension of M.) The metaplectic correction means – in the situation of the holomorphic polarization – taking the spaces

$$Z^m = H^0\left(M, L \otimes K^{\frac{1}{2}}\right) \tag{11.3}$$

as the state spaces replacing $Z = H^0(M, L)$.

In the context of our space of generalized theta functions the metaplectic correction is

$$Z^m = H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k \otimes \mathscr{K}^{\frac{1}{2}}\right),$$

where $\mathscr{K}$ is the canonical bundle of $\mathscr{M}_J^{\mathrm{SU}(r)}$.

Now, the canonical bundle of $\mathscr{M}_J^{\mathrm{SU}(r)}$ turns out to be isomorphic to the dual of $\mathscr{L}^{2r}$, hence a natural metaplectic structure in this case is $\mathscr{K}^{\frac{1}{2}} = \mathscr{L}^{-r}$ (:= dual of $\mathscr{L}^r$). As a result of the metaplectic correction the shift disappears:

$$Z^m = H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k \otimes \mathscr{L}^{-r}\right) = H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^{k-r}\right).$$

The dimension of the corrected state space Z^m is

$$d_k^{m,\mathrm{SU}(r)}(g) = \dim H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k \otimes \mathscr{L}^{-r}\right)$$

and we see

$$d_k^{m,\mathrm{SU}(r)}(g) = d_{k-r}^{\mathrm{SU}(r)}(g).$$

This explanation of the shift is not so accidental as it looks at first sight. A similar shift appears for a general compact simple Lie group G. To explain the shift in this more general context one has to observe first that r is the dual Coxeter number of $\mathrm{SU}(r)$ and that the shift for general G is $k \to k + h^\vee$ where $h^\vee$ is the dual Coxeter number of G (see [Fuc92], [Kac90] for the dual Coxeter number which is the Dynkin index of the adjoint representation of G). Now, the metaplectic correction again explains the shift because the canonical bundle on the corresponding moduli space $\mathscr{M}_J^G$ is isomorphic to $\mathscr{L}^{-2h}$.

Another reason to introduce the metaplectic correction appears in the generalization to higher-dimensional Kähler manifolds X instead of S_J. In order to obtain a general result on the deformation independence of the complex structure generalizing the above independence result it seems that only the metaplectic correction gives an answer at all. This has been shown in [Sche92], [ScSc95].

A different but related explanation of the shift by the dual Coxeter number of a nature closer to mathematics uses the Riemann–Roch formula for the evaluation of the $d_k^G(g)$ where h appears in the Todd genus of $\mathscr{M}_J^G$ because of $\mathscr{L}^{-2h} = \mathscr{K}$.

11.3 Fusion Rules for Surfaces with Marked Points

In this section G is a simple compact Lie group which we assume to be $\mathrm{SU}(2)$ quite often for simplification.

As above, let $S_J =: \Sigma$ be a surface S of genus g with a complex structure J. We fix a *level* $k \in \mathbb{N}$.

Let $P = (P_1, \dots, P_m) \in S^m$ be (pairwise different) points of the surface, which will be called the *marked points*. We choose a labeling $R = (R_1, \dots, R_m)$ of the marked points, that is, we associate to each point P_j an (equivalence class of an) irreducible representation R_j of the group G as a *label*.

From Theorem 11.3 of Kirillov we know that these representations R_j correspond uniquely to quantizable coadjoint orbits $\mathscr{O}_j$ of maximal dimension in $\mathfrak{g}^*$. Using the invariant bilinear form on $\mathfrak{g}$ the $\mathscr{O}_j$s correspond to adjoint orbits in $\mathfrak{g}$ and these, in turn, correspond to conjugacy classes $C_j \subset G$ by exponentiation. The analogue of the moduli space $\mathscr{M}^G$ will be defined as

$$\mathscr{M}^G(P,R) := \left\{\rho \in \operatorname{Hom}(\pi_1(S \setminus P), G) : \rho(c_j) \in C_j\right\}/G.$$

Here, c_j denotes the representative in $\pi_1(S \setminus P)$ of a small positively oriented circle around P_j.

Note that the fundamental group $\pi_1(S \setminus P)$ of $S \setminus P$ is isomorphic to the group generated by

$$a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_m$$

with the relation

$$\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} \prod_{i=1}^{m} c_i = 1. \tag{11.4}$$

In the case of $G = \mathrm{SU}(2)$ the R_j correspond to conjugacy classes C_j generated by

$$\begin{pmatrix} e^{2\pi i\theta_j} & 0 \\ 0 & e^{-2\pi i\theta_j} \end{pmatrix} \quad =: \quad g_j. \tag{11.5}$$

Let us suppose the θ_j to be rational numbers. This condition is no restriction of generality (see [MS80]). Hence, we obtain natural numbers N_j with ${g_j}^{N_j} = 1$ which describe the conjugacy classes C_j. We now define the *orbifold fundamental group* $\pi_1^{orb}(S) = \pi_1(S,P,R)$ as the group generated by

$$a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_m$$

with the relations

$$\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} \prod_{i=1}^{m} c_i = 1 \quad \text{and} \quad c_i^{N_i} = 1 \tag{11.6}$$

for $i = 1, \dots, m$, where N_j depends on θ_j. Then $\mathscr{M}^{\mathrm{SU}(2)}(P,R)$ can be written as

$$\operatorname{Hom}(\pi_1^{orb}(S), \mathrm{SU}(2))/\mathrm{SU}(2).$$

Theorem 11.1 has the following generalization to the case of surfaces with marked points.

Theorem 11.7. *Let S be marked by P with labeling R. The following three moduli spaces are in one-to-one correspondence:*

1. $\mathscr{M}^{\mathrm{SU}(2)}(P,R) = \mathrm{Hom}(\pi_1^{orb}(S), \mathrm{SU}(2))/\mathrm{SU}(2)$.
2. *The set of gauge equivalence classes (that is gauge orbits) of singular* $\mathrm{SU}(2)$*-connections, flat on $S \setminus P$ with holonomy around P_j fixed by the conjugacy class C_j induced by R_j, $j = 1, \ldots, m$.*
3. *The moduli space $\mathscr{M}_J^{\mathrm{SU}(2)}(P,R)$ of semi-stable parabolic vector bundles of rank* 2 *with paradegree* 0 *and paradeterminant $\mathscr{O}_S$ for (P,R).*

We have to explain the theorem. To begin with, the moduli space of singular connections in 2. can again be considered as a phase space of a classical system. The classical phase space $\mathscr{A}_0 / \mathscr{G}$ (cf. 4. in Theorem 11.1) is now replaced with the quotient

$$\mathscr{M} := \mathscr{A}_{\mathscr{O}} / \mathscr{G}.$$

Here, $\mathscr{A}_{\mathscr{O}}$ is the space of *singular unitary connections* A on the trivial vector bundle of rank 2 over the surface S subject to the following conditions: over $S \setminus P$ the curvature of A vanishes and at the marked points P_i the curvature is (up to conjugation) locally given by

$$m(A) = \sum T_i \delta(P_i - x)$$

(with the Dirac δ-functional $\delta(P_i - x)$ in P_i) where $T_i \in \mathfrak{su}(2)$ belongs to the adjoint orbit determined by $\mathscr{O}_j$. Hence, $\mathscr{A}_{\mathscr{O}}$ can be understood as the inverse image $m^{-1}(\mathscr{O})$ of a product $\mathscr{O}$ of suitable coadjoint orbits of the dual $(\mathrm{Lie}\mathscr{G})^*$ of the Lie algebra of the gauge group $\mathscr{G}$. Regarding m as a moment map, $\mathscr{M} = \mathscr{A}_{\mathscr{O}} / \mathscr{G}$ turns out to be a generalized Marsden–Weinstein reduction.

A related interpretation of $\mathscr{M}$ in this context is as follows: the differentiable $\mathrm{SU}(2)$-connections A on the trivial rank 2 vector bundle over $S \setminus P$ define a parallel transport along each closed curve γ in $S \setminus P$. Hence, each A determines a group element $W(A,\gamma)$ in $\mathrm{SU}(2)$ up to conjugacy. If A is flat in $S \setminus P$ one obtains a homomorphism $W(A) : \pi_1(S \setminus P) \to \mathrm{SU}(2)$ up to conjugacy (see (11.4) for $\pi_1(S \setminus P)$) since for a flat connection the parallel transport from one point to another is locally independent of the curve connecting the points. Now, the labels R_j at the marked points P_j fix the conjugacy classes C_j assigned by $W(A)$ to the simple circles (represented by c_j in the description (11.4) of the fundamental group $\pi_1(S \setminus P)$) around the marked points: $W(A)(c_j)$ has to be contained in C_j. Hence, the elements of $\mathscr{M}$ define conjugacy classes of representations in $\mathscr{M}^{\mathrm{SU}(2)}(P,R)$ yielding a bijection.

This explains the first bijection of the theorem. The second bijection has been shown by Mehta and Seshadri [MS80] as a generalization of the theorem of Narasimhan and Seshadri [NS65] (cf. Theorem 11.1). To understand it, we need the following concepts:

Definition 11.8. A *parabolic structure* on a holomorphic vector bundle E of rank r over a marked Riemann surface $\Sigma = S_J$ with points $P_1, \ldots, P_m \in \Sigma$ is given by the following data:

- a *flag* of proper subspaces in every fiber E_i of E over P_i:

$$E_i = F_i^{(0)} \supset \cdots \supset F_i^{(r_i)} \supset \{0\}$$

with $k_i^{(s)} := \dim F_i^{(s)} \big/ F_i^{(s+1)}$ as multiplicities, and
- a sequence of weights $\alpha_i^{(s)}$ corresponding to every flag with

$$0 \leq \alpha_i^{(0)} \leq \ldots \leq \alpha_i^{(r_i)} \leq 1.$$

The *paradegree* of such a parabolic bundle E is

$$\text{paradeg}\, E := \deg(E) + \sum_i d_i \qquad \textit{with } d_i := \sum_s \alpha_i^{(s)} k_i^{(s)}.$$

A parabolic bundle E is semi-stable if for all parabolic subbundles F of E one has:

$$(\text{rg}(F))^{-1}\text{paradeg}\, F \leq (\text{rg}(E))^{-1}\text{paradeg}\, E.$$

E is *stable* if "$\leq$" can be replaced with "$<$".

The *paradeterminant* for this parabolic structure (resp. for these weights at the marked points) is the usual determinant $\det E = \bigwedge^r E$ tensored with the holomorphic line bundle given by $\mathcal{O}_\Sigma(-\sum d_i x_i)$ for the divisor $-\sum d_i P_i$ if d_i is an integer. Otherwise the paradeterminant is undefined.

The second bijection in Theorem 11.7 has the following significance: one collects those equivalence classes of parabolic vector bundles over $\Sigma = S_J$, whose weights $\alpha_i^{(s)}$ are rational and for which all $d_j := \sum_s \alpha_j^{(s)} k_j^{(s)}$ are integers. Then the $\alpha_j^{(s)}$ fix suitable conjugacy classes in $\text{SU}(r)$ and hence a labeling through irreducible representations R_j. Conversely, given the labels R_j attached to the points, only those parabolic bundles are considered where the weights fit the labels. Now the space

$$\mathcal{M}_J^{\text{SU}(r)}(P,R)$$

consists of the equivalence classes of such parabolic vector bundles, which, in addition, are semi-stable with paradegree 0 and trivial paradeterminant. For instance, for $r = 2$ the representation ρ belonging to $[E] \in \mathcal{M}_J^{\text{SU}(2)}(P,R)$ is given on the c_j by

$$\rho(c_j) = \begin{cases} \exp\, 2\pi i \,\text{diag}\left(\alpha_j^{(0)}, \alpha_j^{(0)}\right) & \text{for} \quad k_j^{(0)} = 2 \\ \exp\, 2\pi i \,\text{diag}\left(\alpha_j^{(0)}, \alpha_j^{(1)}\right) & \text{for} \quad k_j^{(0)} = 1 = k_j^{(1)}. \end{cases}$$

The moduli space $\mathcal{M}_J^{\text{SU}(2)}(P,R)$ is according to [MS80] in a one-to-one correspondence to

$$\text{Hom}(\pi_1^{\text{orb}}(S), \text{SU}(2)) \big/ \text{SU}(2)\,.$$

Furthermore,

$$\mathscr{M}_J^{\mathrm{SU}(2)}(P,R)$$

has the structure (depending on J) of a projective variety over $\mathbb{C}$. In this variety, the stable parabolic vector bundles correspond to the regular points. An analogous theorem holds for parabolic vector bundles of rank r (cf. [MS80]).

In the case of $P = \emptyset$ the moduli space

$$\mathscr{M}_{J,g}^{\mathrm{SU}(2)}(P,R) := \mathscr{M}_J^{\mathrm{SU}(2)}(P,R)$$

coincides with the previously introduced moduli space $\mathscr{M}_J^{\mathrm{SU}(2)}$ (cf. Sect. 11.1). Recall that $\mathscr{M}_J^{\mathrm{SU}(2)}$ has a natural line bundle $\mathscr{L}$ which is used to introduce the generalized theta functions or conformal blocks. This has a generalization to the case $P \neq \emptyset$: $\mathscr{M}_{J,g}^{\mathrm{SU}(2)}(P,R)$ possesses a natural line bundle $\mathscr{L}$ – the determinant bundle or the theta bundle – together with a connection whose curvature is $2\pi i \omega_{\mathscr{M}}$. Here, $\omega_{\mathscr{M}}$ is the Kähler form on the regular locus of $\mathscr{M}_{J,g}^{\mathrm{SU}(2)}(P,R)$. Now, the finite-dimensional space of holomorphic sections

$$H^0\left(\mathscr{M}_{J,g}^{\mathrm{SU}(2)}(P,R), \mathscr{L}^k\right)$$

is the *space of generalized theta functions* of level k with respect to (P,R).

For our special case of the group $G = \mathrm{SU}(2)$ let us denote by the number $n \in \mathbb{N}$ the (up to isomorphism) uniquely determined irreducible representation $n : \mathrm{SU}(2) \to \mathrm{GL}(V_n)$ with $\dim_{\mathbb{C}} V_n = n+1$. With respect to the level $k \in \mathbb{N}$ only those labels $R = (n_1, \ldots, n_m)$ are considered in the following which satisfy $n_j \leq k$ for $j = 1, \ldots, m$.

Theorem 11.9. (Fusion Rules)

0. $z_k(g; n_1, \ldots, n_m) := \dim_{\mathbb{C}} H^0(\mathscr{M}_{J,g}^{\mathrm{SU}(2)}(P,R), \mathscr{L}^k)$ *does not depend on J and on the position of the points* $P_1, \ldots, P_m \in S$. *Here,* $R = (n_1, \ldots, n_m)$. *Let* $\mathscr{M}_{g,m}$ *be the moduli space of marked Riemann surfaces of genus g with m points and let* $\overline{\mathscr{M}}_{g,m}$ *be the Deligne–Mumford compactification of* $\mathscr{M}_{g,m}$. *Then, the bundle* $\pi : Z_{g,k}(R) \to \mathscr{M}_{g,m}$ *with fiber*

$$\pi^{-1}(J,P) = H^0\left(\mathscr{M}_{J,g}^{\mathrm{SU}(2)}(P,R), \mathscr{L}^k\right)$$

has a continuation $\overline{Z}_{g,k}(R) \to \overline{\mathscr{M}}_{g,m}$ *to* $\mathscr{M}_{g,m}$ *as a locally free sheaf of rank* $z_k(g; n_1, \ldots, n_m)$.

1. $z_k(g; n_1, \ldots, n_m) = \sum_{n=0}^{k} z_k(g-1; n_1, \ldots, n_m, n, n)$.
2. *For* $1 \leq s \leq m$ *one has*

$$\begin{aligned} &z_k(g' + g''; n_1, \ldots, n_m) \\ &= \sum_{n=0}^{k} z_k(g'; n_1, \ldots, n_s, n) z_k(g''; n, n_{s+1}, \ldots, n_m). \end{aligned}$$

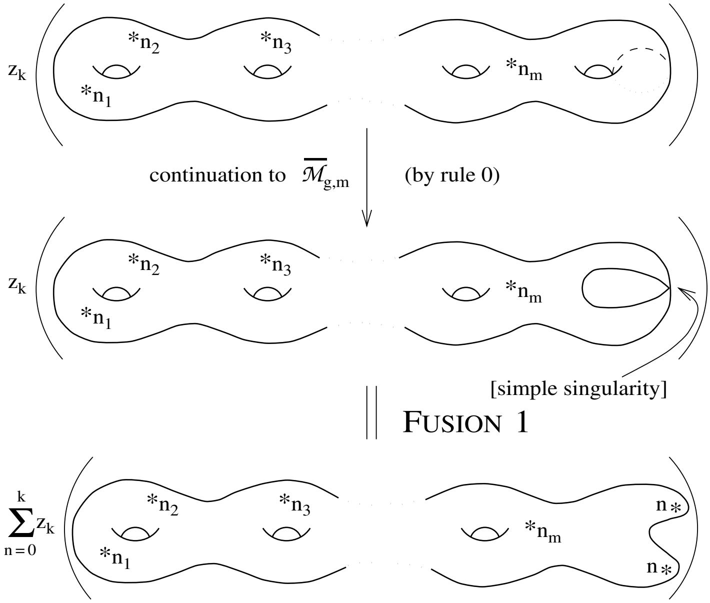

Fig. 11.1 Fusion rule 1

The formulation of the fusion rules for $\mathrm{SU}(2)$ in Theorem 11.9 is special since every representation ρ of the group $\mathrm{SU}(2)$ is equivalent to its conjugate representation ρ^* (Figs. 11.1 and 11.2). For more general Lie groups G instead of $\mathrm{SU}(2)$, one of the two representations (n,n) in the fusion rules has to be replaced with its conjugate.

A proof of the fusion rules 1 and 2 in approximately this form can be found in [NR93] together with [Ram94].

Even in the case of $P = \emptyset$ it is quite difficult to show that the dimensions of $H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k\right)$ do not depend on the complex structure J. This can be deduced from a stronger property which states that the spaces

$$H^0\left(\mathscr{M}_J^{\mathrm{SU}(r)}, \mathscr{L}^k\right)$$

as well as

$$H^0\left(\mathscr{M}_{J,g}^{\mathrm{SU}(r)}(P,R), \mathscr{L}^k\right)$$

are essentially independent of the complex structure. This is in agreement with physical requirements since these spaces are considered to be the result of a quantization which only depends on the topology of S or $S \setminus P$. For this reason the resulting quantum field theory is called a *topological quantum field theory* (cf. [Wit89]). In particular, the state spaces – more precisely their projectivations – should not depend on any metric or complex structure.

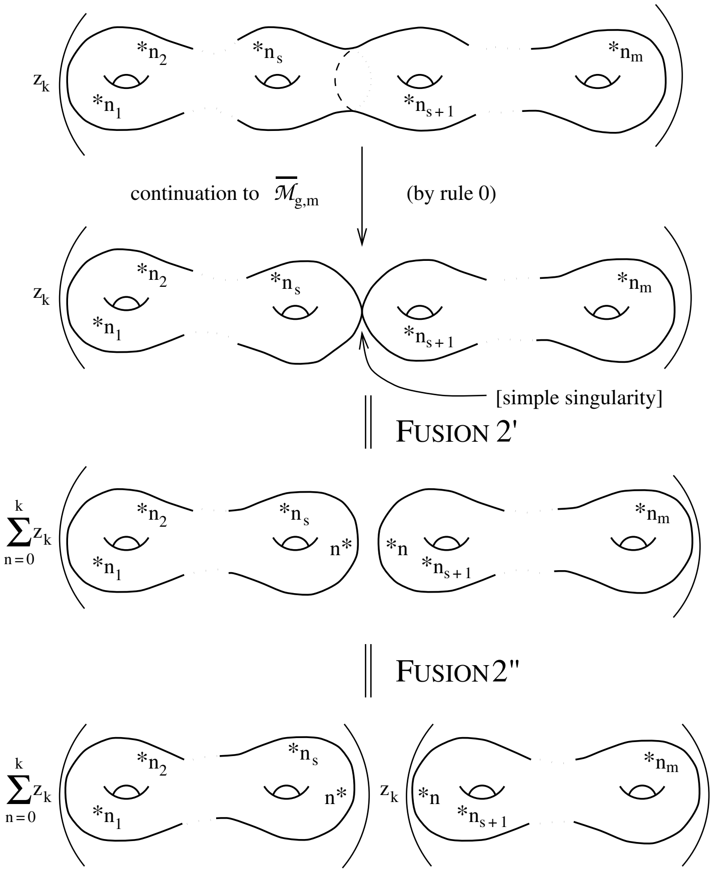

Fig. 11.2 Fusion rule 2 is defined by the successive application of $2'$ and $2''$

That the above state spaces do not depend on the complex structure has been proven in [APW91] and [Hit90] in the case of $P = \emptyset$. Hitchin's methods carry over to the case of $P \neq \emptyset$ using some results of non-abelian Hodge theory [Sche92], [ScSc95]. The strategy of the proof is to consider the bundle $Z_{g,k}(R) \to \mathscr{M}_{g,m}$ over the moduli space $\mathscr{M}_{g,m}$ of Riemann surfaces of genus g and m marked points with fiber $H^0\left(\mathscr{M}_{g,J}^{\mathrm{SU}(r)}(P,R), \mathscr{L}^k\right)$ over $(J,P) \in \mathscr{M}_{g,m}$. On this bundle $Z_{g,k}(R)$ one constructs a natural projectively flat connection. Incidentally, the existence of such a natural projectively flat connection is again motivated by considerations from conformal field theory. Then the fibers of the bundle can be identified in a natural way by parallel transport with respect to this connection up to a constant, that is they are projectively identified. It is remarkable that in the course of the construction in the general case of $P \neq \emptyset$ it seems to be necessary to use the metaplectic correction instead of the uncorrected geometric quantization (see p. 221 and [ScSc95]).

The case $P \neq \emptyset$ is significant for Witten's program, to describe the Jones polynomials of knot theory in the context of quantum field theory. In this picture, the $Z_{k,g}(R)$ are quantum mechanical state spaces, which can be found by path integration [Wit89] or by geometric quantization [Sche92], [ScSc95]. To obtain the knot invariants, one needs, in addition to these state spaces, the corresponding state vectors ("propagators") describing the time development. On the mathematical level this means that one has to assign to a compact three-dimensional manifold M with boundary containing labeled knots a state vector in the state space given by the boundary of M which is a surface with marked points. For instance, one has to assign to such a manifold M with knots $K = (K_1, \dots, K_s)$, labeled by $\mathrm{SU}(r)$-representations and with boundary $\partial M = S_g \cup S'_{g'}$, a vector $Z_k(M,K)$ in

$$Z_{k,g}(R)^* \otimes Z_{k,g'}(R') \cong \mathrm{Hom}(Z_{k,g}(R), Z_{k,g'}(R')).$$

The points in S_g, $S'_{g'}$ and the labels R, R' are induced by the knots $K_1, \dots, K_s$, which may run from boundary to boundary. Only the state spaces together with the state vectors yield a topological quantum field theory. A rigorous construction of these state vectors – which are given by path integration in [Wit89] – is still not known. In the meantime, instead of Witten's original program, other constructions of topological quantum field theories – in some cases by using quantum groups – have been proposed (cf., e.g., [Tur94]) and yield interesting invariants of knots and three manifolds. Related developments are presented in [BK01*].

11.4 Combinatorics on Fusion Rings: Verlinde Algebra

Using the fusion rules of Sect. 11.3, the proof of the Verlinde formula can be reduced to the determination of

$$z_k(0;n), z_k(0;n,m), z_k(0;n,m,l)$$

for $n,m,l \in \{0,\dots,k\}$. This combinatorical reduction has an algebraification, which also has a meaning for more general groups than $\mathrm{SU}(r)$ (cf. [Bea96], [Bea95], [Sze95]).

Definition 11.10 (Fusion Algebra). Let F be a finite-dimensional complex vector space with an element $1 \in F$. For every $g \in \mathbb{Z}, g \geq 0$ and $v_1, \dots, v_m \in F$ let

$$Z(g)_{v_1,\dots,v_m} \in \mathbb{C}$$

be given. $(F,1,Z)$ is a *fusion ring* if the following fusion rules hold:

(F1) $Z(g)_{1,\dots,1} = 1$.
(F2) $Z(g)_{v_1,\dots,v_m} = Z(g)_{1,v_1,\dots,v_m}$ does not depend on the order of the $v_1, \dots, v_m$.
(F3) $v \to Z(0)_{v_1,\dots,v_j,v,v_{j+1},\dots,v_m}$ is $\mathbb{C}$-linear.
(F4) $(v,w) \to Z(0)_{v,w}$ is not degenerated.

We use the notation

$$\int v := Z(0)_v, \quad \langle v,w\rangle := Z(0)_{v,w} \quad \textit{and} \quad \eta(v,w,u) := Z(0)_{v,w,u}.$$

Let $(b_j),(b^j)$ be a pair of bases with $\delta^i_j = \langle b_j, b^i\rangle$. Then, additionally, the following rules hold

(F5) $Z(g)_{v_1,\dots,v_m} = \sum Z(g-1)_{b_j,b^j,v_1,\dots,v_m}$, $g \geq 1$ (Fusion 1).
(F6) $Z(g+g')_{v_1,\dots,v_m,v'_1,\dots,v'_m} = \sum Z(g)_{v_1,\dots,v_m,b_j} Z(g')_{b^j,v'_1,\dots,v'_m}$, (Fusion 2).

One easily proves

Lemma 11.11. *The product $v \cdot w := \sum \eta(v,w,b_j)b^j$ for $v,w \in F$ induces on F the structure of a commutative and associative complex algebra with* 1.

Lemma 11.12. *The bilinear form $\langle , \rangle$ satisfies the trace condition $\langle v\cdot w, x\rangle = \langle v, w\cdot x\rangle$. Therefore, F is a Frobenius algebra.*

Proof. $\langle v\cdot w, x\rangle = \sum \eta(v,w,b_i)\langle b^i, x\rangle$ by definition and linearity. Thus $\langle v\cdot w,x\rangle = \eta(v,w,x)$, since $x = b_i\langle b^i, x\rangle$. In the same way, we obtain $\langle v, w\cdot x\rangle = \langle w\cdot x, v\rangle = \eta(w,x,v) = \eta(v,w,x)$ by (F2). □

Both results need the axioms for $g = 0$ only. With similar arguments one can prove the following version of the Verlinde formula using the fusion rules for general g.

Lemma 11.13. *With $\alpha := \sum b_j b^j = \sum \eta(b_i, b^i, b_k)b^k \in F$ the abstract Verlinde formula holds:*

$$Z(g)_{v_1,\dots,v_m} = \int \alpha^g v_1 \cdot \ldots \cdot v_m.$$

Proof. By induction on m we show

$$Z(g)_{v_1,\dots,v_m} = Z(g)_{v_1\cdot\ldots\cdot v_m}.$$

The case $m = 1$ is trivial. For $m \geq 2$ we have

$$\begin{aligned} Z(g)_{v_1,\dots,v_m} &= \sum Z(0)_{v_1,v_2,b_j} Z(g)_{b^j,v_3,\dots,v_m} \quad \text{by(F6)} \\ &= \sum \eta(v_1,v_2,b_j) Z(g)_{b^j,v_3,\dots,v_m} \\ &= Z(g)_{\sum \eta(v_1,v_2,b_j)b^j,v_3,\dots,v_m} \quad \text{by(F3)} \\ &= Z(g)_{v_1\cdot v_2,v_3,\dots,v_m} \quad \text{by the definition of the product} \\ &= Z(g)_{v_1\cdot v_2\cdot v_3\cdot\ldots\cdot v_m} \quad \text{by the induction hypothesis.} \end{aligned}$$

This implies

$$Z(g)_v = \sum Z(g-1)_{b_j,b^j,v} = Z(g-1)_{\sum b_j b^j v} = Z(g-1)_{\alpha v}$$

and

$$Z(g)_v = Z(g-1)_{\alpha v} = Z(g-2)_{\alpha^2 v} = Z(0)_{\alpha^g v}.$$

Hence for $v = v_1 \cdot \ldots \cdot v_m$ the claimed statement follows. $\square$

For the derivation of the Verlinde formula (Theorem 11.6) from the fusion rules using Lemma 11.13 we refer to [Sze95], where general simple Lie groups instead of $\mathrm{SU}(2)$ are treated.

To indicate the role of the above formula as an abstract Verlinde formula let us represent F as the algebra of functions on the spectrum $\Sigma = \mathrm{Spec}\, F$, that is the finite set of algebra homomorphisms $h : F \to \mathbb{C}$ satisfying, in particular, $h(1) = 1$. With the aid of the Gelfand map $v \mapsto \hat{v}, \hat{v}(h) = h(v)$, we identify F and the function algebra $\mathrm{Map}(\Sigma)$. The structure map $Z(0) : F \to \mathbb{C}$ induces on $F = \mathrm{Map}(\Sigma)$ a complex measure μ which is given by a map $\mu : \Sigma \to \mathbb{C}$. We have $Z(0)_v = \int v d\mu = \sum_{h\in\Sigma} v(h)\mu(h)$ and conclude $1 = Z(0)_1 = \int d\mu = \sum \mu(h)$ and $\mu(h) \neq 0$ for all $h \in \Sigma$.

In order to determine the element $\alpha \in F$ from Lemma 11.13 one uses the characteristic functions e_h of the points $h \in \Sigma$ as a basis: $e_h(k) = \delta_{h,k}$. The dual basis e^h is given by $e^h = \mu(h)^{-1} e_h$ because of

$$\langle e_h, e^h \rangle = Z(0)_{e_h, e^h} = \int e_h e^h d\mu = \mu(h).$$

Therefore, $\alpha = \sum \mu(h)^{-1} e_h$ and $\alpha^g = \sum \mu(h)^{-g} e_h$. Inserting this term into the abstract Verlinde formula in 11.13 gives

$$\int \alpha^g d\mu = \sum \mu(h)^{-g} \mu(h) = \sum \mu(h)^{1-g}.$$

Hence, for $Z(g) = Z(g)_1$ we obtain the following formula which is much closer in its appearance to the Verlinde formula (11.2).

Lemma 11.14.

$$Z(g) = \sum_{h\in\Sigma} (\mu(h))^{g-1}.$$

The fusion rules have their origin in the operator product expansion (cf. p. 168). In the case of the conformal field theory associated to a simple Lie group G (like $\mathrm{SU}(2)$ as considered above) the fusion rules are also related to basic properties of the group and its representations. In fact, the fusion rules have a manifestation in the tensor product of representations of G and the fusion algebras considered above turn out to be isomorphic to certain quotients of the representation ring $\mathrm{R}(G)$. These quotients are called *Verlinde algebras* (cf. [Wit93*]).

We describe the Verlinde algebra $V_k(G)$ explicitly in the case of the group $G = \mathrm{SU}(2)$. The representation ring $\mathrm{R}(G)$, that is the ring of (isomorphism classes of) finite-dimensional representations of G with the tensor product as multiplication, is in the case of $G = \mathrm{SU}(G)$ generated by the standard two-dimensional representation V_1. All other irreducible representations are known to be isomorphic to some V_m where V_m is the symmetric product:

$$V_m := V_1^{\odot m} = V_1 \odot \ldots \odot V_1.$$

V_m is the $(m+1)$-dimensional irreducible representation of $\mathrm{SU}(2)$, unique up to isomorphism, in particular, V_0 is the trivial one-dimensional representation. Let b_n denote the isomorphism class of V_n in $\mathrm{R}(\mathrm{SU}(2))$ (denoted by n in the last section). We regard $\mathrm{R}(\mathrm{SU}(2))$ as a vector space over $\mathbb{C}$ and observe that (b_j) is a basis of $\mathrm{R}(\mathrm{SU}(2))$. In particular, $\mathrm{R}(G)$ is an algebra over $\mathbb{C}$.

The multiplication "$\times$" on $\mathrm{R}(G)$ induced by the tensor product is given by the Clebsch–Gordan formula

$$V_m \otimes V_n \cong V_{m+n} \oplus V_{m+n-2} \oplus \ldots \oplus V_{|m-n|}.$$

Hence, on $\mathrm{R}(G)$ we have

$$b_{m+p} \times b_m = \sum_{j=0}^{m} b_{2m+p-2j}.$$

The truncated multiplication of level $k \in \mathbb{N}$ is

$$b_{m+p} \cdot b_m = b_{m+p} \times b_m, \text{ if } 2m+p \leq k,$$

and

$$b_{m+p} \cdot b_m = \sum_{j \geq 2m+p-k}^{m} b_{2m+p-2j} = b_{2k-2m-p} + \ldots + b_p,$$

if $2m+p > k$ and $m+p \leq k$. The definition implies that no terms b_n with $n > k$ can appear in the summation on the right-hand side. The resulting algebra, the *Verlinde algebra* $V_k(\mathrm{SU}(2))$ *of level* k, is the quotient $\mathrm{R}(G)/(b_{k+1})$ with respect to the ideal (b_{k+1}) generated by $b_{k+1} \in \mathrm{R}(G)$. It is a Frobenius algebra and a fusion algebra in the sense of Definition 11.10. It describes the fusion in the level k case for $\mathrm{SU}(2)$.

The Verlinde algebra has a direct description with respect to the basis $b_0, \ldots, b_k$ in the form

$$b_i \cdot b_j = \sum_{m=0}^{k} N_{ij}^m b_m$$

with coefficients $N_{ij}^m \in \{0,1\}$.

Now, the homomorphisms of $V_k(\mathrm{SU}(2))$ can be determined using the fact that all complex homomorphisms on $R(\mathrm{SU}(2))$ have the form

$$h_z(b_n) = \frac{\sin(n+1)z}{\sin z},$$

where $z \in \mathbb{C}$ is a complex number. Such a homomorphism h_z vanishes on (b_{k+1}) if $\sin(k+2)z = 0$. We conclude that the homomorphisms of $V_k(\mathrm{SU}(2))$ are precisely the $k+1$ maps $h_p : V_k(\mathrm{SU}(2)) \to \mathbb{C}$ satisfying

$$h_p(b_j) = \frac{\sin(j+1)z_p}{\sin z_p}, z_p = \frac{p\pi}{k+2}, p = 1, \ldots, k+1.$$

Using

$$Z(0)_{b_j} = \int b_j = \sum_{n=1}^{k+1} b_j(h_n)\mu(h_n),$$

an elementary calculation yields

$$\mu(h_n) = \frac{2}{k+2}\sin^2\frac{n\pi}{k+2}, n = 1,\ldots,k+1,$$

from which the Verlinde formula (11.2) follows by Lemma 11.14.

Recently, a completely different description of the Verlinde algebra using equivariant twisted K-theory has been developed by Freed, Hopkins, and Teleman [FHT03*] (see also [Mic05*], [HJJS08*]).

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