Chapter 10
Vertex Algebras

In this chapter we give a brief introduction to the basic concepts of vertex algebras. Vertex operators have been introduced long ago in string theory in order to describe propagation of string states. The mathematical concept of a vertex (operator) algebra has been introduced later by Borcherds [Bor86*], and it has turned out to be extremely useful in various areas of mathematics. Conformal field theory can be formulated and analyzed efficiently in terms of the theory of vertex algebras because of the fact that the associativity of the operator product expansion of conformal field theory is already encoded in the associativity of a vertex algebra and also because many formal manipulations in conformal field theory which are not always easy to justify become more accessible and true assertions for vertex algebras. As a result, vertex algebra theory has become a standard way to formulate conformal field theory, and therefore cannot be neglected in an introductory course on conformal field theory.

In a certain way, vertex operators are the algebraic counterparts of field operators investigated in Chap. 8 and the defining properties for a vertex algebra have much in common with the axioms for a quantum field theory in the sense of Wightman and Osterwalder–Schrader. This has been indicated by Kac in [Kac98*] in some detail.

The introduction to vertex algebras in this chapter intends to be self-contained including essentially all proofs. Therefore, we cannot present much more than the basic notions and results together with few examples.

We start with the notion of a formal distribution and familiarize the reader with basic properties of formal series which are fundamental in understanding vertex algebras. Next we study locality and normal ordering as well as fields in the setting of formal distributions and we see how well these concepts from physics are described even before the concept of a vertex algebra has been introduced. In particular, an elementary way of operator expansion can be studied directly after knowing the concept of normal ordering. After the definition of a vertex algebra we are interested in describing some examples in detail which have in parts appeared already at several places in the notes (like the Heisenberg algebra or the Virasoro algebra) but, of course, in a different formulation. In this context conformal vertex algebras are introduced which appear to be the right objects in conformal field theory. Finally, the associativity of the operator product expansion is proven in detail. We conclude this chapter with a section on induced representation of Lie algebras because they

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have been used implicitly throughout the notes and show a common feature in many of our constructions.

The presentation in these notes is based mainly on the course [Kac98*] and to some extent also on the beginning sections of the book [BF01*]. Furthermore, we have consulted other texts like, e.g., [Bor86*], [FLM88*], [FKRW95*], [Hua97*], [Bor00*], and [BD04*].

10.1 Formal Distributions

Let $Z = \{z_1, \ldots, z_n\}$ be a set of indeterminates and let R be a vector space over $\mathbb{C}$. A *formal distribution* is a formal series

$$A(z_1, \ldots, z_n) = \sum_{j \in \mathbb{Z}^n} A_j z^j = \sum_{j \in \mathbb{Z}^n} A_{j_1, \ldots, j_n} z_1^{j_1} \ldots z_n^{j_n}$$

with coefficients $A_j \in R$. The vector space of formal distributions will be denoted by $R[[z_1^{\pm}, \ldots, z_n^{\pm}]] = R\left[\left[z_1, \ldots, z_n, z_1^{-1} \ldots, z_n^{-1}\right]\right]$ or $R\,[[Z^{\pm}]]$ for short. It contains the subspace of *Laurent polynomials*

$$R[z_1^{\pm}, \ldots, z_n^{\pm}] = \{A \in R[[z_1^{\pm}, \ldots, z_n^{\pm}]] \,| \\ \exists k, l : A_j = 0 \text{ except for } k \leq j \leq l\}.$$

Here, the partial order on $\mathbb{Z}^n$ is defined by $i \leq j :\Longleftrightarrow i_\nu \leq j_\nu$ for all $\nu = 1, \ldots, n$. $R[[z_1^{\pm}, \ldots, z_n^{\pm}]]$ also contains the subspace

$$R\,[[z_1, \ldots, z_n]] := \{A : A = \sum_{j \in \mathbb{N}^n} A_{j_1, \ldots, j_n} z_1^{j_1} \ldots z_n^{j_n}\}$$

of *formal power series* (here $\mathbb{N} = \{0, 1, 2, \ldots\}$). The space of *formal Laurent series* will be defined only in one variable

$$R\,((z)) = \{A \in R\left[\left[z^{\pm}\right]\right] | \exists k \in \mathbb{Z}\, \forall j \in \mathbb{Z} : j < k \Rightarrow A_j = 0\}.$$

When R is an algebra over $\mathbb{C}$, the usual Cauchy product for power series

$$AB(z) = A(z)B(z) := \sum_{j \in \mathbb{Z}^n} \left(\sum_{i+k=j} A_i B_k \right) z^j$$

is not defined for all formal distributions. However, given $A, B \in R\,[[Z^{\pm}]]$, the product is well-defined whenever A and B are formal Laurent series or when B is a Laurent polynomial. Moreover, the product $A(z)B(w) \in R\,[[Z^{\pm}, W^{\pm}]]$ is well-defined.

In case of $R = \mathbb{C}$, the ring of formal Laurent series $\mathbb{C}\,((z))$ is a field and this field can be identified with the field of fractions of the ring $\mathbb{C}\,[[z]]$ of formal power series in z. In several variables we define $\mathbb{C}\,((z_1, \ldots, z_n))$ to be the field of fractions of the ring $\mathbb{C}[[z_1,\dots,z_n]]$. This field cannot be identified directly with a field of suitable series. For example, $\mathbb{C}((z,w))$ contains $f=(z-w)^{-1}$, but the following two possible expansions of f,

$$\frac{1}{z}\sum_{n\geq 0} z^{-n}w^n = \sum_{n\geq 0} z^{-n-1}w^n\ ,\ -w\sum_{n\geq 0} z^n w^{-n} = -\sum_{n\geq 0} z^n w^{-n-1},$$

give no sense as elements of $\mathbb{C}((z,w))$. Furthermore, these two series represent two different elements in $\mathbb{C}[[z^{\pm},w^{\pm}]]$. This fact and its precise description are an essential ingredient of vertex operator theory. We come back to these two expansions in Remark 10.16.

Definition 10.1. In the case of one variable $z=z_1$ the *residue* of a formal distribution $A\in R[[z^{\pm}]]$, $A(z)=\sum_{j\in\mathbb{Z}} A_j z^j$, is defined to be

$$\mathrm{Res}_z A(z) = A_{-1}\in R.$$

The *formal derivative* $\partial=\partial_z : R[[z^{\pm}]]\to R[[z^{\pm}]]$ is given by

$$\partial\left(\sum_{j\in\mathbb{Z}} A_j z^j\right) = \sum_{j\in\mathbb{Z}} (j+1)A_{j+1}z^j.$$

One gets immediately the formulas

$$\mathrm{Res}_z A(z)B(z) = \sum_{k\in\mathbb{Z}} A_k B_{-k-1},$$
$$\mathrm{Res}_z \partial A(z)B(z) = -\mathrm{Res}_z A(z)\partial B(z) = \sum_{k\in\mathbb{Z}} kA_k B_{-k}$$

provided the product AB is defined. The following observation explains the name “formal distribution”:

Lemma 10.2. *Every $A\in R[[z^{\pm}]]$ acts on $\mathbb{C}[z^{\pm}]$ as a linear map*

$$\widehat{A}:\mathbb{C}[z^{\pm}]\to R,$$

given by $\widehat{A}\big(f(z)\big):=\mathrm{Res}_z A(z)f(z), \phi\in\mathbb{C}[z^{\pm}]$, thereby providing an isomorphism $R[[z^{\pm}]]\to\mathrm{Hom}(\mathbb{C}[z^{\pm}],R)$.

Proof. Of course, $\widehat{A}\in\mathrm{Hom}(\mathbb{C}[z^{\pm}],R)$, and the map $A\mapsto\widehat{A}$ is well-defined and linear. Due to $\widehat{A}(f)=\sum_{j\in\mathbb{Z}} A_j f_{-(j+1)}$ for $f=\sum f_j z^j$ it is injective. Moreover, any $\mu\in\mathrm{Hom}(\mathbb{C}[z^{\pm}],R)$ defines coefficients $A_j:=\mu(z^{-j-1})\in R$, and the distribution $A:=\sum A_j z^j$ satisfies $\widehat{A}(z^{-j-1})=A_j=\mu(z^{-j-1})$. Hence, $\widehat{A}=\mu$ and the map $A\mapsto\widehat{A}$ is surjective. $\square$

This lemma shows that Laurent polynomials $f\in\mathbb{C}[z^{\pm}]$ can be viewed as to be test functions on which the distributions $A\in R[[z^{\pm}]]$ act.

Definition 10.3. The *formal delta function* is the formal distribution $\delta \in \mathbb{C}[[z^{\pm}, w^{\pm}]]$ in the two variables z, w with coefficients in $\mathbb{C}$ given by

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{n-1}w^{-n} = \sum_{n\in\mathbb{Z}} z^{n}w^{-n-1} = \sum_{n\in\mathbb{Z}} z^{-n-1}w^{n}.$$

Note that δ is the difference of the two above-mentioned expansions of $(z-w)^{-1}$:

$$\delta(z-w) = \sum_{n\geq 0} z^{-n-1}w^{n} - \left(-\sum_{n\geq 0} z^{n}w^{-n-1}\right).$$

We have

$$\delta(z-w) = \sum_{k+n+1=0} z^k w^n = \delta(w-z)$$

and

$$\delta(z-w) = \sum D_{kn} z^k w^n \in \mathbb{C}\left[\left[z^{\pm}, w^{\pm}\right]\right]$$

with coefficients $D_{kn} = \delta_{k,-n-1}$. Hence, for all $f \in R[[z^{\pm}]]$, the product $f(z)\delta(z-w)$ is well-defined and can be regarded as a distribution in $R[[w^{\pm}]])[[z^{\pm}]]$. From the formula

$$f(z)\delta(z-w) = \sum_{n,k\in\mathbb{Z}} f_k z^{k-n-1} w^n = \sum_{k\in\mathbb{Z}} \left(\sum_{n\in\mathbb{Z}} f_{k+n+1} w^n\right) z^k$$

for $f = \sum f_k z^k$ one can directly read off

Lemma 10.4. *For every $f \in R[[z^{\pm}]]$*

$$\mathrm{Res}_z f(z)\delta(z-w) = f(w)$$

and

$$f(z)\delta(z-w) = f(w)\delta(z-w).$$

The last formula implies the first of the following related identities. We use the following convenient abbreviation

$$D_w^j := \frac{1}{j!}\partial_w^j$$

during the rest of this chapter.

Lemma 10.5.

1. $(z-w)\delta(z-w) = 0,$
2. $(z-w)D^{k+1}\delta(z-w) = D^k\delta(z-w)$ *for* $k \in \mathbb{N},$
3. $(z-w)^n D^j\delta(z-w) = D^{j-n}\delta(z-w)$ *for* $j, n \in \mathbb{N}, n \leq j,$
4. $(z-w)^n D^n\delta(z-w) = \delta(z-w)$ *for* $n \in \mathbb{N},$
5. $(z-w)^{n+1}D^n\delta(z-w) = 0$ *for* $n \in \mathbb{N}$, *and therefore*
   $(z-w)^{n+m+1}D^n\delta(z-w) = 0$ *for* $n, m \in \mathbb{N}.$

Proof. 3 and 4 follow from 2, and 5 is a direct consequence of 4 and 1. Hence, it only remains to show 2. One uses $\delta(z-w) = \sum_{m\in\mathbb{Z}} z^{-m-1}w^m$ to obtain the expansion $\partial_w^{k+1}\delta(z-w) = \sum_{m\in\mathbb{Z}} m\dots(m-k)z^{-m-1}w^{m-k-1}$, and one gets

$$(z-w)\partial_w^{k+1}\delta(z-w) = \sum_{m\in\mathbb{Z}} m\dots(m-k)(z^{-m}w^{m-k-1} - z^{-m-1}w^{m-k})$$
$$= \sum_{m\in\mathbb{Z}} ((m+1)m\dots(m-k+1)) - (m\dots(m-k))z^{-m-1}w^{m-k}$$
$$= (k+1)\sum_{m\in\mathbb{Z}} m\dots(m-k+1)z^{-m-1}w^{m-k} = (k+1)\partial_w^k\delta(z-w),$$

which is property 2 of the Lemma. □

As a consequence, for every $N\in\mathbb{N}, N>0$, the distribution $(z-w)^N$ annihilates all linear combinations of $\partial_w^k\delta(z-w)$, $k=0,\dots,N-1$, with coefficients in $R[[w^{\pm}]]$. The next result (due to Kac [Kac98*]) states that these linear combinations already exhaust the subspace of $R[[z^{\pm},w^{\pm}]]$ annihilated by $(z-w)^N$.

Proposition 10.6. *For a fixed $N\in\mathbb{N}$, $N>0$, each*

$$f\in R\left[\left[z^{\pm},w^{\pm}\right]\right] \textit{ with } (z-w)^N f = 0$$

can be written uniquely as a sum

$$f(z,w) = \sum_{j=0}^{N-1} c^j(w)D_w^j\delta(z-w)\ ,\ c^j\in R\left[\left[w^{\pm}\right]\right].$$

Moreover, for such f the formula

$$c^n(w) = \mathrm{Res}_z(z-w)^n f(z,w)$$

holds for $0\le n<N$.

Proof. We have stated already that each such sum is annihilated by $(z-w)^N$ according to the last identity of Lemma (10.5).

The converse will be proven by induction. In the case $N=1$ the condition $(z-w)f(z,w)=0$ for $f(z,w)=\sum f_{nm}z^nw^m\in R[[z^{\pm},w^{\pm}]]$ implies

$$0 = \sum f_{nm}z^{n+1}w^m - f_{nm}z^nw^{m+1} = \sum(f_{n,m+1}-f_{n+1,m})z^{n+1}w^{m+1},$$

and therefore $f_{n,m+1}=f_{n+1,m}$ for all $n,m\in\mathbb{Z}$. As a consequence, $f_{0,m+1}=f_{1,m}=f_{k,m-k-1}$ for all $m,k\in\mathbb{Z}$ which implies

$$f = \sum_{m,k\in\mathbb{Z}} f_{k,m-k-1}z^kw^{m-k-1} = \sum_{m\in\mathbb{Z}} f_{1,m}w^m\sum_{k\in\mathbb{Z}} z^kw^{-k-1} = c^0(w)\delta(z-w)$$

with $c^0(w)=\sum f_{1,m}w^m$. This concludes the proof for $N=1$.

For a general $N\in\mathbb{N}, N>0$, let f satisfy

$$0 = (z-w)^{N+1} f(z,w) = (z-w)^N (z-w) f(z,w).$$

The induction hypothesis gives

$$(z-w) f(z,w) = \sum_{j=0}^{N-1} d^j(w) D^j \delta(z-w),$$

hence, by differentiating with respect to z

$$f + (z-w)\partial_z f = \sum_{j=0}^{N-1} d^j(w) \partial_z D^j \delta(z-w) = -\sum_{j=0}^{N-1} d^j(w)(j+1) D^{j+1} \delta(z-w).$$

Here, we use $\partial_z \delta(z-w) = -\partial_w \delta(z-w)$. Now, applying the induction hypothesis once more to

$$\partial_z((z-w)^{N+1} f) = (z-w)^N ((N+1) f + (z-w)\partial_z f) = 0$$

we obtain

$$(N+1) f + (z-w)\partial_z f = \sum_{j=0}^{N-1} e^j(w) D^j \delta(z-w).$$

By subtracting the two relevant equations we arrive at

$$Nf = \sum_{j=0}^{N-1} e^j(w) D_w^j \delta(z-w) + \sum_{j=1}^{N} j d^{j-1}(w) D^j \delta(z-w),$$

and get

$$f(z,w) = \sum_{j=0}^{N} c^j(w) D^j \delta(z-w)$$

for suitable $c^j(w) \in R[[w^{\pm}]]$.

The uniqueness of this representation of f follows from the formula $c^n(w) = \mathrm{Res}_z (z-w)^n f(z,w)$ which in turn follows from

$$(z-w)^n f(z,w) = c^n(w) f(z,w), 0 \leq n \leq N-1, \text{if} f(z,w) = \sum_{j=0}^{N-1} c^j(w) D^j \delta(z,w)$$

by applying Lemma 10.4. Finally, the identities $(z-w)^n f(z,w) = c^n(w) f(z,w)$ are immediate consequences of

$$(z-w)^n D_w^j \delta(z-w) = 0 \text{ for } n > j$$

and

$$(z-w)^n D_w^j \delta(z-w) = D^{j-n} \delta(z-w)$$

for $n \leq j$ (cf. Lemma 10.5). □

Analytic Aspects. For a rational function $F(z,w)$ in two complex variables z,w with poles only at $z=0, w=0$, or $|z|=|w|$ one denotes the power series expansion of F in the domain $\{|z|>|w|\}$ by $\iota_{z,w}F$ and correspondingly the power series expansion of F in the domain $\{|z|<|w|\}$ by $\iota_{w,z}F$. For example,

$$\iota_{z,w}\frac{1}{(z-w)^{j+1}} = \sum_{m=0}^{\infty}\binom{m}{j} z^{-m-1}w^{m-j},$$
$$\iota_{w,z}\frac{1}{(z-w)^{j+1}} = -\sum_{m=1}^{\infty}\binom{-m}{j} z^{m-1}w^{-m-j}.$$

In particular, as formal distributions

$$\begin{aligned}\iota_{z,w}\frac{1}{(z-w)} - \iota_{w,z}\frac{1}{(z-w)} &= \sum_{m\geq 0} z^{-m-1}w^{m} + \sum_{m>0} z^{m-1}w^{-m}\\ &= \sum_{m\in\mathbb{Z}} z^{-m-1}w^{m} = \delta(z-w)\end{aligned} \tag{10.1}$$

and similarly for the derivatives of δ,

$$D^j\delta(z-w) = \iota_{z,w}\frac{1}{(z-w)^{j+1}} - \iota_{w,z}\frac{1}{(z-w)^{j+1}} = \sum\binom{m}{j} z^{-m-1}w^{m-j}.$$

10.2 Locality and Normal Ordering

Let R be an associative $\mathbb{C}$-algebra. On R one has automatically the *commutator* $[S,T] = ST-TS,$ for $S,T\in R$.

Definition 10.7 (Locality). Two formal distributions $A,B\in R[[z^{\pm}]]$ are *local* with respect to each other if there exists $N\in\mathbb{N}$ such that

$$(z-w)^N[A(z),B(w)] = 0$$

in $R[[z^{\pm},w^{\pm}]]$.

Remark 10.8. Differentiating $(z-w)^N[A(z),B(w)]=0$ and multiplying by $(z-w)$ yields $(z-w)^{N+1}[\partial A(z),B(w)]=0$. Hence, if A and B are mutually local, ∂A and B are mutually local as well.

In order to formulate equivalent conditions of locality we introduce some notations. For $A=\sum A_m z^m$ we mostly write $A=\sum A_{(n)}z^{-n-1}$ such that we have the following convenient formula:

$$A_{(n)} = A_{-n-1} = \mathrm{Res}_z A(z) z^n.$$

We break A into

$$A(z)_- := \sum_{n\geq 0} A_{(n)}z^{-n-1}\ ,\ A(z)_+ := \sum_{n<0} A_{(n)}z^{-n-1}.$$

This decomposition has the property

$$(\partial A(z))_{\pm} = \partial(A(z)_{\pm}),$$

and conversely, this property determines this decomposition.

Definition 10.9. The *normally ordered product* for distributions $A, B \in R[[z^{\pm}]]$ is the distribution

$$\mathbf{:}A(z)B(w)\mathbf{:} := A(z)_{+}B(w) + B(w)A(z)_{-} \in R\left[\left[z^{\pm}, w^{\pm}\right]\right].$$

Equivalently,

$$\mathbf{:}A(z)B(w)\mathbf{:} = \sum_{n\in\mathbb{Z}} \left(\sum_{m<0} A_{(m)}B_{(n)}z^{-m-1} + \sum_{m\geq 0} B_{(n)}A_{(m)}z^{-m-1} \right) w^{-n-1},$$

and the definition leads to the formulas

$$\begin{aligned} A(z)B(w) &= +[A(z)_{-}, B(w)] + \mathbf{:}A(z)B(w)\mathbf{:}, \\ B(w)A(z) &= -[A(z)_{+}, B(w)] + \mathbf{:}A(z)B(w)\mathbf{:}. \end{aligned}$$

With this new notation the result of Proposition 10.6 can be restated as follows.

Theorem 10.10. *The following properties are equivalent for $A, B \in R[[z^{\pm}]]$ and $N \in \mathbb{N}$:*

1. *A, B are mutually local with $(z-w)^N[A(z), B(w)] = 0$.*
2. *$[A(z), B(w)] = \sum_{j=0}^{N-1} C^j(w) D^j \delta(z-w)$ for suitable $C^j \in R[[w^{\pm}]]$.*
3. *$[A(z)_{-}, B(w)] = \sum_{j=0}^{N-1} \iota_{z,w} \frac{1}{(z-w)^{j+1}} C^j(w)$,*
   $-[A(z)_{+}, B(w)] = \sum_{j=0}^{N-1} \iota_{w,z} \frac{1}{(z-w)^{j+1}} C^j(w)$
   for suitable $C^j \in R[[w^{\pm}]]$.
4. *$A(z)B(w) = \sum_{j=0}^{N-1} \iota_{z,w} \frac{1}{(z-w)^{j+1}} C^j(w) + \mathbf{:}A(z)B(w)\mathbf{:}$,*
   $B(w)A(z) = \sum_{j=0}^{N-1} \iota_{w,z} \frac{1}{(z-w)^{j+1}} C^j(w) + \mathbf{:}A(z)B(w)\mathbf{:}$
   for suitable $C^j \in R[[w^{\pm}]]$.
5. *$[A_{(m)}, B_{(n)}] = \sum_{j=1}^{N-1} \binom{m}{j} C^j_{(m+n-j)}$, $m, n \in \mathbb{Z}$, for suitable $C^j = \sum_{k\in\mathbb{Z}} C^j_{(k)} w^{-k-1}$ $\in R[[w^{\pm}]]$.*

The notation of physicists for the first equation in 4 is

$$A(z)B(w) = \sum_{j=0}^{N-1} \frac{C^j(w)}{(z-w)^{j+1}} + {:}A(z)B(w){:}$$

with the implicit assumption of $|z| > |w|$ in order to justify

$$\frac{1}{(z-w)^{j+1}} = \iota_{z,w}\frac{1}{(z-w)^{j+1}}.$$

Another frequently used notation for this circumstance by restricting to the singular part is

$$A(z)B(w) \sim \sum_{j=0}^{N-1} \frac{C^j(w)}{(z-w)^{j+1}}.$$

Here, $\sim$ denotes as before (Sect. 9.2, in particular (9.5)) the asymptotic expansion neglecting the regular part of the series. This is a kind of operator product expansion as in Sect. 9.3, in particular (9.13).

As an example for the operator product expansion in the context of formal distributions and vertex operators, let us consider the Heisenberg algebra H and its generators $a_n, Z \in \mathsf{H}$, with the relations (cf. (4.1) in Sect. 4.1)

$$[a_m, a_n] = m\delta_{m+n}Z\ ,\ [a_m, Z] = 0$$

for $m, n \in \mathbb{Z}$. Let $U(\mathsf{H})$ denote the universal enveloping algebra (cf. Definition 10.45) of H. Then $A(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ defines a formal distribution $a \in U(\mathsf{H})[[z^{\pm}]]$. It is easy to see that

$$[A(z), A(w)] = \partial\delta(z-w)Z,$$

since

$$\sum_{m,n\in\mathbb{Z}} [a_m, a_n] z^{-m-1} w^{-n-1} = \sum_{m\in\mathbb{Z}} m z^{-m-1} w^{m-1} Z.$$

As a result, the distribution A is local with respect to itself. Because of $C^1(w) = Z$ and $C^j(w) = 0$ for $j \neq 1$ in the expansion of $A(z)A(w)$ according to 4 in Lemma 10.5 the operator product expansion has the form

$$A(z)A(w) \sim \frac{Z}{(z-w)^2}.$$

Another example of a typical operator product expansion which is of particular importance in the context of conformal field theory can be derived by replacing the Heisenberg algebra H in the above consideration with the Virasoro algebra Vir. As we know, Vir is generated by $L_n, n \in \mathbb{Z}$, and the central element Z with the relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m}{12}(m^2-1)\delta_{m+n}Z\ ,\ [a_m, Z] = 0,$$

for $m,n \in \mathbb{Z}$. We consider any representation of Vir in a vector space V with $L_n \in \mathrm{End}\ V$ and $Z = c\,\mathrm{id}_V$. Then

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

defines a formal distribution (with coefficients in End V). A direct calculation (see below) shows

$$[T(z),T(w)] = \frac{Z}{12}\partial^3\delta(z-w) + 2T(w)\partial_w\delta(z-w) + \partial_w T(w)\delta(z-w)$$

and, therefore, according to our Theorem 10.5 with $N=4$ the following OPE holds (observe the factor $3! = 6$ in the first equation of property 4 of the theorem):

$$T(z)T(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)}, \tag{10.2}$$

which we have encountered already in (9.5).

In order to complete the derivation of this result let us check the identity for $[T(z),T(w)]$ stated above:

$$[T(z),T(w)] = \sum_{m,n}[L_m,L_n]z^{-m-2}w^{-n-2}$$
$$= \sum_{m,n}(m-n)L_{m+n}z^{-m-2}w^{-n-2} + \sum_m \frac{m}{12}(m^2-1)z^{-m-2}w^{m-2}Z.$$

Substituting $k = m+n$ in the first term and then $l = m+1$ we obtain

$$\begin{aligned}
&\sum_{m,n}(m-n)L_{m+n}z^{-m-2}w^{-n-2}\\
&\quad= \sum_{k,m}(2m-k)L_k z^{-m-2}w^{-k+m-2}\\
&\quad= \sum_{k,l}(2l-k-2)L_k z^{-l-1}w^{-k+l-3}\\
&\quad= 2\sum_{k,l}L_k w^{-k-2} l z^{-l-1}w^{l-1} + \sum_{k,l}(-k-2)L_k w^{-k-3}z^{-l-1}w^{l}\\
&\quad= 2T(w)\partial_w\delta(z-w) + \partial_w T(w)\delta(z-w).
\end{aligned}$$

The second term is (substituting $m+1 = n$)

$$\frac{Z}{12}\sum_n n(n-1)(n-2)z^{-n-1}w^{n-3} = \frac{Z}{12}\partial_w^3\delta(z-w).$$

Note that the expansion (10.2) can also be derived by using property 5 in Lemma 10.5 by explicitly determining the related $C^j_{(n)}$ to obtain $C^j(w)$.

Without proof we state the following result:

Lemma 10.11 (Dong's Lemma). *Assume $A(z), B(z), C(z)$ are distributions which are pairwise local to each other, than the normally ordered product $:A(z)B(z):$ is local with respect to $C(z)$ as well.*

10.3 Fields and Locality

From now on we restrict our consideration to the case of the endomorphism algebra $R = \operatorname{End} V$ of a complex vector space consisting of the linear operators $b : V \to V$ defined on all of V. The value $b(v)$ of b at $v \in V$ is written $b(v) = b.v$ or simply bv.

Definition 10.12. A formal distribution

$$a \in \operatorname{End} V\left[\left[z^{\pm}\right]\right], a = \sum a_{(n)} z^{-n-1},$$

is called a *field* if for every $v \in V$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the condition

$$a_{(n)}(v) = a_{(n)}.v = a_{(n)}v = 0$$

is satisfied.

Equivalently, $a(z).v = \sum (a_{(n)}.v) z^{-n-1}$ is a formal Laurent series with coefficients in V, that is $a(z).v \in V((z))$. We denote the vector space of fields by $\mathscr{F}(V)$. As a general rule, fields will be written in small letters $a, b, \ldots$ in these notes whereas $A, B, \ldots$ are general formal distributions.

We come back to the example given by the Heisenberg algebra and replace the universal enveloping algebra by the Fock space $\mathsf{S} = \mathbb{C}[T_1, T_2, \ldots]$ (cf. (7.12) in Sect. 7.2) in order to have the coefficients in the endomorphism algebra End S and also to relate the example with our previous considerations concerning quantized fields in Sect. 7.2. Hence, we define

$$\Phi(z) := \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

where now the $a_n : \mathsf{S} \to \mathsf{S}$ are given by the representing endomorphisms $a_n = \rho(a_n) \in \operatorname{End} \mathsf{S}$: For a polynomial $P \in \mathsf{S}$ and $n \in \mathbb{N}, n > 0$, we have

$$\begin{aligned}
a_n(P) &= \frac{\partial}{\partial T_n} P, \\
a_0(P) &= 0, \\
a_{-n}(P) &= n T_n P, \\
Z(P) &= P.
\end{aligned}$$

The calculation above shows that Φ is local with respect to itself, and it satisfies the operator product expansion

$$\Phi(z)\Phi(w) \sim \frac{1}{(z-w)^2}$$

with the understanding that a scalar $\lambda \in \mathbb{C}$ (here $\lambda = 1$) as an operator is the operator $\lambda \mathrm{id}_{\mathsf{S}}$. Moreover, Φ is a field: Each polynomial $P \in \mathsf{S}$ depends on finitely many variables T_n, for example on $T_1, \dots, T_k$ and, hence, $a_n P = 0$ for $n > k$. Consequently,

$$\Phi(z)P = \sum_{n\in\mathbb{Z}} a_n(P)z^{-n-1} = \sum_{n\leq k} a_n(P)z^{-n-1} = \sum_{m\geq -k-1} a_{-m-1}(P)z^m$$

is a Laurent series. The field Φ is the quantized field of the infinite set of harmonic oscillators (cf. Sect. 7.2) and thus represents the quantized field of a free boson.

In many important cases the vector space V has a natural $\mathbb{Z}$-grading

$$V = \bigoplus_{n\in\mathbb{Z}} V_n$$

with $V_n = \{0\}$ for $n < 0$ and $\dim V_n < \infty$. An endomorphism $T \in \mathrm{End}\ V$ is called *homogeneous of degree* g if $T(V_n) \subset V_{n+g}$. A formal distribution $a = \sum a_{(n)} z^{-n-1} \in \mathrm{End}\ V[[z^{\pm}]]$ is called *homogeneous of (conformal) weight* $h \in \mathbb{Z}$ if each $a_{(k)} : V \to V$ is homogeneous of degree $h-k-1$. In this case, for a given $v \in V_m$ it follows that $a_{(k)}v \in V_{m+h-k-1}$, and this implies $a_{(k)}v = 0$ for $m+h-k-1 < 0$, that is $k \geq m+h$. Therefore, $\sum_{k\geq m+h} (a_{(k)}v)z^{-k-1}$ is a Laurent series and we have shown the following assertion:

Lemma 10.13. *Any homogeneous distribution* $a \in \mathrm{End}\ V[[z^{\pm}]]$ *is a field.*

In our example of the free bosonic field $\Phi \in \mathrm{End}\ \mathsf{S}[[z^{\pm}]]$ there is a natural grading on the Fock space S given by the degree

$$\deg(\lambda T_{n_1} \dots T_{n_m}) := \sum_{j=1}^{m} n_j$$

of the homogeneous polynomials $P = \lambda T_{n_1} \dots T_{n_m}$:

$$\mathsf{S}_n := \mathrm{span}\{P : P \text{ homogeneous with } \deg(P) = n\}$$

with $\mathsf{S} = \bigoplus \mathsf{S}_n$, $\mathsf{S}_n = \{0\}$ for $n < 0$ and $\dim \mathsf{S}_n < \infty$. Because of $\deg(a_{(k)}P) = \deg(P) - k$ if $a_{(k)}P \neq 0$ ($a_{(k)} = a_k$ in this special example) we see that $a_{(k)}$ is homogeneous of degree $-k$ and the field Φ is homogeneous of weight $h = 1$.

Remark 10.14. The derivative ∂a of a field $a \in \mathscr{F}(V)$ is a field and the normally ordered product $:a(z)b(z):$ of two fields $a(z), b(z)$ is a field as well. Because of $\partial(a(z)_{\pm}) = (\partial a(z))_{\pm}$, the derivative $\partial : \mathscr{F}(V) \to \mathscr{F}(V)$ acts as a derivation with respect to the normally ordered product:

$$\partial(:a(z)b(z):) = :(\partial a(z))b(z): + :a(z)(\partial b(z)):.$$

Moreover, using Dong's Lemma 10.11 we conclude that in the case of three pairwise mutually local fields $a(z), b(z), c(z)$ the normally ordered product $:a(z)b(z):$ is a field which is local with respect to $c(z)$. The corresponding assertion holds for the normally ordered product of more than two fields $a^1(z), a^2(z), \ldots, a^n(z)$ which is defined inductively by

$$:a^1(z)\ldots a^n(z)a^{n+1}(z): := :a^1(z)\ldots :a^n(z)a^{n+1}(z):\ldots:.$$

It is easy to check the following behavior of the weights of homogeneous fields.

Lemma 10.15. *For a homogeneous field a of weight h the derivative ∂a has weight $h+1$, and for another homogeneous field b of weight h' the weight of the normally ordered product $:a(z)b(z):$ is $h+h'$.*

We want to formulate the locality of two fields $a, b \in \mathscr{F}(V)$ by matrix coefficients. For any $v \in V$ and any linear functional $\mu \in V^* = \mathrm{Hom}(V, \mathbb{C})$ the evaluation

$$\langle \mu, a(z).v\rangle = \mu(a(z).v) = \sum \mu(a_{(n)}.v)z^{-n-1}$$

yields a formal Laurent series with coefficients in $\mathbb{C}$, i.e., $\langle \mu, a(z).v\rangle \in \mathbb{C}((z))$. The matrix coefficients satisfy $\langle \mu, a(z)b(w).v\rangle \in \mathbb{C}[[z^{\pm}, w^{\pm}]]$ in any case, since they are formal distributions. A closer inspection regarding the field condition for a and b shows

$$\langle \mu, a(z)b(w).v\rangle = \sum_{n<n_0} \mu(a(z)b_{(n)}.v)w^{-n-1} \in \mathbb{C}((z))((w)).$$

Similarly,

$$\langle \mu, b(w)a(z).v\rangle \in \mathbb{C}((w))((z)).$$

In which sense can such matrix coefficients commute? Commutativity in this context can only mean that the equality

$$\langle \mu, a(z)b(w).v\rangle = \langle \mu, b(w)a(z).v\rangle$$

holds in the intersection of $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$. Consequently, these matrix coefficients of the fields a, b to μ, v commute if and only if the two series are expansions of one and the same element in $\mathbb{C}[[z^{\pm}, w^{\pm}]][z^{-1}, w^{-1}]$. Fields $a, b \in \mathscr{F}(V)$ whose matrix coefficients commute in this sense for all μ, v are local to each other, but locality for fields in general as given in Definition 10.7 is a weaker condition as stated in the following proposition.

Before formulating the proposition we want to emphasize that it is particularly important to be careful with equalities of series regarding the various identifications or embeddings of spaces of series. This is already apparent with our main example, the delta function. Observe that we have two embeddings

$$\mathbb{C}((z,w)) \hookrightarrow \mathbb{C}((z))((w)), \mathbb{C}((z,w)) \hookrightarrow \mathbb{C}((w))((z))$$

of the field of fractions $\mathbb{C}((z,w))$ of $\mathbb{C}[[z,w]]$ induced by the natural embeddings

$$\mathbb{C}[[z,w]] \hookrightarrow \mathbb{C}((z))((w)), \mathbb{C}[[z,w]] \hookrightarrow \mathbb{C}((w))((z))$$

and the universal property of the field of fractions $\mathbb{C}((z,w))$. Moreover, the two spaces $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ both have a natural embedding into $\mathbb{C}[[z^{\pm},w^{\pm}]]$ the full space of formal distributions in the two variables z,w. Now, for a Laurent polynomial $P(z,w) \in \mathbb{C}[z^{\pm},w^{\pm}]$ considered as an element in $\mathbb{C}((z,w))$ the two embeddings of P agree in $\mathbb{C}[[z^{\pm},w^{\pm}]]$. However, this is no longer true for general elements $f \in \mathbb{C}((z,w))$.

Remark 10.16. For example, the element $f = (z-w)^{-1} \in \mathbb{C}((z,w))$ induces the element

$$\delta_-(z-w) = \sum_{n\geq 0} z^{-n-1}w^n = \iota_{z,w}\frac{1}{(z-w)}$$

in $\mathbb{C}((z))((w))$ and the element $-\delta_+(z-w)$ in $\mathbb{C}((w))((z))$ where

$$\delta_+(z-w) = \sum_{n>0} w^{-n}z^{n-1} = \iota_{w,z}\frac{1}{(z-w)}.$$

Hence their embeddings in $\mathbb{C}[[z^{\pm},w^{\pm}]]$ do not agree; the difference $\delta_- - \delta_+$ is, in fact, the delta distribution $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n$, cf. (10.1).

If we now multiply f by $z-w$ we obtain 1 which remains 1 after the embedding into $\mathbb{C}[[z^{\pm},w^{\pm}]]$. Therefore, if we multiply δ_- and $-\delta_+$ by $z-w$ we obtain the same element 1 in $\mathbb{C}[[z^{\pm},w^{\pm}]]$. We are now ready for the content of the proposition.

Proposition 10.17. *Two fields $a,b \in \mathscr{F}(V)$ are local with respect to each other if and only if for all $\mu \in V^*$ and $v \in V$ the matrix coefficients $\langle \mu, a(z)b(w).v\rangle$ and $\langle \mu, b(w)a(z).v\rangle$ are expansions of one and the same element $f_{\mu,v} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1},(z-w)^{-1})]$ and if the order of pole in $z-w$ is uniformly bounded for the $\mu \in V^*, v \in V$.*

Proof. When $N \in \mathbb{N}$ is a uniform bound of the order of pole in $z-w$ of the $f_{\mu,v}$ one has $(z-w)^N f_{\mu,v} \in \mathbb{C}[[z^{\pm},w^{\pm}]][z^{-1},w^{-1}]$ uniformly for all $\mu \in V^*, v \in V$. The expansion condition implies

$$(z-w)^N\langle \mu, a(z)b(w).v\rangle = (z-w)^N f_{\mu,v} = (z-w)^N\langle \mu, b(w)a(z).v\rangle.$$

Consequently, $(z-w)^N\langle \mu, [a(z),b(w)].v\rangle = 0$, and therefore

$$(z-w)^N[a(z),b(w)].v = 0,$$

and finally $(z-w)^N[a(z),b(w)] = 0$.

Conversely, if the fields a,b are local with respect to each other, that is if they satisfy $(z-w)^N[a(z),b(w)] = 0$ for a suitable $N \in \mathbb{N}$, we know already by property 4 of Theorem 10.10 that

$$a(z)b(w) = \sum_{j=0}^{N-1} \iota_{z,w} \frac{1}{(z-w)^{j+1}} c^j(w) + \mathbf{:} a(z)b(w) \mathbf{:}\,,$$

$$b(w)a(z) = \sum_{j=0}^{N-1} \iota_{w,z} \frac{1}{(z-w)^{j+1}} c^j(w) + \mathbf{:} a(z)b(w) \mathbf{:}$$

for suitable fields $c^j \in R[[w^{\pm}]]$ given by $\mathrm{Res}_z (z-w)^j [a(z), b(w)]$. This shows that $\langle \mu, a(z)b(w).v \rangle$ and $\langle \mu, b(w)a(z).v \rangle$ are expansions of

$$\sum_{j=0}^{N-1} \frac{1}{(z-w)^{j+1}} \mu(c^j(w).v) + \mu(\mathbf{:} a(z)b(w) \mathbf{:} v).$$

□

10.4 The Concept of a Vertex Algebra

Definition 10.18. A *vertex algebra* is a vector space V with a distinguished vector Ω (the *vacuum vector*)[1], an endomorphism $T \in \mathrm{End}\ V$ (the *infinitesimal translation operator*)[2], and a linear map $Y : V \to \mathscr{F}(V)$ to the space of fields (the *vertex operator* providing the *state field correspondence*)

$$a \mapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}, a_{(n)} \in \mathrm{End}\ V,$$

such that the following properties are satisfied: For all $a, b \in V$

Axiom V1 (Translation Covariance)

$$[T, Y(a,z)] = \partial Y(a,z),$$

Axiom V2 (Locality)

$$(z-w)^N [Y(a,z), Y(b,w)] = 0$$

for a suitable $N \in \mathbb{N}$ (depending on a, b),

Axiom V3 (Vacuum)

$$T\Omega = 0, Y(\Omega, z) = \mathrm{id}_V, Y(a,z)\Omega|_{z=0} = a.$$

The last condition $Y(a,z)\Omega|_{z=0} = a$ is an abbreviation for $a_{(n)}\Omega = 0, n \geq 0$ and $a_{(-1)}\Omega = a$ when $Y(a,z) = \sum a_{(n)} z^{-n-1}$. In particular,

$$Y(a,z)\Omega = a + \sum_{n<-1} (a_{(n)}\Omega) z^{-n-1} = a + \sum_{k>0} (a_{(-k-1)}\Omega) z^k \in V[[z]].$$

[1] We keep the notation Ω for the vacuum in accordance with the earlier chapters although it is common in vertex algebra theory to denote the vacuum by $|0\rangle$.

[2] Not to be mixed up with the energy–momentum tensor $T(z)$.

Several variants of this definition are of interest.

Remark 10.19. For example, as in the case of Wightman's axioms (cf. Remark 8.12) one can adopt the definition to the supercase in order to include anticommuting fields and therefore the fermionic case. One has to assume that the vector space V is $\mathbb{Z}/2\mathbb{Z}$-graded (i.e., a superspace) and the Locality Axiom V2 is generalized accordingly by replacing the commutator with the anticommutator for fields of different parity. Then we obtain the definition of a *vertex superalgebra.*

Remark 10.20. A different variant concerns additional properties of V since in many important examples V has a natural direct sum decomposition $V = \bigoplus_{n=0}^{\infty} V_n$ into finite-dimensional subspaces V_n. In addition to the above axioms one requires Ω to be an element of V_0 or even $V_0 = \mathbb{C}\Omega$, T to be homogeneous of degree 1 and $Y(a,z)$ to be homogeneous of weight m for $a \in V_m$. We call such a vertex algebra a *graded vertex algebra.*

Remark 10.21. The notation in the axioms could be reduced, for example, the infinitesimal translation operator T can equivalently be described by $Ta = a_{(-2)}\Omega$ for all $a \in V$.

Proof. In fact, the Axiom V1 reads for $Y(a,z) = \sum a_{(n)} z^{-n-1}$:

$$\sum [T, a_{(n)}] z^{-n-1} = \sum (-n-1) a_{(n)} z^{-n-2} = \sum -n a_{(n-1)} z^{-n-1}.$$

Hence, $[T, a_{(n)}] = -n a_{(n-1)}$. Because of $T\Omega = 0$, this implies $Ta_{(n)}\Omega = -na_{(n-1)}\Omega$. For $n = -1$ we conclude $a_{(-2)}\Omega = Ta_{(-1)}\Omega = Ta$, where $a_{(-1)}\Omega = a$ is part of the Vacuum Axiom V3. □

Vertex Algebras and Quantum Field Theory. To bring the new concept of a vertex algebra into contact to the axioms of a quantum field theory as presented in the last two chapters we observe that the postulates for a vertex algebra determine a structure which is similar to axiomatic quantum field theory.

In fact, on the one hand a field in Chap. 8 is an operator-valued distribution

$$\Phi_a : \mathscr{S} \to \text{End } V$$

indexed by $a \in I$ with $V = D$ a suitable common domain of definition for all the operators $\Phi(f), f \in \mathscr{S}$. On the other hand, a field in the sense of vertex algebra theory is a formal series $Y(a,z) \in \text{End } V[[z^{\pm}]]$, $a \in V$, which acts as a map

$$\widehat{Y}(a, \ ) : \mathbb{C}\left[\left[z^{\pm}\right]\right] \to \text{End } V$$

as has been shown in Lemma 10.2. This map resembles an operator-valued distribution with $\mathbb{C}[[z^{\pm}]]$ as the space of test functions.

Locality in the sense of Chap. 9 is transferred into the locality condition in Axiom V2. The OPE and its associativity is automatically fulfilled in vertex algebras (see Theorem 10.36 below). However, the reflection positivity or the spectrum condition has no place in vertex algebra theory since we are not dealing with a Hilbert space. Moreover, the covariance property is not easy to detect due to the absence of an inner product except for the translation covariance in Axiom V2. Finally, the existence of the energy–momentum tensor as a field and its properties according to the presentation in Chap. 9 is in direct correspondence to the existence of a conformal vector in the vertex algebra as described below in Definition 10.30.

Under suitable assumptions a two-dimensional conformally invariant field theory in the sense of Chap. 9 determines a vertex algebra as is shown below (p. 190).

We begin now the study of vertex algebras with a number of consequences of the Translation Covariance Axiom V1. Observe that it splits into the following two conditions:

$$[T, Y(a,z)_{\pm}] = \partial Y(a,z)_{\pm}.$$

The significance of Axiom V1 is explained by the following:

Proposition 10.22. *Any element $a \in V$ of a vertex algebra V satisfies*

$$\begin{aligned} Y(a,z)\Omega &= e^{zT} a, \\ e^{wT} Y(a,z) e^{-wT} &= Y(a,z+w), \\ e^{wT} Y(a,z)_{\pm} e^{-wT} &= Y(a,z+w)_{\pm}, \end{aligned}$$

where the last equalities are in End $V[[z^{\pm}]][[w]]$ *which means that $(z+w)^n$ is replaced by its expansion $\iota_{z,w}(z+w)^n = \sum_{k \geq 0} \binom{n}{k} z^{n-k} w^k \in \mathbb{C}[[z^{\pm}]][[w]]$.*

For the proof we state the following technical lemma which is of great importance in the establishment of equalities.

Lemma 10.23. *Let W be a vector space with an endomorphism $S \in$* End W. *To each element $f_0 \in W$ there corresponds a uniquely determined solution*

$$f = \sum_{n \geq 0} f_n z^n \in W[[z]]$$

of the initial value problem

$$\frac{d}{dz} f(z) = S f(z), f(0) = f_0.$$

In fact, $f(z) = e^{Sz} f_0 = \sum \frac{1}{n!} S^n f_0 z^n$.

Proof. The differential equation means $\sum (n+1) f_{n+1} z^n = \sum S f_n z^n$, and therefore $(n+1) f_{n+1} = S f_n$ for all $n \geq 0$, which is equivalent to $f_n = \frac{1}{n!} S^n f_0$. □

Proof. (Proposition 10.22) By the translation covariance we obtain for $f(z) = Y(a,z)\Omega$ ($\in V[[z]]$ by the Vacuum Axiom) the differential equation $\partial f(z) = T f(z)$. Applying Lemma 10.23 to $W = V$ and $S = T$ yields $f(z) = e^{Tz} a = e^{zT} a$. This proves the first equality. To show the second, we apply Lemma 10.23 to $W =$

End $V[[z^{\pm}]]$ and $S = \mathrm{ad}T$. We have $\partial_w(e^{wT}Y(a,z)e^{-wT}) = [T, e^{wT}Y(a,z)e^{-wT}] = \mathrm{ad}T(e^{wT}Y(a,z)e^{-wT})$ by simply differentiating, and $\partial_w Y(a,z+w) = [T, Y(a,z+w)]$ by translation covariance. Because of $Y(a,z) = Y(a,z+w)|_{w=0}$ the two solutions of the differential equation $\partial_w f = (\mathrm{ad}T)(f)$ have the same initial value $f_0 = Y(a,z) \in$ End $V[[z^{\pm}]]$ and thus agree. The last equalities follow by observing the splitting $[T, Y(a,z)_{\pm}] = \partial Y(a,z)_{\pm}$. □

In order to describe examples the following existence result is helpful:

Theorem 10.24 (Existence). *Let V be a vector space with an endomorphism T and a distinguished vector $\Omega \in V$. Let $(\Phi_a)_{a\in I}$ be a collection of fields*

$$\Phi_a(z) = \sum a_{(k)} z^{-k-1} = a(z) \in \mathrm{End}\ V\left[\left[z^{\pm}\right]\right]$$

indexed by a linear independent subset $I \subset V$ such that the following conditions are satisfied for all $a, b \in I$:

1. $[T, \Phi_a(z)] = \partial\Phi_a(z)$.
2. $T\Omega = 0$ *and* $\Phi_a(z)\Omega|_{z=0} = a$.
3. Φ_a *and* Φ_b *are local with respect to each other.*
4. *The set* $\{a^1_{(-k_1)} a^2_{(-k_2)} \dots a^n_{(-k_n)}\Omega : a^j \in I, k_j \in \mathbb{Z}, k_j > 0\}$ *of vectors along with* Ω *forms a basis of* V.

Then the formula

$$Y(a^1_{(-k_1)} \dots a^n_{(-k_n)}\Omega, z) \colon= \colon D^{k_1-1}\Phi_{a^1}(z) \dots D^{k_n-1}\Phi_{a^n}(z)\colon \qquad (10.3)$$

together with $Y(\Omega, z) = \mathrm{id}_V$ defines the structure of a unique vertex algebra with translation operator T, vacuum vector Ω, and

$$Y(a,z) = \Phi_a(z) \text{ for all } a \in I.$$

Proof. First of all, we note that the requirement $\Phi_a(z)\Omega|_{z=0} = a$ in condition 2, that is $\sum a_{(n)}(\Omega) z^{-n-1}|_{z=0} = a$, implies that $a = a_{(-1)}\Omega$ for each $a \in I$. Therefore, $Y(a,z) = Y(a_{(-1)}\Omega, z) = \colon D^0\Phi_a(z)\colon = \Phi_a(z)$ for $a \in I$ if everything is well-defined. According to condition 4 the fields $Y(a,z)$ will be well-defined by formula (10.3).

To show the Translation Axiom V1 one observes that for any endomorphism $T \in$ End V the adjoint $\mathrm{ad}T : \mathscr{F}(V) \to \mathscr{F}(V)$ acts as a derivation with respect to the normal ordering:

$$[T, \colon a(z)b(z)\colon] = \colon [T, a(z)]b(z)\colon + \colon a(z)[T, b(z)]\colon.$$

Moreover, $\mathrm{ad}T$ commutes with $D^k, k \in \mathbb{N}$. Since the derivative ∂ is a derivation with respect to the normal ordering as well (cf. Remark 10.14) commuting with D^k, and since $\mathrm{ad}T$ and ∂ agree on all $\Phi_a, a \in I$, by condition 1, they agree on all repeated normally ordered products of the fields $D^k\Phi_a(z)$ for all $a \in I, k \in \mathbb{N}$, and hence on all $Y(b,z), b \in V$, because of condition 4 and the formula (10.3).

To check the Locality Axiom V2 one observes that all the fields

$$D^k\Phi_a(z),\ a\ \in I, k \in \mathbb{N},$$

are pairwise local to each other by condition 3 and Remark 10.8. As a consequence, this property also holds for arbitrary repeated normally ordered products of the $D^k\phi_a(z)$ by and Dong's Lemma 10.11 and Remark 10.14.

Finally, the requirements of the Vacuum Axiom V3 are directly satisfied by assumption 2 and the definition of Y. □

The condition of being a basis in Theorem 10.24 can be relaxed to the requirement that $\{a^1_{(-k_1)}a^2_{(-k_2)}\dots a^n_{(-k_n)}\Omega : a^j \in I, k_j \in \mathbb{Z}, k_j > 0\} \cup \{\Omega\}$ spans V (cf. [FKRW95*]). With this result one can deduce that in a vertex algebra the field formula (10.3) holds in general.

Heisenberg Vertex Algebra. Let us apply the Existence Theorem 10.24 to determine the vertex algebra of the free boson. In Sect. 10.3 right after the Definition 10.12 we have already defined the generating field

$$\Phi(z) = \sum a_n z^{-n-1}$$

with $a_n \in \mathrm{End}\ \mathsf{S}$. We use the representation $\mathsf{H} \to \mathrm{End}\ \mathsf{S} = \mathbb{C}[T_1, T_2, \dots]$ of the Heisenberg Lie algebra H in the Fock space S which describes the canonical quantization of the infinite dimensional harmonic oscillator (cf. p. 114). The vacuum vector is $\Omega = 1$, as before, and the definition of the action of the a_n on S yields immediately $a_n\Omega = 0$ for $n \in \mathbb{Z}, n \geq 0$. It follows

$$\Phi(z)\Omega = \sum_{n<0}(a_n\Omega)z^{-n-1} = \sum_{k\geq 0}(a_{-k-1}\Omega)z^k.$$

Consequently, $\Phi(z)\Omega|_{z=0} = a_{-1}\Omega$. Hence, to apply Theorem 10.24 we set $\Phi_a = \Phi$ with $a := a_{-1}\Omega = T_1 \in \mathsf{S}$ and $I = \{a\}$. We know that the properties 3 and 4 of the theorem are satisfied.

In order to determine the infinitesimal translation operator T we observe that T has to satisfy

$$[T, a_n] = -na_{n-1}, T\Omega = 0,$$

by property 1 and the first condition of property 2. This is a recursion for T determining T uniquely. We can show that

$$T = \sum_{m>0} a_{-m-1}a_m. \tag{10.4}$$

In fact, the endomorphism

$$T' = \sum_{m>0} a_{-m-1}a_m \in \mathrm{End}\ \mathsf{H}$$

is well-defined and has to agree with T since $T'\Omega = 0$ and T' satisfies the same recursion $[T', a_n] = -na_{n-1}$: If $n > 0$ then $a_m a_n = a_n a_m$ and $[a_{-m-1}, a_n] = (-m-1)\delta_{n-m-1}$ for $m > 0$, hence $[a_{-m-1}a_m, a_n] = [a_{-m-1}, a_n]a_m = -(m+1)\delta_{n-m-1}a_m$, and therefore

$$[T', a_n] = \sum_{m>0} -(m+1)\delta_{n-m-1} a_m = -n a_{n-1}.$$

Similarly, if $n < 0$ we have $[a_m, a_n] = m\delta_{m+n}$ and $a_{-m-1}a_n = a_n a_{-m-1}$ for $m > 0$, hence $[a_{-m-1}a_m, a_n] = m\delta_{m+n}a_{-m-1}$, and therefore again $[T', a_n] = -na_{n-1}$.

Now, the theorem guarantees that with the definition of the vertex operation as

$$Y(a,z) := \Phi(z) \text{ for } a = T_1 \text{ and}$$

$$Y(T_{k_1} \dots T_{k_n}, z) \mathbf{:} = \mathbf{:} D^{k_1 - 1}\Phi(z) \dots D^{k_n - 1}\Phi(z) \mathbf{:}$$

for $k_j > 0$ we have defined a vertex algebra structure on S, the vertex algebra associated to the Heisenberg algebra H. This vertex algebra will be called the *Heisenberg vertex algebra* S.

In the preceding section we have introduced the natural grading of the Fock space $\mathsf{S} = \bigoplus \mathsf{S}_n$ and we have seen that $\Phi(z)$ is homogeneous of degree 1. Using Lemma 10.15 on the weight of the derivative of a homogeneous field it follows that $D^{k-1}\Phi(z)$ is homogeneous of weight k for $k > 0$ and therefore, again using Lemma 10.15 on the weight of a normally ordered product of homogeneous fields, that $Y(T_{k_1} \dots T_{k_n}, z)$ has weight $k_1 + \dots + k_n = \deg(T_{n_1} \dots T_{k_n})$. As a consequence, for $b \in \mathsf{S}_m$ the vertex operator $Y(b,z)$ is homogeneous of weight m and thus the requirements of Remark 10.20 are satisfied. The Heisenberg vertex algebra is a graded vertex algebra.

Vertex Algebras and Osterwalder–Schrader Axioms. Most of the models satisfying the six axioms presented in Chap. 9 can be transformed into a vertex algebra thereby yielding a whole class of examples of vertex algebras. To sketch how this can be done we start with a conformal field theory given by a collection of correlation functions satisfying the six axioms in Chap. 9. According to the reconstruction in Theorem 9.3 there is a collection of fields Φ_a defined as endomorphisms on a common dense subspace $D \subset \mathbb{H}$ of a Hilbert space $\mathbb{H}$ with $\Omega \in D$.

Among the fields Φ_a in the sense of Definition 9.3 we select the primary fields $(\Phi_a)_{a \in B_1}$. We assume that the asymptotic states $a := \Phi_a(z)\Omega|_{z=0} \in D$ exist. Without loss of generality we can assume, furthermore that $\{a : a \in B_1\}$ is linearly independent. Otherwise, we delete some of the fields.

The operator product expansion (Axiom 6 on p. 168) of the primary fields allows to understand the fields Φ_a as fields

$$\Phi_a(z) = \sum a_{(n)} z^{-n-1} \in \text{End } D\left[\left[z^{\pm}\right]\right]$$

in the sense of vertex algebras. We define $V \subset D$ to be the linear span of the set

$$E := \{a^1_{(-k_1)} a^2_{(-k_2)} \dots a^n_{(-k_n)} \Omega : a^j \in B_1, k_j \in \mathbb{Z}, k_j > 0\} \cup \{\Omega\}$$

and obtain the fields $\Phi_a, a \in B_1$, as fields in V by restriction

$$\Phi_a(z) = \sum a_{(n)} z^{-n-1} \in \text{End } V\left[\left[z^{\pm}\right]\right].$$

Now, using the properties of the energy–momentum tensor $T(z) = \sum L_n z^{-n-2}$ we obtain the endomorphism $L_{-1} : V \to V$ with the properties $[L_{-1}, \Phi_a] = \partial\Phi_a$ (the condition of primary fields (9.6) for $n = -1$) and $L_{-1}\Omega = 0$. Moreover, the fields Φ are mutually local according to the locality Axiom 1 on p. 155.

We have thus verified the requirements 1–3 of the Existence Theorem where L_{-1} has the role of the infinitesimal translation operator. If the set $E \subset V$ is a basis of V we obtain a vertex algebra V with $\Phi_a(z) = Y(a,z)$ according to the Existence Theorem reflecting the properties of the original correlation functions. If D is not linear independent we can use the above-mentioned generalization of the Existence Theorem (cf. [FKRW95*]) to obtain the same result.

We conclude this section by explaining in which sense vertex algebras are natural generalizations of associative and commutative algebras with unit.

Remark 10.25. The concept of a vertex algebra can be viewed to be a generalization of the notion of an associative and commutative algebra A over $\mathbb{C}$ with a unit 1. For such an algebra the map

$$Y : A \to \operatorname{End} A,\ Y(a).b := ab \text{ for all } a,b \in A,$$

is $\mathbb{C}$-linear with $Y(a)1 = a$ and $Y(a)Y(b) = Y(b)Y(a)$. Hence, $Y(a,z) = Y(a)$ defines a vertex algebra A with $T = 0$ and $\Omega = 1$.

Conversely, for a vertex algebra V without dependence on z, that is $Y(a,z) = Y(a)$, we obtain the structure of an associative and commutative algebra A with 1 in the following way. The multiplication is given by

$$ab := Y(a).b, \text{ for } a,b \in A := V.$$

Hence, Ω is the unit 1 of multiplication by the Vacuum Axiom. By locality $Y(a)Y(b) = Y(b)Y(a)$, and this implies $ab = Y(a)b = Y(a)Y(b)\Omega = Y(b)Y(a)\Omega = ba$. Therefore, A is commutative. In the same way we obtain $a(cb) = c(ab)$:

$$a(cb) = Y(a)Y(c)Y(b)\Omega = Y(c)Y(a)Y(b)\Omega = c(ab),$$

and this equality suffices to deduce associativity using commutativity: $a(bc) = a(cb) = c(ab) = (ab)c$.

Another close relation to associative algebras is given by the concept of a holomorphic vertex algebra.

Definition 10.26. A vertex algebra is *holomorphic* if every $Y(a,z)$ is a formal power series $Y(a,z) \in \operatorname{End} V\,[[z]]$ without singular terms.

The next result is easy to check.

Proposition 10.27. *A holomorphic vertex algebra is commutative in the sense that for all $a,b \in V$ the operators $Y(a,z)$ and $Y(b,z)$ commute with each other. Conversely, this kind of commutativity implies that the vertex algebra is holomorphic.*

For a holomorphic vertex algebra the constant term $a_{(-1)} \in \operatorname{End} V$ in the expansion

$$Y(a,z) = \sum_{n<0} a_{(n)} z^{-n-1} = \sum_{k\geq 0} a_{(-(k+1))} z^k = a_{(-1)} + \sum_{k>0} a_{(-(k+1))} z^k$$

determines a multiplication by $ab := a_{(-1)}b$. Now, for $a,b \in V$ one has $[Y(a,z), Y(b,z)] = 0$ and this equality implies $a_{(-1)}b_{(-1)} = b_{(-1)}a_{(-1)}$. In the same way as above after Remark 10.25 the multiplication turns out to be associative and commutative with Ω as unit.

The infinitesimal translation operator T acts as a derivation. By Axiom V1 $[T, a_{(-1)})] = a_{(-2)}$. Because of $(Ta)_{(-1)} = a_{(-2)}$ which can be shown directly but also follows from a more general formula proven in Proposition 10.34 we obtain

$$T(ab) = Ta_{(-1)}b = a_{(-1)}Tb + (Ta)_{(-1)}b = a(Tb) + (Ta)b.$$

Proposition 10.28. *The holomorphic vertex algebras are in one-to-one correspondence to the associative and commutative unital algebras with a derivation.*

Proof. Given such an algebra V with derivation $T : V \to V$ we only have to construct a holomorphic vertex algebra in such a way that the corresponding algebra is V. We take the vacuum Ω to be the unit 1 and define the operators $Y(a,z)$ by

$$Y(a,z) := e^{zT} a = \sum_{n\geq 0} \frac{T^n a}{n!} z^n.$$

The axioms of a vertex algebra are easy to check. Moreover,

$$Y(a,z) = a + \sum_{n>0} \frac{T^n a}{n!} z^n,$$

hence $a_{(-1)} = a$ which implies that by $ab = a_{(-1)}b$ we get back the original algebra multiplication. □

Note that T may be viewed as the generator of infinitesimal translations of z on the formal additive group. Thus, holomorphic vertex algebras are associative and commutative unital algebras with an action of the formal additive group. As a consequence, general vertex algebras can be regarded to be "meromorphic" generalizations of associative and commutative unital algebras with an action of the formal additive group. This point of view can be found in the work of Borcherds [Bor00*] and has been used in another generalization of the notion of a vertex algebra on the basis of Hopf algebras [Len07*].

10.5 Conformal Vertex Algebras

We begin this section by completing the example of the generating field

$$L(z) = \sum L_n z^{-n-2}$$

associated to the Virasoro algebra for which we already derived the operator product expansion (10.2) in Sect. 10.2:

$$L(z)L(w) \sim \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{(z-w)}. \tag{10.5}$$

(We have changed the notation from $T(z)$ to $L(z)$ in order not to mix up the notation with the notation for the infinitesimal translation operator T.)

Now, we associate to Vir another example of a vertex algebra.

Virasoro Vertex Algebra. In analogy to the construction of the Heisenberg vertex algebra in Sect. 10.4 we use a suitable representation V_c of Vir where $c \in \mathbb{C}$ is the central charge. This is another induced representation, cf. Definition 10.49. V_c is defined to be the vector space with basis

$$\{v_{n_1 \dots n_k} : n_1 \geq \dots n_k \geq 2, n_j \in \mathbb{N}, k \in \mathbb{N}\} \cup \{\Omega\}$$

(similar to the Verma module $M(c,0)$ in Definition 6.4 and its construction in Lemma 6.5) together with the following action of Vir on V_c ($n, n_j \in \mathbb{Z}, n_1 \geq \dots n_k \geq 2, k \in \mathbb{N}$):

$$Z := c\,\mathrm{id}_{V_c},$$
$$L_n \Omega := 0 \ , \ n \geq -1 \ , \ n \in \mathbb{Z},$$
$$L_0 v_{n_1 \dots n_k} := (\sum_{j=1}^{k} n_j) v_{n_1 \dots n_k},$$
$$L_{-n}\Omega := v_n, n \geq 2, \text{ and } L_{-n} v_{n_1 \dots n_k} := v_{n n_1 \dots n_k} \ , \ n \geq n_1.$$

The remaining actions $L_n v, v \in V_c$, are determined by the Virasoro relations, for example $L_{-1} v_n = (n-1) v_{n+1}$ or $L_n v_n = \frac{1}{12} c n (n^2-1)\Omega$ if $n > 1$, in particular $L_2 v_2 = \frac{1}{2} c\Omega$, since

$$L_{-1} v_n = L_{-1} L_{-n}\Omega = L_{-n} L_{-1}\Omega + (-1+n) L_{-1-n}\Omega = (n-1) v_{n+1},$$

and $L_n v_n = L_n L_{-n}\Omega = L_{-n} L_n \Omega + 2n L_0 \Omega + \frac{c}{12} n(n^2-1)\Omega$ with $L_n\Omega = L_0\Omega = 0$. The definition $L(z) = \sum L_n z^{-n-2}$ directly yields that $L(z)$ is a field, since for every $v \in V_c$ there is N such that $L_n v = 0$ for all $n \geq N$.

Observe that V_c as a vector space can be identified with the space $\mathbb{C}[T_2, T_3, \dots]$ of polynomials in the infinitely many indeterminates $T_j, j \geq 2$.

To apply Theorem 10.24 with $L(z)$ as generating field we evaluate, first of all, the "asymptotic state" $L(z)\Omega|_{z=0} =: a \in \mathsf{S}$. Because of $L_n\Omega = 0$ for $n > -2$ and $L_{-n}\Omega = v_n$ for $n \geq 2$ we obtain

$$a = L(z)\Omega|_{z=0} = \sum_{m \leq -2} L_m \Omega z^{-m-2}|_{z=0} = L_{-2}\Omega = v_2.$$

We set $I = \{a\} = \{v_2\}$ and $\Phi_a(z) := L(z)$ in order to agree with the notation in Theorem 10.24.

Proposition 10.29. *The field $\Phi_a(z) = L(z), a = v_2$, generates the structure of a vertex algebra on V_c with L_{-1} as the infinitesimal translation operator. V_c is called the* Virasoro vertex algebra with central charge c.

Proof. Property 3 of Theorem 10.24 is satisfied, since the field $\Phi_a = L$ is local with itself according to (10.5), and property 4 holds because of the definition of V_c. As the infinitesimal translation operator T we take $T := L_{-1}$, so that property 2 is satisfied as well. Finally, $[L_{-1}, L(z)] = \partial L(z)$ (which is $[T, \Phi(z)] = \partial\Phi(z)$) can be checked directly: $[L_{-1}, L(z)] = \sum[L_{-1}, L_n]z^{-n-2} = \sum(-1-n)L_{n-1}z^{-n-2} = \sum(-n-2)L_n z^{-n-3} = \partial L(z)$.

As a consequence,

$$Y(v_2, z) = L(z),$$
$$Y(v_{n_1 \dots n_k}, z) = \boldsymbol{:}D^{n_1-2}T(z)\dots D^{n_k-2}T(z)\boldsymbol{:}$$

define the structure of a vertex algebra which will be called the Virasoro vertex algebra with central charge c. The central charge can be recovered by $L_2 a = \frac{1}{2}c\Omega$. □

V_c has the grading $V_c = \bigoplus V_N$ with V_N generated by the basis elements $\{v_{n_1 \dots n_k} : \sum n_j = N\}$ ($\sum n_j = N = \deg v_{n_1 \dots n_k}$), $V_0 = \mathbb{C}\Omega$. The finite-dimensional vector subspace V_N can also be described as the eigenspace of L_0 with eigenvalue N: $V_N = \{v \in V_c : L_0 v = Nv\}$. The translation operator $T = L_{-1}$ is homogeneous of degree 1 and the generating field has weight 2 since each $L_{n-1} = T_{(n)}$ has degree $2-n-1$. Hence, V_c is a graded vertex algebra and L_0 is the degree.

This example of a vertex algebra motivates the following definition:

Definition 10.30. ***(Conformal Vertex Algebra)*** A field $L(z) = \sum L_n z^{-n-2}$ with the operator expansion as in (10.5) will be called a *Virasoro field with central charge c*.

A *conformal vector* with *central charge c* is a vector $\nu \in V$ such that $Y(\nu, z) = \sum \nu_{(n)} z^{-n-1} = \sum L_n^\nu z^{-n-2}$ is a Virasoro field with central charge c satisfying, in addition, the following two properties:

1. $T = L_{-1}^\nu$
2. L_0^ν is diagonalizable.

Finally, a *conformal vertex algebra* (of *rank c*) is a vertex algebra V with a distinguished conformal vector $\nu \in V$ (with central charge c). In that case, the field $Y(\nu, z)$ is also called the *energy–momentum tensor* or energy–momentum field of the vertex algebra V.

Examples. 1. The Virasoro vertex algebras V_c are clearly conformal vertex algebras of rank c with conformal vector $\nu = v_2 = L_{-2}\Omega$. $L(z) = Y(v_2, z)$ is the energy–momentum tensor.

2. The vertex algebra associated to an axiomatic conformal field theory in the sense of the last chapter (cf. p. 190 under the assumptions made there) has $L_{-2}\Omega$ as a conformal vector and T is the energy–momentum tensor.

3. The Heisenberg vertex algebra S has a one-parameter family of conformal vectors
$$v_\lambda := \frac{1}{2}T_1^2 + \lambda T_2 \ , \ \lambda \in \mathbb{C}.$$

To see this, we have to check that the field $Y(v_\lambda, z) = \sum L_n^\lambda z^{-n-2}$ is a Virasoro field, that $T = L_{-1}^\lambda$, and that L_0^λ is diagonalizable.

That the L_n^λ satisfy the Virasoro relations and therefore determine a Virasoro field can be checked by a direct calculation which is quite involved. We postpone the proof because we prefer to obtain the Virasoro field condition as an application of the associativity of the operator product expansion, which will be derived in the next section (cf. Theorem 10.40).

The other two conditions are rather easy to verify. By the definition of the vertex operator we have $Y(T_1^2, z) = {:}\Phi(z)\Phi(z){:}$ and $Y(T_2, z) = \partial\Phi(z)$, hence

$$Y(T_1^2, z) = \sum_{k\neq 0}\sum_{n+m=k} a_n a_m z^{-k-2} + 2\sum_{n>0} a_{-n}a_n z^{-2},$$

where $\Phi(z) = \sum a_n z^{-n-1}$ with the generators a_n of the Heisenberg algebra H acting on the Fock space S, and

$$Y(T_2, z) = \sum(-k-1)a_k z^{-k-2},$$

and therefore,

$$Y(v_\lambda, z) = \frac{1}{2}\sum_{k\neq 0}\left(\sum_{n+m=k} a_n a_m - \lambda(k+1)a_k\right) z^{-k-2} + \sum_{n>0} a_{-n}a_n z^{-2}. \qquad (10.6)$$

(Recall that we defined a_0 to satisfy $a_0 = 0$ in this representation of H.) Consequently,
$$L_0 = L_0^\lambda = \sum_{n>0} a_{-n}a_n$$
and
$$L_{-1} = L_{-1}^\lambda = \sum_{n>0} a_{-n-1}a_n,$$
and both these operators turn out to be independent of λ. Now, on the monomials $T_{n_1}\dots T_{n_k}$ we have $L_0(T_{n_1}\dots T_{n_k}) = \sum_{j=1}^{k} n_j = \deg(T_{n_1}\dots T_{n_k})$ and L_0 is diagonalizable with $L_0 v = \deg(v)v = nv$ for $v \in V_n$. Finally, we have already seen in (10.4) that the infinitesimal translation operator is $\sum_{n>0} a_{-n-1}a_n = L_{-1}$.

4. A fourth example of a conformal vertex algebra is given by the Sugawara vector as a conformal vector of the vertex algebra associated to a Lie algebra $\mathfrak{g}$.

(This example appears also in the context of associating a vertex algebra to a conformal field theory with $\mathfrak{g}$-symmetry in the sense of Chap. 9, but there we have not introduced the related example of a conformal field theory corresponding to a Kac–Moody algebra.)
At first, we have to describe the corresponding vertex algebra.

Affine Vertex Algebra. As a fourth example of applying the Existence Theorem 10.24 to describe vertex algebras we now come to the case of a finite-dimensional simple Lie algebra $\mathfrak{g}$ and its associated vertex algebra $V_k(\mathfrak{g}), k \in \mathbb{C}$, which will be called affine vertex algebra.

In the list of examples of central extensions in Sect. 4.1 we have introduced the affinization

$$\hat{\mathfrak{g}} = \mathfrak{g}[T, T^{-1}] \oplus \mathbb{C}Z$$

of a general Lie algebra $\mathfrak{g}$ equipped with an invariant bilinear form $(\, , \,)$ as the central extension of the loop algebra $L\mathfrak{g} = \mathfrak{g}[T^{\pm}]$ with respect to the cocycle

$$\Theta(a_m, b_n) = m(a,b)\delta_{m+n}Z,$$

where we use the abbreviation $a_m = T^m a = T^m \otimes a, b_n = T^n b$ for $a, b \in \mathfrak{g}$ and $n \in \mathbb{Z}$. The commutation relations for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$ are therefore

$$[a_m, b_n] = [a,b]_{m+n} + m(a,b)\delta_{m+n}Z, \; [a_m, Z] = 0.$$

In the case of a finite-dimensional simple Lie algebra $\mathfrak{g}$ any invariant bilinear symmetric form $(\, , \,)$ is unique up to a scalar (it is in fact a multiple of the Killing form κ) and the resulting affinization of $\mathfrak{g}$ is called the affine Kac–Moody algebra of $\mathfrak{g}$ where the invariant form is normalized in the following way: The Killing form on $\mathfrak{g}$ is $\kappa(a,b) = \mathrm{tr}(\mathrm{ad}\, a \; \mathrm{ad}\, b)$ for $a, b \in \mathfrak{g}$, where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}, \; \mathrm{ad}\, a(x) = [a,x]$ for $x \in \mathfrak{g}$ is the adjoint representation. The normalization in question now is

$$(a,b) := \frac{1}{2h^{\vee}} \kappa(a,b),$$

where $h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$ (see p. 221).

As before, we need to work in a fixed representation of the Kac–Moody algebra $\hat{\mathfrak{g}}$. Let $\{J^{\rho} : \rho \in \{1, \dots, r\}\}$ be an ordered basis of $\mathfrak{g}$. Then $\{J^{\rho}_n : 1 \le \rho \le r = \dim \mathfrak{g}, n \in \mathbb{Z}\} \cup \{Z\}$ is a basis for $\hat{\mathfrak{g}}$.

We define the representation space $V_k(\mathfrak{g}), k \in \mathbb{C}$, to be the complex vector space with the basis

$$\{v^{\rho_1 \cdots \rho_m}_{n_1 \dots n_m} : n_1 \ge \dots n_m \ge 1, \rho_1 \le \dots \le \rho_m\} \cup \{\Omega\},$$

and define the action of $\hat{\mathfrak{g}}$ on $V = V_k(\mathfrak{g})$ by fixing the action as follows ($n > 0$):

$$Z = k\mathrm{id}_V, \; J^{\rho}_n \Omega = 0,$$
$$J^{\rho}_{-n}\Omega = v^{\rho}_n, \; J^{\rho}_{-n} v^{\rho_1 \cdots \rho_m}_{n_1 \dots n_m} = v^{\rho\rho_1 \cdots \rho_m}_{nn_1 \dots n_m},$$

if $n \geq n_1$ and $\rho \leq \rho_1$. The remaining actions of the J_n^ρ on the basis of $V_k(\mathfrak{g})$ are determined by the commutation relations

$$[J_m^\rho, J_n^\sigma] = [J^\rho, J^\sigma]_{m+n} + m(J^\rho, J^\sigma)k\delta_{m+n}.$$

The resulting representation is called the *vacuum representation of rank k*. It is again an induced representation, cf. Sect. 10.7.

The generating fields are

$$J^\rho(z) = \sum J_n^\rho z^{-n-1} \in \text{End } V_k(\mathfrak{g})\left[\left[z^\pm\right]\right], 1 \leq \rho \leq r.$$

In view of the commutation relations one has

$$J_n^\rho v_{n_1 \ldots n_m}^{\rho_1 \ldots \rho_m} = 0$$

if $n > n_1$. Therefore, these formal distributions are in fact fields. Because of $J_n^\rho \Omega = 0$ for every $n \in \mathbb{Z}, n \geq 0$, we obtain

$$J^\rho(z)\Omega = \sum_{n<0} J_n^\rho \Omega z^{-n-1} = \sum_{m \geq 0} v_{m+1} z^m,$$

and thus $J^\rho(z)\Omega|_{z=0} = v_1^\rho$. Hence, to match the notation of the Existence Theorem 10.24 we should set $I = \{v_1^\rho : 1 \leq \rho \leq r\}$ and

$$\Phi_a(z) := J^\rho(z) \text{ if } a = v_1^\rho.$$

Proposition 10.31. *The fields $\Phi_a(z), a \in I$, resp. $J^\rho(z), 1 \leq \rho \leq r$, generate a vertex algebra structure on $V_k(\mathfrak{g})$. $V_k(\mathfrak{g})$ is the* affine vertex algebra of rank *k*.

Proof. In order to check locality we calculate $[J^\rho(z), J^\sigma(w)]$:

$$\begin{aligned}
[J^\rho(z), J^\sigma(w)] &= \sum_{m,n} [J_m^\rho, J_n^\sigma] z^{-m-1} w^{-n-1} \\
&= \sum_{m,n} [J^\rho, J^\sigma]_{m+n} z^{-m-1} w^{-n-1} + \sum_m m(J^\rho, J\sigma)kz^{-m-1}w^{m-1} \\
&= \sum_l [J^\rho, J^\sigma]_l w^{-l-1} \sum_m z^{-m-1} w^m + (J^\rho, J^\sigma)k \sum_m m z^{-m-1} w^{m-1} \\
&= [J^\rho, J^\sigma](w)\delta(z-w) + (J^\rho, J^\sigma)k\partial\delta(z-w).
\end{aligned}$$

This equality implies by Theorem 10.5 that the operator product expansion is

$$J^\rho(z)J^\sigma(w) \sim \frac{[J^\rho, J^\sigma](w)}{z-w} + \frac{(J^\rho, J^\sigma)k}{(z-w)^2}, \tag{10.7}$$

and that the fields $J^\rho(z), J^\sigma(z)$ are pairwise local with respect to each other. We thus have established property 3 of the Existence Theorem 10.24, and by the construction of the space $V_k(\mathfrak{g})$ and the definition of the action of the J^ρ_n property 4 is satisfied as well.

It remains to determine the infinitesimal translation operator T which will again be defined recursively by

$$T\Omega = 0, [T\ ,\ J^\rho_n] = -nJ^\rho_{n-1}.$$

$T \in \mathrm{End}\ V_k(\mathfrak{g})$ is well-defined and satisfies evidently $[T, J^\rho(z)] = \partial J^\rho(z)$. Therefore, the Existence Theorem applies yielding a vertex algebra structure given by

$$Y(v^{\rho_1\dots\rho_m}_{n_1\dots n_m}, z) = \mathbf{:} D^{n_1-1}J^{\rho_1}(z)\dots D^{n_m-1}J^{\rho_m}(z)\mathbf{:}.$$

□

In order to determine a conformal vector of the affine vertex algebra $V_k(\mathfrak{g})$ by the Sugawara construction we denote the elements of the dual basis with respect to $\{J^1,\dots J^r\}$ by $J_\rho \in \mathfrak{g}$ satisfying $(J_\sigma, J^\rho) = \delta^\rho_\sigma$. Then it can be shown that the vector

$$S := \frac{1}{2}\sum_{\rho=1}^{r} J_{\rho,-1}J^\rho_{-1}\Omega \in V_k(\mathfrak{g})$$

is independent of the choice of the basis. We call

$$\nu := \frac{1}{k+h^\vee}S$$

the *Sugawara vector*.

Proposition 10.32. *Assume $k \neq -h^\vee$. Then the Sugawara vector ν is a conformal vector of $V_k(\mathfrak{g})$ with central charge*

$$c = c(k) = \frac{k\dim\mathfrak{g}}{k+h^\vee}.$$

Proof. (sketch) Using the associativity of the OPE (see Theorem 10.36 in the next section) one can deduce for $Y(\nu,z) = L(z) = \sum L_n z^{-n-2}$ $(L_n = L^\nu_n)$ the OPE

$$L(z)J^\rho(w) \sim \frac{J^\rho(w)}{(z-w)^2} + \frac{\partial J^\rho(w)}{z-w},$$

and hence the following commutation relations

$$[L_m, J^\rho_n] = -nJ^\rho_{m+n}, m,n \in \mathbb{Z}, 1 \leq \rho \leq r.$$

These relations imply $L_{-1} = T$ and the diagonalizability of L_0 immediately. Moreover, $L_n\nu = 0$ for $n > 2$. Therefore, according to the above-mentioned criterion in Theorem 10.40 v is a conformal vector of central charge c where c is determined by $L_2 v = \frac{1}{2}c\Omega$. Finally,

$$\begin{aligned} L_2 v &= \frac{1}{2(k+h^\vee)} L_2 \sum J_{\rho,-1} J^\rho_{-1}\Omega \\ &= \frac{1}{2(k+h^\vee)} \sum J_{\rho,1} J^\rho_{-1}\Omega \\ &= \frac{k \dim \mathfrak{g}}{2(k+h^\vee)}\Omega. \end{aligned}$$

We conclude $c = \frac{k\dim\mathfrak{g}}{k+h^\vee}$. Details are in [Kac98*] and [BF01*]. □

Altogether, the coefficients L_n of the Virasoro field

$$Y(v,z) = \frac{1}{2(k+h^\vee)} \sum_{\rho=1}^{r} {:}J_\rho(z)J^\rho(z){:}$$

yield an action of the Virasoro algebra with central charge $c(k)$ on the space $V_k(\mathfrak{g})$.

Many more vertex algebras are known and many of them are not constructed by using a Lie algebra representation. It is not in the scope of this book to survey other interesting classes of vertex algebras. Instead we refer to the course of Kac [Kac98*] where the last third of the book is devoted to describe such vertex algebras as lattice vertex algebras, coset constructions, W-algebras, various $\mathbb{Z}/2\mathbb{Z}$-graded (or super) vertex algebras to include also the anticommutator in the considerations, and many more examples.

Examples are presented in the book of Frenkel and Ben-Zvi [BF01*], too, where the vertex algebras are related to algebraic curves. The first step in doing this is to formulate a theory of vertex algebras being invariant against coordinate changes $z \mapsto w(z)$. This leads eventually to vertex algebra bundles and moduli spaces as well as chiral algebras. In contrast to this local approach to algebraic curves in [Lin04*] an attempt has been made to study "global" vertex algebras on Riemann surfaces which turns out to be connected to Krichever–Novikov algebras.

Let us mention also the approach of Huang [Hua97*] who relates the algebraic approach to vertex algebras as presented here to the more geometrically and topologically inspired description of conformal field theory of Segal [Seg88a], [Seg91].

10.6 Associativity of the Operator Product Expansion

We begin with the uniqueness result of Goddard.

Theorem 10.33 (Uniqueness). *Let V be a vertex algebra and let $f \in \operatorname{End} V[[z^{\pm}]]$ be a field which is local with respect to all fields $Y(a,z)$, $a \in V$. Moreover, assume that*

$$f(z)\Omega = e^{zT}b$$

for a suitable $b \in V$. Then $f(z) = Y(b,z)$.

Proof. By locality we have $(z-w)^N[f(z),Y(a,w)] = 0$, in particular,

$$(z-w)^N f(z)Y(a,w)\Omega = (z-w)^N Y(a,w)f(z)\Omega.$$

We insert the assumption $f(z)\Omega = e^{zT}b$, and the equalities $Y(a,w)\Omega = e^{wT}a$ and $Y(b,z)\Omega = e^{zT}b$ (according to Proposition 10.22), and we obtain

$$(z-w)^N f(z)e^{wT}a = (z-w)^N Y(a,w)e^{zT}b = (z-w)^N Y(a,w)Y(b,z)\Omega.$$

Since $Y(a,z)$ and $Y(b,z)$ are local to each other we have (for sufficiently large N)

$$(z-w)^N f(z)e^{wT}a = (z-w)^N Y(b,z)Y(a,w)\Omega = (z-w)^N Y(b,z)e^{wT}a.$$

Letting $w=0$ we conclude $z^N f(z)a = z^N Y(b,z)a$ for all $a \in V$ which implies $f(z)a = Y(b,z)a$ and hence $f(z) = Y(b,z)$. □

The Uniqueness Theorem yields immediately the following result:

Proposition 10.34. *The identity*

$$Y(Ta,z) = \partial Y(a,z)$$

holds in a vertex algebra.

Proof. For $f(z) = \partial Y(a,z)$ we have

$$f(z)\Omega = \sum_{n\geq 0}(n+1)a_{(-n-2)}\Omega z^n$$

and therefore $f(z)\Omega|_{z=0} = a_{(-2)}\Omega = Ta$. Using translation covariance we have $\partial(f(z)\Omega) = \partial TY(a,z)\Omega = T(f(z)\Omega)$ and we conclude $f(z)\Omega = e^{zT}Ta$ by Lemma 10.23. By Theorem 10.33 it follows that $f(z) = Y(Ta,z)$. □

In a similar way as the Uniqueness Theorem 10.33 one can prove the following:

Proposition 10.35 (Quasisymmetry). *The equality*

$$Y(a,z)b = e^{zT}Y(b,-z)a$$

holds in $V((z))$.

Proof. Since $Y(a,z), Y(b,z)$ are local to each other by the Locality Axiom there exists $N \in \mathbb{N}$ with

$$(z-w)^N Y(a,z)Y(b,z)\Omega = (z-w)^N Y(b,z)Y(a,z)\Omega.$$

By $Y(a,z)\Omega)e^{zT}a$ (Proposition 10.22) and analogously for b this implies

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N Y(b,z)e^{zT}a.$$

By Proposition 10.22 we also have $e^{zT}Y(b,w)e^{-Tz} = Y(b,z+w)$, hence, $e^{zT}Y(b,w-z) = Y(b,z)e^{zT}$. Consequently,

$$(z-w)^N Y(a,z)e^{wT}b = (z-w)^N e^{zT}Y(b,w-z)a,$$

where $(w-z)^{-1}$ has to replaced by the expansion $(w-z)^{-1} = \sum_{n\geq 0} z^n w^{-n-1}$. Let N be large enough such that on the right-hand side of the above formula there appear no negative powers of $(w-z)$. Then it becomes an equality in $V((z))[[w]]$, and we can put $w=0$ again and divide by z^N to obtain the desired identity of quasisymmetry.□

We now come to the associativity of the operator product expansion (OPE for short). To motivate the result we apply Proposition 10.22 repeatedly to obtain

$$Y(a,z)Y(b,w)\Omega = Y(a,z)e^{wT}b = e^{wT}Y(a,z-w)b, \text{ and}$$
$$e^{wT}Y(a,z-w)b = Y(Y(a,z-w)b,w)\Omega,$$

where the last expression $Y(Y(a,z-w)b,w)\Omega$ is defined by

$$Y(Y(a,z-w)b,w) := \sum_{n\in\mathbb{Z}} Y(a_{(n)}b,w)(z-w)^{-n-1}.$$

One is tempted to apply the Uniqueness Theorem 10.33 to the equality

$$Y(a,z)Y(b,w)\Omega = Y(Y(a,z-w)b,w)\Omega$$

to deduce

$$Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w)$$

which is the desired "associativity" of the OPE. However, the theorem cannot be applied directly: we first have to make precise where the equality should hold. Observe that for $b\in V$ there exists n_0 such that $a_{(n)}b = 0$ for $n\geq n_0$. Consequently, $Y(Y(a,z-w)b,w) = \sum Y(a_{(n)}b,w)(z-w)^{-n-1}$ is a series in End $V[[w^{\pm}]]((z-w))$. Replacing

$$(z-w)^{-k} \mapsto \delta_-^k = (\sum_{n\geq 0} z^{-n-1}w^n)^k, k>0,$$

we obtain an embedding

$$\text{End } V\left[\left[w^{\pm}\right]\right]((z-w)) \hookrightarrow \text{End } V\left[\left[w^{\pm},z^{\pm}\right]\right].$$

The following equalities have to be understood as identities in End $V[[w^{\pm},z^{\pm}]]$ using this embedding.

Theorem 10.36 (Associativity of the OPE). *For any vertex algebra V the following associativity property is satisfied:*

$$Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w) = \sum_{n\in\mathbb{Z}} Y(a_{(n)}b,w)(z-w)^{-n-1}$$

for all $a,b \in V$. More specifically,

$$Y(a,z)Y(b,w) = \sum_{n\geq 0} Y(a_{(n)}b,w)(z-w)^{-n-1} + {:}Y(a,z)Y(b,w){:},$$

and, equivalently,

$$[Y(a,z),Y(b,w)] = \sum_{n\geq 0} D_w^n \delta(z-w)Y(a_{(n)}b,w).$$

Proof. We use the attempt described earlier and start with

$$Y(a,z)Y(b,w)\Omega = e^{wT}Y(a,z-w)b = Y(Y(a,z-w)b,w)\Omega,$$

where the last equality can be shown in a similar way as the corresponding equality in the proof of Proposition 10.22. For arbitrary $c \in V$ we obtain the equality

$$Y(c,t)Y(a,z)Y(b,w)\Omega = Y(c,t)Y(Y(a,z-w)b,w)\Omega$$

in End $[[z^{\pm},w^{\pm}]]$. For sufficiently large $M,N \in \mathbb{Z}$ we have by locality

$$\begin{aligned}(t-z)^M(t-w)^N&Y(a,z)Y(b,w)Y(c,t)\Omega \\ &= (t-z)^M(t-w)^N Y(c,t)Y(a,z)Y(b,w)\Omega\end{aligned}$$

and

$$\begin{aligned}(t-z)^M(t-w)^N&Y(c,t)Y(Y(a,z-w)b,w)\Omega \\ &= (t-z)^M(t-w)^N Y(Y(a,z-w)b,w)Y(c,t)\Omega.\end{aligned}$$

Consequently,

$$\begin{aligned}(t-z)^M(t-w)^N&Y(a,z)Y(b,w)Y(c,t)\Omega \\ &= (t-z)^M(t-w)^N Y(Y(a,z-w)b,w)Y(c,t)\Omega,\end{aligned}$$

and by the Vacuum Axiom $Y(c,t)\Omega|_{t=0} = c$ we obtain

$$z^M w^N Y(a,z)Y(b,w)c = z^M w^N Y(Y(a,z-w)b,w)c,$$

which implies

$$Y(a,z)Y(b,w) = Y(Y(a,z-w)b,w).$$

The other two equalities follow by using the fundamental Theorem 10.5. $\square$

Corollary 10.37. *Each of the expansion in Theorem 10.36 is equivalent to each of the following commutation relations due to Borcherds*

$$[a_{(m)}, b_{(n)}] = \sum_{k\geq 0} \binom{m}{k} (a_{(k)})_{(m+n-k)}$$

or, equivalently,

$$[a_{(m)}, Y(b,z)] = \sum_{k\geq 0} \binom{m}{k} Y(a_{(k)}b, z) z^{m-k}.$$

We conclude that the subspace of all coefficients $a_{(n)} \in \mathrm{End}\ V, a \in V, n \in \mathbb{Z}$, is a Lie algebra Lie V with respect to the commutator.

Another direct consequence of the associativity of the OPE is the following: Note that a vertex subalgebra of a vertex algebra V is a vector subspace $U \subset V$ containing Ω such that $a_{(n)}U \subset U$ for all $a \in U$ and $n \in \mathbb{Z}$. Of course, a vertex subalgebra is itself a vertex algebra by restricting $a_{(n)}$ to U:

$$a^U_{(n)} = a_{(n)}|_U : U \to U$$

with vertex operators $Y^U(a,z) = \sum a^U_{(n)} z^{-n-1}$.

Corollary 10.38. *Let V be a vertex algebra.*

1. $a_{(0)}b = 0 \Longleftrightarrow [a_{(0)}, Y(b,z)] = 0$.
2. $\forall k \geq 0 : a_{(k)}b = 0 \Longleftrightarrow [Y(a,z), Y(b,w)] = 0$.
3. $a_{(0)}$ *is a derivation* $V \to V$ *for each* $a \in V$*, and thus* $\ker a_{(0)}$ *is a vertex subalgebra of* V.
4. *The centralizer of the field $Y(a,z)$–the subspace*

$$C(a) = \{b \in V : [Y(a,z), Y(b,w)] = 0\} \subset V$$

–is a vertex subalgebra of V.

5. *The fixed point set of an automorphism of V with respect to the vertex algebra structure is vertex subalgebra.*

Proof. The first two properties follow from the second equality in Corollary 10.37. Property 3 follows from the first equality in the above Corollary 10.37 for $m = 0$. 4 is implied by 2, and 5 is obvious. □

Remark 10.39. Through Corollary 10.38 the associativity of the OPE provides the possibility of obtaining new vertex algebras as subalgebras of a given vertex algebra V which are related to some important constructions of vertex algebra in physics and in mathematics.

1. The centralizer of a vector subspace $U \subset V$

$$C_V(U) = \{b \in V | \forall a \in U : [Y(a,z), Y(b,w)] = 0\}$$

is a vertex subalgebra of V by property 4 of Corollary 10.38 called the *coset model*.

2. For any subset $A \subset V$ the intersection

$$\bigcap \{\ker a_{(0)} : a \in A\}$$

is a vertex subalgebra by property 3 of Corollary 10.38 called a *W-algebra*.

3. For a subset $I \subset V$ the linear span of all the vectors

$$a^1_{(n_1)} a^2_{(n_2)} \dots a^k_{(n_k} \Omega, a^j \in I, n_j \in \mathbb{Z}, k \in \mathbb{N},$$

is a vertex subalgebra of V generated by the fields $Y(a,z), a \in I$.

4. Given a group G of automorphisms of a vertex algebra, the fixed point set V^G is a vertex subalgebra of V by property 5 of Corollary 10.38 called an *orbifold model* in case G is a finite group.

We finally come to the application of the associativity of the OPE to check the Virasoro field condition for the Heisenberg vertex algebra and the affine vertex algebras.

Theorem 10.40. *For a vector $\nu \in V$ denote $L(z) := Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, that is $L_n = L_n^\nu = \nu_{(n+1)}$. Suppose, $L(z)$ and $c \in \mathbb{C}$ satisfy*

$$L_{-1} = T \ , \ L_2 \nu = \frac{c}{2} \Omega \ , \ L_n \nu = 0 \text{ for } n > 2 \ , \ L_0 \nu = 2\nu.$$

Then $L(z)$ is a Virasoro field with central charge c. If, in addition, L_0 is diagonalizable on V, then ν is a conformal vector with central charge c.

Proof. By the OPE (Theorem 10.36)

$$Y(\nu,z)Y(\nu,w) \sim \sum_{n \geq 0} \frac{Y(\nu_{(n)}\nu, w)}{(z-w)^{n+1}} = \sum_{n \geq -1} \frac{Y(L_n \nu, w)}{(z-w)^{n+2}}.$$

By the assumptions on $L_n \nu$ we obtain

$$L(z)L(w) \sim \frac{1}{2} c \frac{Y(\Omega, w)}{(z-w)^4} + \frac{Y(L_1 \nu, w)}{(z-w)^3} + \frac{Y(2\nu, w)}{(z-w)^2} + \frac{Y(T\nu, w)}{(z-w)}.$$

It remains to show that the term $Y(L_1 \nu, z)$ vanishes, because in that case by inserting $Y(T\nu, z) = \partial Y(\nu, w)$ (according to Corollary 10.34) and using $Y(\Omega, w) = \mathrm{id}_V$, one obtains the desired expansion

$$L(z)L(w) \sim \frac{1}{2} c \frac{1}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{(z-w)}.$$

In order to show $a(z) := Y(L_1 \nu, z) = 0$ one interchanges z and w and obtains

$$L(w)L(z) \sim \frac{1}{2} c \frac{1}{(z-w)^4} - \frac{a(z)}{(z-w)^3} + \frac{2L(z)}{(z-w)^2} - \frac{\partial L(z)}{(z-w)},$$

hence, by Taylor expansion

$$L(w)L(z) \sim \frac{1}{2}c\frac{1}{(z-w)^4} - \frac{a(w)+Da(w)(z-w)+D^2a(w)(z-w)^2}{(z-w)^3} \\ +2\frac{L(w)+DL(w)(z-w)}{(z-w)^2} - \frac{\partial L(w)}{(z-w)}.$$

By locality, the two expansions of $L(z)L(w)$ and $L(w)L(z)$ have to be equal and this implies

$$\frac{a(w)}{(z-w)^3} = 0$$

and thus $a(z) = 0$. □

We are now in the position to apply the associativity of the OPE in order to show that the vectors v_λ resp. v_k are conformal vectors in our examples of the Heisenberg vertex algebra S resp. of the affine vertex algebra $V_k(\mathfrak{g})$.

We focus on the Heisenberg case since the corresponding equalities for the affine vertex algebra have been established already on page 198. We already know that $L_0 = \deg$ and $L_1 = T$. It remains to show that $L(z) = \sum L_n z^{-n-2}, L_n = L_n^v$, is a Virasoro field which means by Theorem 10.40 that only $L_2 v_\lambda = \frac{1}{2}c\Omega$ and $L_n v_\lambda = 0$ for $n \geq 3$ have to be checked. By using the expansion (10.6) of $Y(v_\lambda, z)$ we obtain

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{n-m}a_m - \lambda(n+1)a_n.$$

Now, $a_2(v_\lambda) = 2\lambda\Omega$ and $a_{n-m}a_m(v_\lambda) = 0$ for $m > 2$ or $m < n-2$ (because then $n-m > 2$). In the case of $n > 2$ we have $a_n(v_\lambda) = 0$ and only for $n = 3, n = 4$ there exist m with $n-2 \leq m \leq 2$. It follows that $L_n v_\lambda = 0$ for $n \geq 5$. Because of $a_2a_1v_\lambda = 0$ and $a_2a_2v_\lambda = 0$ we also have $L_3v_\lambda = 0 = L_4v_\lambda$. For $n = 2$ we get $L_2v_\lambda = \frac{1}{2}a_1a_1(v_\lambda) + a_2a_0(v_\lambda) - 6\lambda^2\Omega = (\frac{1}{2} - 6\lambda^2)\Omega$, and the central charge is $c = 1 - 12\lambda^2$. □

Remark 10.41. The Fock space representations of the Virasoro algebra which we have studied in the context of the quantization of the bosonic string on p. 116 are in perfect analogy with the observation that the $\frac{1}{2}T_1^2 + \lambda T_2$ are conformal vectors. We can show that

$$L_{-2}\Omega = \frac{1}{2} + 2\mu T_2$$

for

$$L_{-2} = \frac{1}{2}a_{-1}^2 + \sum_{k>0} a_{-k-1}a_{k-1},$$

where μ is the eigenvalue of a_0 to Ω. This yields another way of constructing a vertex algebra from the Heisenberg algebra using the calculations made there.

Indeed, $a_{-1}^2\Omega = T_1^2$ and $\sum_{k>0} a_{-k-1}a_{k-1}\Omega = a_{-2}a_0\Omega = 2\mu T_2$, hence

$$L_{-2}\Omega = \frac{1}{2}T_1^2 + 2\mu T_2. \qquad \square$$

Primary Fields. The conformal vector v of a conformal vertex algebra V provides, in particular, the diagonalizable endomorphism $L_0 : V \to V$. For each eigenvector $a \in V$ of L_0 with $L_0 a = ha$ the OPE (cf. Theorem 10.36) yields

$$Y(v,z)Y(a,w) \sim \sum_{n \geq -1} \frac{Y(L_n a, w)}{(z-w)^{n+2}},$$

and therefore begins with the following terms

$$Y(v,z)Y(a,w) \sim \frac{\partial Y(a,w)}{(z-w)} + \frac{hY(a,w)}{(z-w)^2} + \dots.$$

Here, we use $L_{-1} = T$ and $Y(Ta,w) = \partial Y(a,w)$ (according to Corollary 10.34) and $L_0 a = ha$.

Definition 10.42 (Primary Field). A field $Y(a,z)$ of a conformal vertex algebra V with conformal vector v is called *primary* of *(conformal) weight h* if there are no other terms in the above OPE, that is

$$Y(v,z)Y(a,w) \sim \frac{\partial Y(a,w)}{(z-w)} + \frac{hY(a,w)}{(z-w)^2}.$$

Equivalently, $Y(L_n a, z) = 0$ for all $n > 0$.

The following is in accordance with Definition 9.7.

Corollary 10.43. *The field $Y(a,z)$ is primary of weight h if and only if one of the following equivalent conditions holds:*

1. $L_0 a = ha$ *and* $L_n a = 0$ *for all* $n > 0$.
2. $[L_n, Y(a,z)] = z^{n+1}\partial Y(a,z) + h(n+1)z^n Y(a,z)$ *for all* $n \in \mathbb{Z}$.
3. $[L_n, a_{(m)}] = \big((h-1)n - m\big)a_{(m+n)}$ *for all* $n, m \in \mathbb{Z}$.

Proof. We have already stated the equivalence with 1. To show the second property for a primary field $Y(a,z)$ we compare

$$[Y(v,z)Ya,w)] = \sum_{n \in \mathbb{Z}} [L_n, Y(a,w)]z^{-n-2}$$

with

$$\begin{aligned}
[Y(\nu,z)Ya,w)] &= \partial Y(a,w)\delta(z-w) + hY(a,w)\partial\delta(z-w) = \\
&= \sum_{m\in\mathbb{Z}}(-m-1)a_{(m)}w^{-m-2}\sum_{n\in\mathbb{Z}}z^{-n-1}w^{n} \\
&\quad +h\sum_{m\in\mathbb{Z}}a_{(m)}w^{-m-1}\sum_{n\in\mathbb{Z}}nz^{-n-1}w^{n-1} \\
&= \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(-m-1+h(n+1))a_{(m)}w^{n-m-1}z^{-n-2},
\end{aligned}$$

and obtain for all $n\in\mathbb{Z}$

$$\begin{aligned}
[L_n,Y(a,w)] &= (-m-1+h(n+1))a_{(m)}w^{n-m-1} \\
= w^{n+1}\sum_{m\in\mathbb{Z}}&(-m-1)a_{(m)}w^{-m-2} + w^{n}h(n+1)\sum_{m\in\mathbb{Z}}a_{(m)}w^{-m-1} \\
&= w^{n+1}\partial Y(a,w) + z^{n}h(n+1)Y(a,z).
\end{aligned}$$

Hence, a primary field $Y(a,z)$ satisfies 2, and the converse is true since the implications above can be reversed.

To deduce 3 from 2 we use

$$\begin{aligned}
[L_n,Y(a,z)] &= \sum_{m\in\mathbb{Z}}[L_n,a_{(m)}]z^{-m-1} \\
&= z^{n+1}\sum_{m\in\mathbb{Z}}a_{(m)}z^{-m-2} + z^{n}h(n+1)\sum_{m\in\mathbb{Z}}a_{(m)}z^{-m-1} \\
&= \sum_{m\in\mathbb{Z}}(-m-n-1+h(n+1))a_{(m+n)}z^{-m-1}
\end{aligned}$$

to obtain $[L_n,a_{(m)}] = ((h-1)(n-1)-m)a_{(m+n)}$ by comparing coefficients. Hence, 2 implies 3 and vice versa. □

Correlation Functions. Let us end this short introduction to vertex algebra theory by presenting the fundamental properties of correlation functions of a vertex algebra which have not been discussed so far although they play an important role in the axiomatic theory of quantum field theory and of conformal field theory as explained in Sections 8 and 9.

Let V^* denote the dual of V that is the space of linear functions $\mu : V\to\mathbb{C}$. Given $a_1,\dots,a_n\in V$ and $v\in V$ we consider

$$\langle\mu, Y(a_1,z_1)\dots Y(a_n,z_n)v\rangle := \mu(Y(a_1,z_1)\dots Y(a_n,z_n)v)$$

as a formal power series in $\mathbb{C}\left[\left[z_1^{\pm},\dots,z_m^{\pm}\right]\right]$. These series are called n-point functions or *correlation functions*. Since $v = Y(v,z)|_{z=0}\Omega$ it is enough to study the case of $v=\Omega$ only.

Theorem 10.44. *Let (V,Y,T,Ω) be a vertex algebra and let $\mu\in V^*$ be a linear functional on V. For any $a_1,\dots,a_n\in V$ there exists a series*

$$f^{\mu}_{a_1 \dots a_n}(z_1, \dots, z_n) \in \mathbb{C}[[z_1 \dots z_n]]\,[(z_i - z_j)^{-1}, i \neq j]$$

such that the following properties are satisfied:

1. *For any permutation π of $\{1, \dots, n\}$ the correlation function*
$$\langle \mu, Y(\pi(a_1), z_{\pi(1)}) \dots Y(\pi(a_n), z_{\pi(n)})\Omega \rangle$$
is the expansion in $\mathbb{C}\left(\left(z_{\pi(1)}\right)\right) \dots \left(\left(z_{\pi(n)}\right)\right)$ of $f^{\mu}_{a_1 \dots a_n}(z_1, \dots, z_n)$.
2. *For $i < j$ we have*
$$f^{\mu}_{a_1 \dots a_n}(z_1, \dots z_n) = f_{(Y(a_i, z_i - z_j)a_j)a_1 \dots \hat{a}_i \dots \hat{a}_j \dots a_n}(z_1 \dots \hat{z}_i \dots z_j \dots z_n),$$
where $(z_i - z_j)^{-1}$ has to be replaced by its expansion $\sum_{k \geq 0} z_i^{-k-1} z_j^k$ into positive powers of $\frac{z_j}{z_i}$.
3. *For $1 \leq j \leq n$ we have*
$$\partial_{z_j} f^{\mu}_{a_1 \dots a_n}(z_1, \dots z_n) = f^{\mu}_{a_1 \dots Ta_j \dots a_n}(z_1, \dots z_n).$$

Proof. Since $Y(a,z)$ is a field by the defining properties of a vertex algebra we have $\langle \mu, Y(a,z)v \rangle \in \mathbb{C}((z))$ for all $a, v \in V$, and by induction

$$\langle \mu, Y(\pi(a_1), z_{\pi(1)}) \dots Y(\pi(a_n), z_{\pi(n)})\Omega \rangle \in \mathbb{C}\left(\left(z_{\pi(1)}\right)\right) \dots \left(\left(z_{\pi(n)}\right)\right).$$

By the Locality Axiom V2 there exist integers $N_{ij} > 0$ such that

$$(z_i - z_j)^{N_{ij}} [Y(a_i, z_i), Y(a_j, z_j)] = 0.$$

Hence, the series

$$\prod_{i<j} (z_i - z_j)^{N_{ij}} \langle \mu, Y(\pi(a_1), z_{\pi(1)}) \dots Y(\pi(a_n), z_{\pi(n)})\Omega \rangle$$

is independent of the permutation π. Moreover, it contains only non-negative powers of all the variables $z_i, 1 \leq i \leq n$, because of $Y(a,z)\Omega \in V[[z]]$ (Vacuum Axiom V3). Consequently,

$$\prod_{i<j} (z_i - z_j)^{N_{ij}} \langle \mu, Y(\pi(a_1), z_{\pi(1)}) \dots Y(\pi(a_n), z_{\pi(n)})\Omega \rangle$$

coincides with

$$\prod_{i<j} (z_i - z_j)^{N_{ij}} \langle \mu, Y(a_1, z_1) \dots Y(a_n, z_n)\Omega \rangle \in \mathbb{C}[[z_1, \dots, z_n]]$$

as a series in $\mathbb{C}[[z_1, \dots, z_n]]$. Dividing this series by $\prod_{i<j}(z_i - z_j)^{N_{ij}}$ yields the series $f^{\mu}_{a_1 \dots a_n} \in \mathbb{C}[[z_1 \dots z_n]]\,[(z_i - z_j)^{-1}, i \neq j]$ with property 1.

The second property follows directly from 1 and the associativity of the OPE (Theorem 10.36). For example, in the case of $n = 2$ it has the form

$$f^{\mu}_{a_1a_2}(z_1,z_2) = f_{(Y(a_1,z_1-z_2)a_2)}(z_2)$$

and this equality is the same as

$$\langle \mu, Y(a_1,z_1)Y(a_2,z_2)\Omega \rangle = \langle \mu, Y(Y(a_1,z_1-z_2)a_2,z_2)\Omega \rangle.$$

The third property is a consequence of the equality $Y(Ta,z) = \partial Y(a,z)$ proven in Corollary 10.34. $\square$

10.7 Induced Representations

In the course of these notes we have used Fock spaces and representation spaces for Lie algebras which all look very similar to each other and mostly have been given as vector spaces of polynomials. The unifying principle behind this observation is that all these representation spaces can be understood as certain induced representations which are mostly induced by a one-dimensional representation of a Lie subalgebra of the Lie algebra in question. This has to do with the fact that our representation spaces are cyclic in the sense that they can be generated by a suitable vector.

In order to describe induced representations we use the concept of a universal enveloping algebra. For any associative algebra A let $L(A)$ denote the Lie algebra with A as the underlying vector space and with the commutator as the Lie bracket.

Definition 10.45. A *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ is a pair (U,i) of an associative algebra U with unit 1 and a Lie algebra homomorphism $i : \mathfrak{g} \to L(U)$, such that the following universal property is fulfilled. For any associative algebra A with unit 1 and any Lie algebra homomorphism $j : \mathfrak{g} \to L(A)$ there exists a unique algebra homomorphism $h : U \to A$ with $h(1) = 1$ such that $h \circ i = j$.

Observe that a representation of the Lie algebra $\mathfrak{g}$, that is a Lie algebra homomorphism $\mathfrak{g} \to L(\text{End } W)$ (where End W is considered as an associative algebra) has a natural extension to $U(\mathfrak{g})$ as a homomorphism of associative algebras by the universal property. Conversely, a homomorphism $U(\mathfrak{g}) \to \text{End } W$ of associative algebras can be restricted to $\mathfrak{g}$ in order to obtain a Lie algebra homomorphism, that is a representation. We have shown:

Lemma 10.46. *The representations* $\mathfrak{g} \to \text{End } W$ *are in one-to-one correspondence with the representations* $U(\mathfrak{g}) \to \text{End } W$.

Lemma 10.47. *To each Lie algebra there corresponds a universal enveloping algebra unique up to isomorphism.*

Proof. The uniqueness of such a pair (U,i) is easy to show. In order to establish the existence let

$$T(W) = \bigoplus_{n=0}^{\infty} W^{\otimes n}$$

be the tensor algebra of a vector space W, where $W^{\otimes n}$ is n-fold tensor product of W with itself. The tensor algebra has the universal property that every linear map $W \to A$ into an associative algebra A with unit has a unique extension $T(W) \to A$ as an algebra homomorphism sending 1 to 1. Let $J \subset T(\mathfrak{g})$ be the two-sided ideal generated by the elements of the form $a \otimes b - b \otimes a - [a,b],\ a,b \in \mathfrak{g}$. Let $U(\mathfrak{g}) := T(\mathfrak{g})/J$ be the quotient algebra with projection $p : T(\mathfrak{g}) \to U(\mathfrak{g})$. The map i is then defined by the restriction of p to $\mathfrak{g}$ with respect to its natural embedding $\mathfrak{g} \subset U(\mathfrak{g})$.

To show that $(U(\mathfrak{g}), i)$ fulfills the universal property, let A be an associative algebra with unit 1 and let $j : \mathfrak{g} \to L(A)$ be a Lie algebra homomorphism. Then, by the universal property of the tensor algebra $T(\mathfrak{g})$, there exists a unique algebra homomorphism $H : T(\mathfrak{g}) \to A$ extending the linear map j and satisfying $H(1) = 1$. Each generating element $a \otimes b - b \otimes a - [a,b]$ of J is annihilated by H since $H(a \otimes b - b \otimes a) = H(a)H(b) - H(b)H(a) = j(a)j(b) - j(b)j(a) = j([a,b]) = H([a,b])$. Hence, the ideal J is contained in the kernel of H. Consequently, H has a factorization h through p, that is there is an algebra homomorphism $h : U(\mathfrak{g}) \to A$ respecting the units with $H = h \circ p$ and thus $j = H|_{\mathfrak{g}} = h \circ p|_{\mathfrak{g}} = h \circ i$. □

Neither the definition nor the above proof yields the injectivity of i. However, using the construction of $U(\mathfrak{g})$ this follows from the Poincaré–Birkhoff–Witt theorem which can be found in many books, e.g., [HN91]. We state one essential consequence of this theorem which is of special interest regarding the various descriptions of representation spaces.

Proposition 10.48 (Poincaré–Birkhoff–Witt). *Let $(a_i)_{i \in I}$ be an ordered basis of the Lie algebra $\mathfrak{g}$. Then the elements $p(a_{i_1} \otimes \ldots \otimes a_{i_m}), m \in \mathbb{N}, i_1 \leq \ldots \leq i_m$, together with 1 form a basis of $U(\mathfrak{g})$.*

As a consequence we obtain an isomorphism of vector spaces from the symmetric algebra

$$S(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\odot n} \longrightarrow U(\mathfrak{g})$$

to $U(\mathfrak{g})$, where $W^{\odot n}$ is the n-fold symmetric product of a vector space, that is the subspace of symmetric tensors in $W^{\otimes n}$. $S(W)$ can also be understood as the quotient $T(W)/S$ with respect to the two-sided ideal $S \subset T(W)$ generated by all elements of the form $v \otimes w - w \otimes v,\ v, w \in W$. So far $S(\mathfrak{g})$ is the enveloping algebra of an abelian Lie algebra $\mathfrak{g}$.

Note that the symmetric algebra $S(W)$ can be identified with the algebra of polynomials $\mathbb{C}[T_i : i \in I]$ whenever $(a_i)_{i \in I}$ is an ordered basis of the vector space W.

Consequently, as a vector space the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is isomorphic to the vector space $\mathbb{C}[T_i : i \in I]$ of polynomials:

$$1 \mapsto 1,\ T_{i_1} \ldots T_{i_m} \mapsto p(a_{i_1} \otimes \ldots \otimes a_{i_m}),\ m \in \mathbb{N},\ i_1 \leq \ldots \leq i_m,$$

provides an isomorphism.

Now, let $\mathfrak{b}$ be a Lie subalgebra of the Lie algebra $\mathfrak{g}$ and let $\pi : \mathfrak{b} \to \mathrm{End}\ W$ a Lie algebra homomorphism, that is a representation of $\mathfrak{b}$ in the vector space W.

Definition 10.49. The *induced representation* (induced by π) is given by the *induced* $\mathfrak{g}$*-module*

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} W,$$

that is

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} = (U(\mathfrak{g}) \otimes W)/U(\mathfrak{g})\{b \otimes w - 1 \otimes \pi(b)w : (b,w) \in \mathfrak{b} \times W\},$$

where $\mathfrak{g}$ acts by left multiplication in the first factor.

It is straightforward to check that this prescription defines a representation. In fact, the action of $a \in U(\mathfrak{g})$ on $U(\mathfrak{g}) \otimes W$, $x \otimes w \mapsto ax \otimes w$, descends to a linear action $\rho(a) \in \mathrm{End}\ (\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}})$ since $J_\pi := U(\mathfrak{g})\{b \otimes w - 1 \otimes \pi(b)w : (b,w) \in \mathfrak{b} \times W\}$ is a left ideal, in particular $a(J_\pi) \subset J_\pi$. In addition, $\rho(a)([x \otimes w]) = [ax \otimes w]$ defines a homomorphism $a \mapsto \rho(a)$ of associative algebras, again since J_π a left ideal in $U(\mathfrak{g}) \otimes W$. The restriction of ρ to $\mathfrak{g}$ is therefore a Lie algebra homomorphism.

An elementary example is the Fock space representation of the Heisenberg algebra described on p. 114. The Heisenberg algebra H is generated by $a_n, n \in \mathbb{Z}$, and the central element Z. The inducing representation π is defined on the abelian Lie subalgebra $\mathsf{B} \subset \mathsf{H}$ generated by the $a_n, n \geq 0$ and Z, with $W = \mathbb{C}$, and this representation $\pi : \mathsf{P} \to \mathrm{End}\ \mathbb{C} \cong \mathbb{C}$ is determined by

$$\rho(Z) = \mathrm{id}_{\mathbb{C}} = 1, \rho(a_0) = \mu \mathrm{id}_{\mathbb{C}} = \mu,\ \rho(a_n) = 0 \text{ for } n > 0.$$

Let $\Omega := 1 \otimes 1$. Then $a_n \in J_p i$ for $n > 0$, since $a_n\Omega = a_n \otimes 1 = 1 \otimes \pi(a_n)1 = 0$, $a_0\Omega = a_0 \otimes 1 = 1 \otimes \mu = \mu\Omega$, and $Z(\Omega) = 1 \otimes \pi(Z) = \Omega$. Hence, $a_n \in J_\pi, n > 0$, and a_0, Z depend on Ω modulo J_π.

Consequently, $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C})$ is generated by the classes

$$[a_{i_1} \otimes \ldots \otimes a_{i_m}\Omega], m \in \mathbb{N}, i_1 \leq \ldots \leq i_m < 0,$$

and Ω according to Proposition 10.48. These elements remain linearly independent, since the a_{-n}, a_{-m} commute with each other for $m, n \geq 0$, so that $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C})$ is isomorphic to the vector space $\mathbb{C}[T_n : n > 1]$ with the action $\rho(a_{-n})\Omega = T_n$ for $n > 0$, and, more generally,

$$\rho(a_{-n})P = T_n P,$$

for any polynomial $P \in \mathbb{C}[T_n : n > 1]$. Similarly, because of the other commutation relations, for $n > 0$ we obtain $\rho(a_n)T_m = 0$ if $n \neq m$ and $\rho(a_n)T_n = n\Omega$, and, more generally, $\rho(a_n)P = n\partial_{T_n} P$. This, of course, is exactly the representation on p. 114.

The example is typical, in the cases considered in these notes, we have $W = \mathbb{C}$ and an ordered basis $(a_i)_{i \in I}$ with a division $I = I_+ \cup I_-$ such that $a_i, i \in I_+$ is a basis of J_π and $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C})$ is isomorphic to the space of polynomials $\mathbb{C}[T_n : n \in I_-]$. The action of the $a_i, i \in I$, is then essentially determined by $a_i\Omega = T_i$ if $i \in I_-$ and the commutation relations of all the a_i.

In this way we obtain similarly the description of a Verma module with respect to given numbers $c, h \in C$ on p. 94, the representation of the string algebra on p. 119, the representation V_c of the Virasoro algebra Vir used for the Virasoro vertex algebras on p. 193, the representation of the Kac–Moody algebras on p. 196, and in a certain sense even the free boson representation on p. 136 where, however, the Hilbert space structure has to be respected as well. Analogously, the fermionic Fock space on p. 52 can be described as an induced representation. To do this, we have to extend the consideration to the case of Lie superalgebras in order to include the anticommutation relations.

References

[BD04*] A. Beilinson and V. Drinfeld. *Chiral Algebras*. AMS Colloquium Publications **51**, AMS, Providence, RI, 2004.

[BF01*] D. Ben-Zvi and E. Frenkel. *Vertex Algebras and Algebraic Curves*. AMS, Providence, RI, 2001.

[Bor86*] R.E. Borcherds. Vertex algebras, Kac-Moody algebra and the monster. *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.

[Bor00*] R.E. Borcherds, Quantum vertex algebras. In: *Taniguchi Conference on Mathematics Nara '98*. Advanced Studies in Pure Mathematics **31**, 51–74. Math. Soc. Japan, 2000.

[FKRW95*] E. Frenkel, V. Kac, A. Radul, and W. Wang. $\mathscr{W}_{1+\infty}$ and $W(gl_N)$ with central charge N. *Commun. Math. Phys*. **170** (1995), 337–357.

[FLM88*] I. Frenkel, J. Lepowsky, and A. Meurman. *Vertex Operator Algebras and the Monster*. Academic Press, New York, 1988.

[HN91] J. Hilgert and K.-H. Neeb. *Lie Gruppen und Lie Algebren*. Vieweg, Braunschweig, 1991.

[Hua97*] Y-Z. Huang. *Two-Dimensional Conformal Geometry and Vertex Operator Algebras*. Progress in Mathematics 148, Birkhuser, Basel, 1997.

[Kac98*] V. Kac. *Vertex Algebras for Beginners*. University Lecture Series 10, AMS, Providencs, RI, 2nd ed., 1998.

[Len07*] S. Lentner. *Vertex Algebras Constructed from Hopf Algebra Structures*. Diplomarbeit, LMU München, 2007.

[Lin04*] K. Linde. Global vertex operators on Riemann surfaces. Dissertation. LMU München, 2004.

[Seg88a] G. Segal. The definition of conformal field theory. Unpublished Manuscript, 1988. Reprinted in *Topology, Geometry and Quantum Field Theory*, U. Tillmann (Ed.), 432–574, Cambridge University Press, Cambridge, 2004.

[Seg91] G. Segal. Geometric aspects of quantum field theory. *Proc. Intern. Congress Kyoto 1990, Math. Soc*. Japan, 1387–1396, 1991.