Chapter 9
Foundations of Two-Dimensional Conformal Quantum Field Theory

In this chapter we study *two-dimensional conformally invariant quantum field theory* (*conformal field theory* for short) by some basic concepts and postulates – that is using a system of axioms as presented in [FFK89] and based on the work of Osterwalder and Schrader [OS73], [OS75]. We will assume the Euclidean signature $(+,+)$ on $\mathbb{R}^2$ (or on surfaces), as it is customary because of the close connection of conformal field theory to statistical mechanics (cf. [BPZ84] and [Gin89]) and its relation to complex analysis.

We do not use the results of Chap. 8 where the axioms of quantum field theory are investigated in detail and for arbitrary spacetime dimensions nor do we assume the notations to be known in order to keep this chapter self-contained. However, the preceding chapter may serve as a motivation for several concepts and constructions. In particular, the presentation of the axioms explains why locality for the correlation functions in Axiom 1 below is expressed as the independence of the order of the indices, and why the covariance in Axiom 2 does not refer to the unitary representation of the Poincaré group. Moreover, in the light of the results of the preceding chapter the reconstruction used below on p. 158 is a general principle in quantum field theory relating the formulation based on field operators with an equivalent formulation based on correlation functions.

9.1 Axioms for Two-Dimensional Euclidean Quantum Field Theory

The basic objects of a two-dimensional quantum field theory (cf. [BPZ84], [IZ80], [Gaw89], [Gin89], [FFK89], [Kak91], [DMS96*]) are the fields Φ_i, $i \in B_0$, subject to a number of properties. These fields are also called field operators or operators. They are defined as maps on open subsets M of the complex plane $\mathbb{C} \cong \mathbb{R}^{2,0}$ (or on Riemann surfaces M). They take their values in the set $\mathscr{O} = \mathscr{O}(\mathbb{H})$ of (possibly unbounded and mostly self-adjoint) operators on a fixed Hilbert space $\mathbb{H}$. To be precise, these field operators are usually defined only on spaces of test functions on M, e.g. on the Schwartz space $\mathscr{S}(M)$ of rapidly decreasing functions or on other

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suitable spaces of test functions. Hence, they can be regarded as operator-valued distributions (cf. Definition 8.8).

The *matrix coefficients* $\langle v|\Phi_i(z)|w\rangle$ of the field operators are supposed to be well-defined for $v,w \in D$ in a dense subspace $D \subset \mathbb{H}$. Here, $\langle v,w\rangle$, $v,w \in \mathbb{H}$, denotes the inner product of $\mathbb{H}$ and $\langle v|\Phi_i(z)|w\rangle$ is the same as $\langle v,\Phi_i(z)w\rangle$.

The essential parameters of the theory, which connect the theory with experimental data, are the *correlation functions*

$$G_{i_1\ldots i_n}(z_1,\ldots,z_n) := \langle\Omega|\Phi_{i_1}(z_1)\ldots\Phi_{i_n}(z_n)|\Omega\rangle.$$

These functions are also called *n-point function*s or *Green's function*s. Here, $\Omega \in \mathbb{H}$ is the vacuum vector. These correlation functions have to be interpreted as *vacuum expectation value*s of time-ordered products $\Phi_{i_1}(z_1)\ldots\Phi_{i_n}(z_n)$ of the field operators (*time ordered* means $\operatorname{Re} z_n > \ldots > \operatorname{Re} z_1$, or $|z_n| > \ldots > |z_1|$ for the radial quantization). They usually can be analytically continued to

$$M_n := \{(z_1,\ldots,z_n) \in \mathbb{C}^n : z_i \neq z_j \quad \text{for } i \neq j\},$$

the space of configurations of n points. (To be precise, they have a continuation to the universal covering $\widetilde{M}_n$ of M_n and thus they are no longer single valued on M_n, in general. In this manner, the pure braid group P_n appears, which is the fundamental group $\pi_1(M_n)$ of M_n.) For simplification we will assume in the formulation of the axioms that the $G_{i_1\ldots i_n}$ are defined on M_n.

The positivity of the hermitian form, that is the inner product of $\mathbb{H}$, can be expressed by the so-called reflection positivity of the correlation functions. This property is defined by fixing a reflection axis – which typically is the imaginary axis in the simplest case – and requiring the correlation of operator products of fields on one side of the axis with their reflection on the other side to be non-negative (cf. Axiom 3 below).

Now, the two-dimensional quantum field theory can be described completely by the properties of the correlation functions using a system of axioms (Axiom 1–6 in these notes, see below). The field operators and the Hilbert space do not have to be specified a priori, they are determined by the correlation functions (cf. Lemma 9.2 and Theorem 9.3).

To state the axioms we need a few notations:

$$\begin{aligned}
M_n^+ &:= \{(z_1,\ldots,z_n) \in M_n : \operatorname{Re} z_j > 0 \quad \text{for } j = 1,\ldots,n\},\\
\mathscr{S}_0^+ &:= \mathbb{C},\\
\mathscr{S}_n^+ &:= \{f \in \mathscr{S}(\mathbb{C}^n) : \operatorname{Supp}(f) \subset M_n^+\}.
\end{aligned}$$

Here, $\mathscr{S}(\mathbb{C}^n)$ is the Schwartz space of rapidly decreasing smooth functions, that is the complex vector space of all functions $f \in C^\infty(\mathbb{C}^n)$ for which

$$\sup_{|\alpha|\leq p}\ \sup_{x\in\mathbb{R}^{2n}} |\partial^\alpha f(x)|(1+|x|^2)^k < \infty,$$

for all $p,k \in \mathbb{N}$. We have identified the spaces $\mathbb{C}^n$ and $\mathbb{R}^{2n}$ and have used the real coordinates $x = (x_1,\dots,x_{2n})$ as variables. ∂^α is the partial derivative for the multi-index $\alpha \in \mathbb{N}^{2n}$ with respect to x. $\mathrm{Supp}(f)$ denotes the support of f, that is the closure of the set $\{x \in \mathbb{R}^{2n} : f(x) \neq 0\}$.

It makes sense to write $z \in \mathbb{C}$ as $z = t + iy$ with $t,y \in \mathbb{R}$, and to interpret $\overline{z} = t - iy$ as a quantity not depending on z. In this sense one sometimes writes $G(z,\overline{z})$ instead of $G(z)$, to emphasize that $G(z)$ is not necessarily holomorphic. In the notation $z = t + iy$, y is the "space coordinate" and t is the (imaginary) "time coordinate".

The group $E = E_2$ of Euclidean motions, that is the Euclidean group (which corresponds to the Poincaré group in this context), is generated by the rotations

$$r_\alpha : \mathbb{C} \to \mathbb{C}, \quad z \mapsto e^{i\alpha} z, \quad \alpha \in \mathbb{R},$$

and the translations

$$t_a : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z + a, \quad a \in \mathbb{C}.$$

Further Möbius transformations are the dilatations

$$d_\tau : \mathbb{C} \to \mathbb{C}, \quad z \mapsto e^\tau z, \quad \tau \in \mathbb{R},$$

and the inversion

$$i : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z^{-1}, \quad z \in \mathbb{C} \setminus \{0\}.$$

These conformal transformations generate the Möbius group Mb (cf. Sect. 2.3). All other global conformal transformations (cf. Definition 2.10) of the Euclidean plane (with possibly one singularity) are generated by Mb and the *time reflection*

$$\theta : \mathbb{C} \to \mathbb{C}, \quad z = t + iy \mapsto -t + iy = -\overline{z}.$$

(cf. Theorems 1.11 and 2.11 and the discussion after Definition 2.12)

Osterwalder–Schrader Axioms ([OS73], [OS75], [FFK89])
Let B_0 be a countable index set. For multi-indices $(i_1,\dots,i_n) \in B_0^n$ we also use the notation $i = i_1 \dots i_n = (i_1,\dots,i_n)$. Let $B = \bigcup_{n \in \mathbb{N}_0} B_0^n$. The quantum field theory is described by a family $(G_i)_{i \in B}$ of continuous and polynomially bounded correlation functions

$$G_{i_1 \dots i_n} : M_n \to \mathbb{C}, \quad G_\emptyset = 1,$$

subject to the following axioms:

Axiom 1 (Locality) *For all $(i_1,\dots,i_n) \in B_0^n, (z_1,\dots,z_n) \in M_n$, and every permutation $\pi : \{1,\dots,n\} \to \{1,\dots,n\}$ one has*

$$G_{i_1,\dots,i_n}(z_1,\dots,z_n) = G_{i_{\pi(1)} \dots i_{\pi(n)}}(z_{\pi(1)},\dots,z_{\pi(n)}).$$

Axiom 2 (Covariance) *For every $i \in B_0$ there are* conformal weights $h_i, \overline{h}_i \in \mathbb{R}$ *($\overline{h}_i$ is not the complex conjugate of h_i, but completely independent of h_i), such that for all $w \in E$ and $n \geq 1$ one has*

$$\begin{aligned} &G_{i_1\dots i_n}(z_1,\bar{z}_1,\dots,z_n,\bar{z}_n) \\ &= \prod_{j=1}^{n} \left(\frac{dw}{dz}(z_j)\right)^{h_j} \left(\overline{\frac{dw}{dz}(z_j)}\right)^{\bar{h}_j} G_{i_1\dots i_n}(w_1,\bar{w}_1,\dots,w_n,\bar{w}_n), \end{aligned} \tag{9.1}$$

with $w_j := w(z_j)$, $\overline{w}_j := \overline{w(z_j)}$, $h_j := h_{i_j}$.

Here, $s_i := h_i - \bar{h}_i$ is called the *conformal spin* for the index i and $d_i := h_i + \bar{h}_i$ is called the *scaling dimension*.
Furthermore, we assume

$$h_i - \bar{h}_i, h_i + \bar{h}_i \in \mathbb{Z}, \quad i \in B_0.$$

As a consequence, there do not occur any ambiguities concerning the exponents. In particular, this is satisfied whenever

$$h_i, \bar{h}_i \in \frac{1}{2}\mathbb{Z}.$$

See Hawley/Schiffer [HS66] for a discussion of this condition.

The covariance of the correlation functions formulated in Axiom 2 corresponds to the transformation behavior of tensors or generalized differential forms under change of coordinates when extended to more general conformal transformations (see also p. 164).

The covariance conditions severely restricts the form of 2-point functions and 3-point functions. Because of the covariance with respect to translations, all correlation functions $G_{i_1\dots i_n}$ for $n \geq 2$ depend only on the differences $z_{ij} := z_i - z_j$, $i \neq j$, $i,j \in \{1,\dots,n\}$. Typical 2-point functions $G_{i_1 i_1} = G$, which satisfy Axiom 2, are

$$\begin{aligned} G &= \text{const.} && \text{with} \quad h = \bar{h} = 0, \\ G(z_1,\bar{z}_1,z_2,\bar{z}_2) &= C z_{12}^{-2} \bar{z}_{12}^{-2} && \text{with} \quad h = \bar{h} = 1, \\ G(z_1,z_2) &= C z_{12}^{-4} && \text{with} \quad h = 2, \bar{h} = 0. \end{aligned}$$

A general example is

$$G(z_1,z_2) = C z_{12}^{-2h} \bar{z}_{12}^{-2\bar{h}} \quad \text{with} \quad h, \bar{h} \in \frac{1}{2}\mathbb{Z}.$$

Hence, for the case $h = \bar{h}$,

$$G(z_1,\bar{z}_1,z_2,\bar{z}_2) = C|z_{12}|^{-4h} = C|z_{12}|^{-2d}.$$

Typical 2-point functions $G = G_{i_1 i_2}$ with $i_1 \neq i_2$, for which Axiom 2 is valid, are

$$G(z_1,\bar{z}_1,z_2,\bar{z}_2) = C z_{12}^{-h_1} z_{12}^{-h_2} \bar{z}_{12}^{-\bar{h}_1} \bar{z}_{12}^{-\bar{h}_2}.$$

All these examples are also Möbius covariant.

For the function $F = G_{i_1 i_1}$ with

$$F(z_1, \bar{z}_1, z_2, \bar{z}_2) = -\log |z_{12}|^2$$

Axioms 1 and 2 hold as well (with arbitrary $h, \bar{h}$, $h = \bar{h}$). However, this function is not Möbius covariant because one has e.g., for $w(z) = e^{\tau} z$, $\tau \neq 0$, and in the case $h = \bar{h} \neq 0$,

$$\prod_{j=1}^{2} \left(\frac{dw}{dz}(z_j) \right)^h \left(\overline{\frac{dw}{dz}(z_j)} \right)^{\bar{h}} F(w_1, w_2)$$
$$= (e^{\tau})^{2h+2\bar{h}} (-\log e^{2\tau} |z_{12}|^2) \neq -\log |z_{12}|^2.$$

In particular, F is not scaling covariant in the sense of Axiom 4 (see below). A typical 3-point function is

$$G(z_1, \bar{z}_1, z_2, \bar{z}_2, z_3, \bar{z}_3)$$
$$= z_{12}^{-h_1 - h_2 + h_3} z_{23}^{-h_2 - h_3 + h_1} z_{13}^{-h_3 - h_1 + h_2}$$
$$\bar{z}_{12}^{-\bar{h}_1 - \bar{h}_2 + \bar{h}_3} \bar{z}_{23}^{-\bar{h}_2 - \bar{h}_3 + \bar{h}_1} \bar{z}_{13}^{-\bar{h}_3 - \bar{h}_1 + \bar{h}_2}, \tag{9.2}$$

as can be checked easily. It is not difficult to see that this 3-point function is also Möbius covariant, hence conformally covariant.

We describe a rather simple example involving all correlation functions.

Example 9.1. Let $B_0 = \{1\}$ and $n := (1, \dots 1) \in B_0^n = \{n\}$. The functions G_n are supposed to be zero if n is odd and

$$G_{2n}(z_1, \dots, z_{2n}) = \frac{k^n}{2^n n!} \sum_{\sigma \in S_{2n}} \prod_{j=1}^{n} \frac{1}{(z_{\sigma(j)} - z_{\sigma(n+j)})^2},$$

where S_N is the group of permutations of N elements and where $k \in \mathbb{C}$ is a constant. The weights are $h_1 = 1$, $\bar{h}_1 = 0$.

If the exponent "2" in the denominator is replaced with $2m$ we get another example with conformal weight $h = m$ instead of 1 and $\bar{h} = 0$.

The dependence in z and $\bar{z}$ can be treated independently, as in the example. The example can be extended by defining $F_{2n}(z, \bar{z}) = G_{2n}(z) G_{2n}(\bar{z})$, and the resulting theory has the weights $h_1 = 1 = \bar{h}_1$.

Note that the correlation functions in Example 9.1 are covariant with respect to general Möbius transformations, even if the $\bar{z}$-dependence is included. Möbius covariance (and hence conformal covariance) holds as well if the exponent 2 is replaced by $2m$.

In the following, we mostly treat only the dependence in z in order to simplify the formulas. The general case can easily be derived from the formulas respecting only the dependence on z (see p. 88 for an explanation).

Next, we formulate reflection positivity (cf. Sect. 8.6). Let $\underline{\mathscr{S}}^+$ be the space of all sequences $\underline{f} = (f_i)_{i\in B}$ with $f_i \in \mathscr{S}_n^+$ for $i \in B_0^n$ and $f_i \neq 0$ for at most finitely many $i \in B$.

Axiom 3 (Reflection Positivity) *There is a map* $* : B_0 \to B_0$ *with* $*^2 = \mathrm{id}_{B_0}$ *and a continuation* $* : B \to B$, $i \mapsto i^*$, *so that*

1. $G_i(z) = G_{i^*}(\theta(z)) = G_{i^*}(-z^*)$ *for* $i \in B$, *where* z^* *is the complex conjugate of* z.
2. $\langle \underline{f}, \underline{f} \rangle \geq 0$ *for all* $\underline{f} \in \underline{\mathscr{S}}^+$.

Here, $\langle \underline{f}, \underline{f} \rangle$ *is defined by*

$$\sum_{i,j\in B}\sum_{n,m}\int_{M_{n+m}} G_{i^*j}(\theta(z_1),\ldots,\theta(z_n),w_1,\ldots,w_m)f_i(z)^* f_j(w)d^nzd^mw.$$

In the Example 9.1 for $*1 = 1$ the two conditions of Axiom 3 are satisfied.

Lemma 9.2 (Reconstruction of the Hilbert Space). *Axiom 3 yields a positive semi-definite hermitian form* H *on* $\underline{\mathscr{S}}^+$ *and hence the Hilbert space* $\mathbb{H}$ *as the completion of* $\underline{\mathscr{S}}^+ \big/ \ker H$ *with the inner product* $\langle\ ,\ \rangle$.

We now obtain the field operators by using a multiplication in $\underline{\mathscr{S}}^+$ in the same way as in the proof of the Wightman Reconstruction Theorem 8.18. Indeed, Φ_j for $j \in B_0$ shall be defined on the space $\mathscr{S}^+ = \mathscr{S}_1^+$ of distributions with values in a space of operators on $\mathbb{H}$. Given $f \in \mathscr{S}_1^+$ and $\underline{g} \in \underline{\mathscr{S}}^+$, $\underline{g} = (g_i)_{i\in B}$, we define $\Phi_j(f)([\underline{g}])$ to be the equivalence class (with respect to $\ker H$) of $\underline{g} \times f$ (the expected value of Φ_j at f), with

$$\underline{g} \times f = ((\underline{g} \times f)_{i_1\ldots i_{n+1}})_{i_1\ldots i_{n+1}\in B},$$

where

$$(\underline{g} \times f)_{i_1\ldots i_{n+1}}(z_1,\ldots,z_{n+1}) := g_{i_1\ldots i_n}(z_1,\ldots,z_n)f(z_{n+1})\delta_{ji_{n+1}}.$$

It can be shown (cf. [OS73], [OS75]) that this construction yields a unitary representation U of the group E of Euclidean motions of the plane in $\mathbb{H}$. Moreover, there exists a dense subspace $D \subset \mathbb{H}$ left invariant by the unitary representation such that the maps $\Phi_j(f) : [\underline{g}] \mapsto [\underline{g} \times f]$ are defined on D for all $j \in B_0$ and $\Phi_j(f)(D) \subset D$. In addition, with the vacuum $\Omega \in \mathbb{H}$ (namely $\Omega = [\underline{f}]$, with $f_\emptyset = 1$ and $f_i = 0$ for $i \neq \emptyset$) the following properties are satisfied:

Theorem 9.3. (Reconstruction of the Field Operators)

1. *For all* $j \in B_0$ *the mapping* $\Phi_j : \mathscr{S}^+ \to \mathrm{End}(D)$ *is linear, and* Φ_j *is a field operator. Moreover,* $\Phi_j(D) \subset D$, $\Omega \in D$, *and the unitary representation* U *leaves* Ω *invariant.*
2. *The fields* Φ_j *transform covariantly with respect to the representation* U:

$$U(w)\Phi_j(z)U(w)^* = \left(\frac{\partial w}{\partial z}\right)^{h_i} \Phi_j(w(z)).$$

3. *The matrix coefficients* $\langle \Omega | \Phi_i(f) | \Omega \rangle$ *can be represented by analytic functions and for* $\mathrm{Re}\, z_n > \ldots > \mathrm{Re}\, z_1 > 0$ *the correlation functions agree with the given functions*
$$\langle \Omega | \Phi_{i_1}(z_1) \ldots \Phi_{i_n}(z_n) | \Omega \rangle = G_{i_1 \ldots i_n}(z_1, \ldots, z_n).$$

Furthermore, if the dependence on z and $\bar{z}$ is taken into account the corresponding correlation functions $G_{i_1 \ldots i_n}(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n)$ are holomorphic in $M_n^> \times M_n^>$, where

$$M_n^> := \{z \in M_n^+ : \mathrm{Re}\, z_n > \ldots > \mathrm{Re}\, z_1 > 0\}.$$

They can be analytically continued into a larger domain $N \subset \mathbb{C}^n \times \mathbb{C}^n$. A general description of the largest domain (the domain of holomorphy for the $G_{i_1 \ldots i_n}$) is not known.

Similar results are true for other regions in $\mathbb{C}$ instead of the right half plane

$$\{w \in \mathbb{C} : \mathrm{Re}\, w > 0\},$$

e.g., for the disc (radial quantization). In this case the points $z \in \mathbb{C}$ are parameterized as $z = e^{\tau + i\alpha}$ with the time variable τ and the space variable α, which is cyclic. The time order becomes $|z_n| > \ldots > |z_1|$.

The Axioms 1–3 describe essentially a general two-dimensional Euclidian field theory as in Sect. 8.6 where no conformal invariance is required.

9.2 Conformal Fields and the Energy–Momentum Tensor

A two-dimensional quantum field theory with field operators

$$(\Phi_i)_{i \in B_0},$$

satisfying Axioms 1–3, is a *conformal field theory* if the following conditions hold:

- the theory is covariant with respect to dilatations (Axiom 4),
- it has a divergence-free energy–momentum tensor (Axiom 5), and
- it has an associative operator product expansion for the primary fields (Axiom 6).

Axiom 4 (Scaling Covariance) *The correlation functions*

$$G_i, i \in B,$$

satisfy (34) also for the dilatations $w(z) = e^\tau z$, $\tau \in \mathbb{R}$. *Hence*

$$G_i(z_1, \ldots, z_n) = (e^\tau)^{h_1 + \ldots + h_n + \bar{h}_1 + \ldots + \bar{h}_n} G_i(e^\tau z_1, \ldots, e^\tau z_n)$$

for $(z_1, \ldots, z_n) \in M$, $i = (i_1, \ldots, i_n)$ *and* $h_j = h_{i_j}$.

The correlation functions in the Example 9.1 are scaling covariant.

Lemma 9.4. *In a quantum field theory satisfying Axioms 1–4, any 2-point function G_{ij} has the form*

$$G_{ij}(z_1,z_2) = C_{ij} z_{12}^{-(h_i+h_j)} \overline{z}_{12}^{-(\overline{h}_i+\overline{h}_j)} \quad (z_{12} = z_1 - z_2)$$

with a suitable constant $C_{ij} \in \mathbb{C}$. Hence, for $i = j$,

$$G_{ii}(z_1,z_2) = C_{ii} z_{12}^{-2h} \overline{z}_{12}^{-2\overline{h}}.$$

Similarly, any 3-point function G_{ijk} is a constant multiple of the function G in (9.1):

$$G_{ijk} = C_{ijk} G, \text{ with } C_{ijk} \in \mathbb{C}.$$

In particular, the 2- and 3-point functions are completely determined by the constants C_{ij}, C_{ijk}.

Proof. As a consequence of the covariance with respect to translations, $G := G_{ij}$ depends only on $z_{12} = z_1 - z_2$, that is $G(z_1,z_2) = G_{ij}(z_1 - z_2, 0)$. For $z = re^{i\alpha} = e^{\tau} e^{i\alpha}$ one has $G(z,0) = G(e^{\tau+i\alpha}1,0)$. From Axioms 2 and 4 it follows

$$G(1,0) = (e^{\tau+i\alpha})^{h_i} (e^{\tau-i\alpha})^{\overline{h}_i} (e^{\tau+i\alpha})^{h_j} (e^{\tau-i\alpha})^{\overline{h}_j} G(e^{\tau+i\alpha}1,0).$$

This implies $G(z,0) = z^{-(h_i+h_j)} \overline{z}^{-(\overline{h}_i+\overline{h}_j)} G(1,0)$, $C := G(1,0)$.
A similar consideration leads to the assertion on 3-point functions. □

The 4-point functions are less restricted, but they have a specific form for all theories satisfying Axioms 1–3 where the correlation functions are Möbius covariant. To show this, one can use the following differential equations:

Proposition 9.5 (Conformal Ward Identities). *Under the assumption that the correlation function $G = G_{i_1 \dots i_n}(z_1, \dots, z_n)$ satisfies the covariance condition (9.1) for all Möbius transformations the following Ward identities hold:*

$$0 = \sum_{j=1}^{n} \partial_{z_j} G(z_1, \dots, z_n),$$

$$0 = \sum_{j=1}^{n} (z_j \partial_{z_j} + h_j) G(z_1, \dots, z_n),$$

$$0 = \sum_{j=1}^{n} (z_j^2 \partial_{z_j} + 2h_j z_j) G(z_1, \dots, z_n)$$

Proof. These identities are shown in the same way as Lemma 9.4. We focus on the third identity. The Möbius covariance applied to the conformal transformation

$$w = w(z) = \frac{z}{1 - \zeta z}$$

with a complex parameter ζ yields

$$G(z_1,\dots,z_n) = \prod_{i=1}^{n}\left(\frac{1}{1-\zeta z_i}\right)^{2h_i} G(w_1,\dots,w_n)$$

because of

$$\frac{\partial w}{\partial z} = \frac{1}{(1-\zeta z)^2},$$

where $w_j = w(z_j)$. The derivative of this equality with respect to ζ is

$$0 = \prod_{i=1}^{n}\left(\frac{1}{1-\zeta z_i}\right)^{2h_i} \sum_{j=1}^{n} 2h_j \frac{1}{1-\zeta z_j} z_j G(w_1,\dots,w_n)$$
$$+ \prod_{i=1}^{n}\left(\frac{1}{1-\zeta z_i}\right)^{2h_i} \sum_{j=1}^{n} \frac{z_j^2}{(1-\zeta z_j)^2} \partial_{z_j} G(w_1,\dots,w_n),$$

from which the identity follows by setting $\zeta = 0$. □

It can be seen that the solutions of these differential equations in the case of $n = 4$ are of the following form:

$$G(z_1,z_2,z_3,z_4) = F(r(z),\overline{r(z)}) \prod_{i<j} z_{ij}^{-(h_i+h_j)+\frac{1}{3}h} \prod_{i<j} \bar{z}_{ij}^{-(\bar{h}_i+\bar{h}_j)+\frac{1}{3}\bar{h}},$$

where $h = h_1 + h_2 + h_3 + h_4$ and correspondingly for $\bar{h}$, and where F is a holomorphic function in the cross-ratio

$$r(z) := (z_{12}z_{34})/(z_{13}z_{24})$$

of the $z_{12}, z_{34}, z_{13}, z_{24}$ and in $\overline{r(z)}$.

Analogous statements hold for the n-point functions, $n \geq 5$. As an essential feature of conformal field theory we observe that the form of the n-point functions can be determined by using the global conformal symmetry. They turn out to be Laurent monomials in the $z_{ij}, \bar{z}_{ij}$ up to a factor similar to F.

Axiom 5 (Existence of the Energy–Momentum Tensor)
Among the fields $(\Phi_i)_{i\in B_0}$ there are four fields $T_{\mu\nu}$, $\mu,\nu \in \{0,1\}$, with the following properties:

- $T_{\mu\nu} = T_{\nu\mu}$, $T_{\mu\nu}(z)^* = T_{\nu\mu}(\theta(z))$,
- $\partial_0 T_{\mu 0} + \partial_1 T_{\mu 1} = 0$ *with* $\partial_0 := \frac{\partial}{\partial t}$, $\partial_1 := \frac{\partial}{\partial y}$,
- $d(T_{\mu\nu}) = h_{\mu\nu} + \bar{h}_{\mu\nu} = 2$, $s(T_{00} - T_{11} \pm 2iT_{01}) = \pm 2$.

Theorem 9.6 (Lüscher–Mack). [LM76] *The Axioms 1–5 imply*

- $\mathrm{tr}(T_{\mu\nu}) = T^{\mu}_{\mu} = T_{00} + T_{11} = 0$.
  Therefore, $T := T_{00} - iT_{01} = \frac{1}{2}(T_{00} - T_{11} - 2iT_{01})$ *is independent of* $\bar{z}$*, that is* $\overline{\partial} T = 0$*. Hence,* T *is holomorphic . In the same way* $\overline{T} := T_{00} + iT_{01}$ *is independent of* z*, and therefore antiholomorphic. For the corresponding conformal weights we have* $h(T) = \overline{h}(\overline{T}) = 2$ *and* $\overline{h}(T) = h(\overline{T}) = 0$.
- *By*
$$L_{-n} := \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{T(\zeta)}{\zeta^{n+1}} d\zeta, \quad \overline{L}_{-n} := \frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{\overline{T}(\zeta)}{\zeta^{n+1}} d\zeta \tag{9.3}$$
  the operators $L_n, \overline{L}_n$ *on* $D \subset \mathbb{H}$ *are defined, which satisfy the commutation relations of two commuting Virasoro algebras with the same central charge* $c \in \mathbb{C}$*:*
$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m},\\
[\overline{L_n}, \overline{L_m}] &= (n-m)\overline{L_{n+m}} + \frac{c}{12}n(n^2-1)\delta_{n+m},\\
[L_n, \overline{L}_m] &= 0.
\end{aligned}$$
- *The representations of the Virasoro algebra defined by* L_n *and* $\overline{L}_n$*, respectively, are unitary:* $L_n{}^* = L_{-n}$ *and* $\overline{L}_n{}^* = \overline{L}_{-n}$.

Incidentally, the proof given in [LM76] is based on the Minkowski signature.

The L_n, $\overline{L}_n$ can be interpreted as Fourier coefficients of T, $\overline{T}$, since

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-(n+2)}, \quad \overline{T}(z) = \sum_{n\in\mathbb{Z}} \overline{L}_n \bar{z}^{-(n+2)}. \tag{9.4}$$

This is how conformal symmetry in the sense of the representation theory of the Virasoro algebra (cf. Sect. 6) appears in the axiomatic presentation of conformal field theory. The operators L_n, $\overline{L}_n$ define a unitary representation of $\mathsf{Vir} \times \overline{\mathsf{Vir}}$. In general, this representation decomposes into unitary highest-weight representations as follows:

$$\bigoplus W(c,h) \otimes W(c,\overline{h}),$$

where one has to sum over a suitable collection of central charges c and conformal weights $h, \overline{h}$. The theory is called *minimal*, if this sum is finite.

An important tool in conformal field theory is the operator product expansion of two operators A and B of the form $A = \Phi(z_1)$ and $B = \Psi(z_2)$, where Φ, Ψ are field operators. Before we treat operator product expansions in the next section (and also in the next chapter on vertex algebras) let us briefly note that in the case of $\Phi = \Psi = T$ the product $T(z_1)T(z_2)$ has the operator product expansion

$$T(z_1)T(z_2) \sim \frac{c}{2}\frac{1}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{dT}{dz_2}(z_2)\frac{1}{(z_1-z_2)}. \tag{9.5}$$

The symbol "$\sim$" signifies asymptotic expansion, that is "$=$" modulo a regular function $R(z_1, z_2)$.

The validity of (9.5) turns out to be equivalent to the commutation relations of the L_n, $\overline{L}_n$ (see also Theorem 9.6 and the formula (10.2) in Sect. 10.2).

9.3 Primary Fields, Operator Product Expansion, and Fusion

The primary fields are distinguished by the property that their correlation functions have the covariance property as in Axiom 2 for arbitrary local (that is defined on open subsets of $\mathbb{C}$) holomorphic transformations $w = w(z)$ as well. This covariance expresses the full conformal symmetry. However, the covariance property (9.1) for general w only holds "infinitesimally". This infinitesimal version of (9.1) leads to the following concept of a primary field.

Definition 9.7 (Primary Field). A conformal field Φ_i, $i \in B_0$, is called a *primary field* if

$$[L_n, \Phi_i(z)] = z^{n+1}\partial\Phi_i(z) + h_i(n+1)z^n\Phi_i(z) \tag{9.6}$$

for all $n \in \mathbb{Z}$, where $\partial = \frac{\partial}{\partial z}$ (and correspondingly for the $\overline{z}$-dependence, which we shall not consider in the following).

The primary field property can be characterized in the following way: the primary fields are precisely those field operators $\Phi_i, i \in B_0$, which have the following *operator product expansion* (OPE) with the energy–momentum tensor T (cf. Corollary 10.43):

$$T(z_1)\Phi_i(z_2) \sim \frac{h_i}{(z_1 - z_2)^2}\Phi_i(z_2) + \frac{1}{z_1 - z_2}\frac{\partial}{\partial z_2}\Phi_i(z_2). \tag{9.7}$$

(Note that this condition and other formulas used in physics as well as several calculations and formal manipulations become clearer within the formalism of vertex algebras which we introduce in the next chapter.)

The invariance required by (9.6) can also be interpreted as a formal infinitesimal version of (9.1) in Axiom 2 for the transformation $w = w(z) = z + z^{n+1}$. Assume that there would exist a Virasoro group, that is Lie group for Vir with a reasonable exponential map (which is not the case, cf. Sect. 5.4), and assume that we would have a corresponding unitary representation of this symmetry group (or of a central extension of $\mathrm{Diff}_+(\mathbb{S})$ according to Chap. 3) denoted by U. This would imply the formal identity

$$U(e^{tL_n})\Phi_i(z)U(e^{-tL_n}) = \left(\frac{dw_t}{dz}\right)^{h_i}\Phi_i(w_t(z)) \tag{9.8}$$

for $w_t(z) = z + tz^{n+1}$ (here we take $L_n = -(z^{n+1})\frac{d}{dz}$, cf. Sect. 5.2). Since U is unitary, the globalized formal analogue of (9.8) for holomorphic transformations leads to (9.1) for w_t:

$$G_i(z) = \left(\frac{dw_t}{dz}\right)^{h_i} G_i(w_t(z)).$$

Applying $\frac{d}{dt}\Big|_{t=0}$ to the equation (9.8) we obtain

$$[L_n, \Phi_i(z)]$$

on the left-hand side and

$$\begin{aligned}&\frac{d}{dt}(1+t(n+1)z^n)^{h_i}\Phi_i(z)\Big|_{t=0} + \frac{d}{dt}\Phi_i(w_t(z))\Big|_{t=0}\\ &= h_i(n+1)z^n\Phi_i(z) + z^{n+1}\frac{\partial}{\partial z}\Phi_i(z)\end{aligned}$$

on the right-hand side. This discussion motivates the notion of a primary field, and in particular (9.6).

The correlation functions of primary fields satisfy more than the three identities in Proposition 9.5.

Proposition 9.8 (Conformal Ward Identities). *For every correlation function $G = G_{i_1\dots i_n}(z_1,\dots,z_n)$ where all the fields Φ_{i_j} are primary the Ward identities*

$$0 = \sum_{j=1}^{n}(z_j^{m+1}\partial_{z_j} + (m+1)h_j z_j^m)G(z_1,\dots,z_n)$$

are satisfied for all $m \in \mathbb{Z}$.

To show these identities one proceeds as in the proof of Proposition 9.5, but with the conformal transformation $w(z) = z + \zeta z^{m+1}$.

The energy–momentum tensor T is not a primary field, as one can see by comparing the expansions (9.5) and (9.7), except for the special case of $c = 0$ and $h = 2$. The deviation from T being primary can be described by the Schwarzian derivative.

From a more geometrical point of view, a primary field with $h = 1, \overline{h} = 0$ or better its matrix coefficient $G_i = \langle \Omega, \Phi_i \Omega \rangle$ corresponds to a meromorphic differential form. In general, it has the transformation property of a quantity like

$$G_i(z,\overline{z})(dz)^h(d\overline{z})^{\overline{h}} = G_i(w,\overline{w})(dw)^h(d\overline{w})^{\overline{h}},$$

where $w = w(z)$ is a local conformal transformation. In geometric terms such a G_i could be understood as a meromorphic section in the vector bundle $K^h \otimes \overline{K}^{\overline{h}}$ where K is the canonical bundle of the respective Riemann surface.

Let $\Phi_i = \Phi$ be a primary field of conformal weight $h_i = h$ and assume that the asymptotic state $v = \lim_{z\to 0}\Phi(z)\Omega$ exists as a vector in the Hilbert space $\mathbb{H}$ of states (v is often denoted by $|h\rangle$).

We have $[L_0, \Phi(z)]\Omega = L_0\Phi(z)\Omega$ and $[L_0, \Phi(z)]\Omega = z\partial\Phi(z)\Omega + h\Phi(z)\Omega$. Therefore v is an eigenvector of L_0 with eigenvalue h. Moreover, for $n > 0$ we deduce in the same way $L_n v = 0$ by using $L_n\Phi(z)\Omega = [L_n, \Phi(z)]\Omega = z^{n+1}\partial\Phi(z)\Omega + h(n+1)z^n\Phi\Omega$. Consequently,

$$L_0 v = hv, L_n v = 0, n > 0.$$

According to our exposition on Virasoro modules in Chapt. 6 we come to the following conclusion:

Remark 9.9. The asymptotic state $v = \lim_{z\to 0}\Phi(z)\Omega$ of a primary field defines a Virasoro module

$$\{L_{-n_1}\dots L_{-n_k}v : n \geq 0, k \in \mathbb{N}\} \subset \mathbb{H}$$

with highest-weight vector v.

The states $L_{-n_1}\dots L_{-n_k}v$ can be viewed as excited states of the ground state and they are called *descendants* of v.

It is in general required that the collection of all descendants of the asymptotic states belonging to the primary fields has a dense span in the Hilbert space $\mathbb{H}$ of states. In this case, we obtain a decomposition of $\mathbb{H}$ into Virasoro modules as described above but more concretely given by the primary fields.

Definition 9.10. In a quantum field theory satisfying Axioms 1–5 let

$$B_1 := \{i \in B_0 : \Phi_i \text{ is a primary field}\}.$$

The associated *conformal family* $[\Phi_i]$ for $i \in B_1$ is the complex vector space generated by

$$\Phi_i^\alpha(z) := L_{-\alpha_1}(z)\dots L_{-\alpha_N}(z)\Phi_i(z) \tag{9.9}$$

for $\alpha = (\alpha_1,\dots,\alpha_N) \in \mathbb{N}^N$, $\alpha_1 \geq \dots \geq \alpha_N > 0$, where

$$L_{-n}(z) := \frac{1}{2\pi i}\oint \frac{T(\zeta)}{(\zeta - z)^{n+1}}d\zeta$$

for $z \in \mathbb{C}$. The operators $\Phi_i^\alpha(z)$ are called *secondary fields* or *descendants*.

The operators $L_{-n}(z)$ are in close connection with the Virasoro generators L_n because of

$$L_{-n} = \frac{1}{2\pi i}\oint \frac{T(\zeta)}{\zeta^{n+1}}d\zeta = L_{-n}(0)$$

(cf. Theorem 9.6). The secondary fields Φ_i^α can be expressed as integrals as well. For instance, for Φ_i^k, $k \in \mathbb{N}$,

$$\Phi_i^k(z) = L_{-k}(z)\Phi_i(z) = \frac{1}{2\pi i}\oint \frac{T(\zeta)}{(\zeta - z)^{k+1}}\Phi_i(z)d\zeta.$$

Moreover, the correlation functions of the secondary fields can be determined in terms of correlation functions of primary fields by means of certain specific linear differential equations. It therefore suffices for many purposes to know the correlation functions of the primary fields and in particular the constants C_{ijk} for $i,j,k \in B_1$.

For any fixed $z \in \mathbb{C}$ the conformal family $[\Phi_i]$ of a given primary field Φ_i defines a highest-weight representation with weight (c_i,h_i) (cf. Sect. 6) in a natural manner. $v := \Phi_i(z)$ is the highest-weight vector, $L_0(v) = h_i v$, $L_n(v) := 0$ for $n \in \mathbb{N}$, and $L_{-n}(v) := \Phi_i^n(z)$ for $n \in \mathbb{N}$.

Remark 9.11 (State Field Correspondence). Assume that the asymptotic states of the primary fields together with their descendants generate a dense subspace V of $\mathbb{H}$. Then to each state $a \in V$ there corresponds a field Φ such that $\lim_{z\to 0}\Phi(z)\Omega = a$.

To show this property we only have to observe that for a descendant state of the form $w = L_{-\alpha_1}\dots L_{-\alpha_N}\Phi_i(0)\Omega$ with respect to a primary field Φ_i one has

$$w = \lim_{z\to 0}\Phi_i^{\alpha}(z)\Omega = \lim_{z\to 0} L_{-\alpha_1}(z)\dots L_{-\alpha_N}(z)\Phi_i(z)\Omega.$$

Of course, the remark does not assert that a field corresponding to a state is already of the form Φ_i with $i \in B_0$. It rather means that there is always a suitable field among the descendants of the primary fields.

Note that the state field correspondence is one of the basic requirements in the definition of vertex algebras (see Sect. 10.4). If we denote the field $\Phi(z)$ in the last remark by $Y(a,z)$ we are close to a vertex algebra, where $Y(a,z)$ is supposed to be a formal series with coefficients in $\operatorname{End} V$.

Operator Product Expansion. For the primary fields of a conformal field theory it is postulated (according to the fundamental article of Belavin, Polyakov, and Zamolodchikov [BPZ84]) that they obey the following operator product expansion (OPE)

$$\Phi_i(z_1)\Phi_j(z_2) \sim \sum_{k\in B_0} C_{ijk}(z_1 - z_2)^{h_k - h_i - h_j}\Phi_k(z_2) \tag{9.10}$$

with the constants C_{ijk} that occur already in the expression (9.2) of the 3-point functions (cf. Lemma 9.4). Similar expansions hold for the descendants.

The **central object of conformal field theory** is the determination of

- the scaling dimensions $d_i = h_i + \overline{h}_i$,
- the *central charge* c_i for the family $[\Phi_i]$, and
- the coefficients C_{ijk} (structure constants)

from the operator product expansion (9.10) using the conformal symmetry. When all these constants are calculated one has a complete solution.

Proposition 9.12 (Bootstrap Hypothesis). *This can be achieved if the OPE (9.10) is required in addition to be associative. (See also Axiom 6 below.)*

Some comments are due concerning the use of terms like "operator product" and its "associativity". First of all, the expansion (9.10) can only be valid for the corresponding matrix coefficients or better for the vacuum expectation values. In particular, we do not have an algebra of operators with a nice expansion of the product. Therefore the associativity constraint does not refer to the associativity of a true multiplication in a ring as the term suggests from the mathematical viewpoint, but simply means that the respective behavior of the expansions of the product of three or more primary fields is independent of the order the expansions are executed. And this equality concerns again only the vacuum expectation values and it is restricted to the singular terms in the expansions.

Note that in the language of vertex algebras the "associativity" constraint has a nice and clear formulation, cf. Theorem 10.36. Furthermore, the associativity is a consequence of the basic properties of a vertex algebra and not an additional postulate.

In any case, the associativity of the OPE (9.10) in this sense is strong enough to determine all generic 4-point functions

$$G_{i_1 i_2 i_3 i_4}(z_1,z_2,z_3,z_4,\bar{z}_1,\bar{z}_2,\bar{z}_3,\bar{z}_4),(i_1,i_2,i_3,i_4) \in B_1^4.$$

This can be done by using the associativity of the OPE to obtain several expansions of $G_{i_1 i_2 i_3 i_4}$ differing by the order in which we expand. For instance, one can first expand with respect to the indices i_1, i_2 and i_3, i_4 and then expand the resulting two expansions to obtain a series $\sum_m \alpha_m G_m$ or one expands first with respect to the indices i_1, i_4 and i_2, i_3 (here we need locality) and then expand the resulting expansions to obtain another series $\sum_m \beta_m G_m$. Associativity means that the resulting two expansions are the same. This gives infinitely many equations for the structure constants C_{ijk} of the 3-pointfunctions and allows in turn to determine $G_{i_1 i_2 i_3 i_4}$.

We know already that such a function depends only on the cross-ratios $r(z) := (z_{12}z_{34})/(z_{13}z_{24})$ and $\overline{r(z)}$ (see p. 161). Since these ratios are invariant under global conformal transformations on the extended plane we can set $z_1 = \infty, z_2 = 1, z_3 = z$, and $z_4 = 0$. The above correlation function reduces under this change of coordinates to

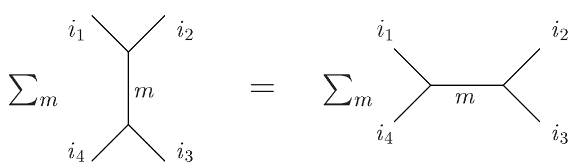
$$G(z,\bar{z}) = \lim_{z_1,\bar{z}_1 \to \infty} G_{i_1 i_2 i_3 i_4}(z_1,1,z,0,\bar{z}_1,1,\bar{z},0).$$

The associativity of the OPE (9.10) allows to represent G with the aid of so-called (holomorphic and antiholomorphic, respectively) "conformal blocks" $\mathscr{F}^r$, $\overline{\mathscr{F}}^s$:

$$G(z,\bar{z}) = \sum_{k \in B_1} C_{i_1 i_2 k} C_{i_3 i_4 k} \mathscr{F}^k(z) \overline{\mathscr{F}}^k(\bar{z}),$$

where the $C_{i_1 i_2 k}, C_{i_3 i_4 k} \in \mathbb{C}$ are the coefficients of the 3-point functions in Lemma 9.4.

The associativity can be indicated schematically in diagrammatic language:



The diagram has a physical interpretation as *crossing symmetry*.

Note that there is an additional way applying the associativity of the OPE in case of the 4-point function leading to another diagram and two further equalities.

A conformal field theory can also be defined on arbitrary Riemann surfaces instead of $\mathbb{C}$. Then the $\mathscr{F}^r$, $\overline{\mathscr{F}}^s$ depend only on the complex structure of the surface. Finally, they can be considered as holomorphic sections on the appropriate moduli spaces with values in suitable line bundles (cf. [FS87], [TUY89], [KNR94], [Uen95], [Sor95], [Bea95], [Tyu03*] and Chap. 11).

In any case a conformal field theory has to satisfy – in addition to the Axioms 1–5 – the following axiom:

Axiom 6 (Operator Product Expansion) *The primary fields have the OPE (9.10). This OPE is associative.*

Concluding Remarks:

1. All n-point functions of the primary fields can be derived from the G_i for $i \in B_1^4$.
2. The expansions (9.10) are the fusion rules, which can be written formally as

$$[\Phi_i] \times [\Phi_j] = \sum_{l \in B_1} [\Phi_l],$$

   or, carrying more information, as

$$\Phi_i \times \Phi_j = \sum_l N_{ij}^l \Phi_l,$$

   where $N_{ij}^l \in \mathbb{N}_0$ is the number of occurrences of elements of the family $[\Phi_l]$ in the OPE of $\Phi_i(z)\Phi_j(0)$. The coefficients N_{ij}^k define the structure of a fusion ring, cf. Sect. 11.4.
3. We have sometimes passed over to radial quantization, e.g., by using Cauchy integrals in Sect. 9.2, for instance

$$L_{-n}(z) = \frac{1}{2\pi i} \oint \frac{T(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

4. To construct interesting examples of conformal field theories satisfying Axioms 1–6 it is reasonable to begin with string theory. On a more algebraic level this amounts to study Kac–Moody algebras (cf. pp. 65 and 196). This subject is surveyed, e.g., in [Uen95] where an interesting connection with the presentation of conformal blocks as sections in certain holomorphic vector bundles is described (cf. also [TUY89] or [BF01*]). For other examples, see [FFK89].

9.4 Other Approaches to Axiomatization

In order to lay down the foundations of conformal field theory introduced in [BPZ84], Moore and Seiberg proposed the following axioms for a conformal field theory in [MS89]:

A conformal field theory is a Virasoro module

$$V = \bigoplus_{i \in B_1} W(c_i, h_i) \otimes W(\overline{c}_i, \overline{h}_i)$$

with unitary highest-weight modules $W(c_i, h_i)$, $W(\overline{c}_i, \overline{h}_i)$ (cf. Sect. 6), subject to the following axioms:

P 1. There is a uniquely determined *vacuum vector* $\Omega = |0\rangle \in V$ with $\Omega \in W(c_{i_0}, h_{i_0}) \otimes W(\overline{c}_{i_0}, \overline{h}_{i_0})$, $h_{i_0} = \overline{h}_{i_0} = 0$. Ω is $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$-invariant.

P 2. To each vector $\alpha \in V$ there corresponds a field Φ_α, i.e. an operator $\Phi_\alpha(z)$ on V, $z \in \mathbb{C}$. Moreover, there exists a conjugate $\Phi_{\alpha'}$ such that the OPE of $\Phi_\alpha \Phi_{\alpha'}$ contains a descendant of the unit operator.

P 3. The highest-weight vectors $\alpha = i = v_i$ of $W(c_i, h_i)$ determine primary fields Φ_i. Similarly for the highest-weight vectors of $W(\overline{c}_i, \overline{h}_i)$.

P 4. $G_i(z) = \langle \Omega | \Phi_{i_1}(z_1) \ldots \Phi_{i_n}(z_n) | \Omega \rangle$, $|z_1| > \ldots > |z_n|$, always has an analytical continuation to M_n.

P 5. The correlation functions and the one-loop partition functions are modular invariant (cf. [MS89]).

Another axiomatic description of conformal field theory was proposed by Segal in [Seg91], [Seg88b], [Seg88a]. The basic object in this ansatz is the set of equivalence classes of Riemann surfaces with boundaries, which becomes a semi-group by defining the product of two such Riemann surfaces by a suitable fusion or sewing (cf. Sect. 6.5).

Friedan and Shenker introduced in [FS87] a different, interesting system of axioms, which also uses the collection of all Riemann surfaces as a starting point.

All these approaches can be formulated in the language of vertex algebras which seems to be the right theory to describe conformal field theory. In the next chapter we present a short introduction to vertex algebras and their relation to conformal field theory.

Along these lines, the course of V. Kac [Kac98*] describes the structure of conformal field theories as well as the book of E. Frenkel and D. Ben-Zvi [BF01*]. A more general point of view is taken by Beilinson and Drinfeld in their work on chiral algebras [BD04*] where the theory of vertex algebras turns out to be a special case of a much wider theory of chiral algebras.

A comprehensive account of different developments in conformal field theory is collected in the Princeton notes on strings and quantum field theory of Deligne and others [Del99*].

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