Part II
First Steps Toward Conformal Field Theory

The term “conformal field theory” stands for a variety of different formulations and slightly different structures. The aim of the second part of these notes is to describe some of these formulations and structures and thereby contribute to answering the question of what conformal field theory is.

Conformal field theories are best described either by the way they appear and are constructed or by properties and axioms which provide classes of conformal field theories. The most common theories by examples are

- free bosons or fermions (σ-models on a torus),
- WZW-models[1] for compact Lie groups and gauged WZW-models,
- coset and orbifold constructions of WZW-models.

Systematic descriptions of conformal field theory emphasizing the fundamental structures and properties comprise

- various combinatorical approaches like the axioms of Moore–Seiberg [MS89], Friedan–Shenker [FS87], or Segal [Seg88a].
- the Osterwalder–Schrader axioms with conformal invariance [FFK89],
- the vertex algebras or chiral algebras [BD04*] as their generalizations,

A common feature and essential point of all these approaches to conformal field theory is the appearance of representations of the Virasoro algebra which play a central role. The simple reason for this major role of the Virasoro is based on the fact that the elements of the Virasoro algebra are symmetries of the quantum system and these elements are regarded as the most important observables in conformal field theory. In this context the generators L_n can be compared in their physical significance to the momentum or angular momentum in conventional one-particle quantum mechanics.

Since the Witt algebra W is a generating subalgebra of the infinitesimal classical conformal transformations of the Minkowski plane in each of the two light cone variables (cf. Corollary 2.15 and Sect. 5.1), the set of all observables of conformal field theory contains the direct product $\mathsf{Vir} \times \overline{\mathsf{Vir}}$ of two copies of the Virasoro algebra. (Note that after quantization, the Witt algebra has to be replaced by its nontrivial central extension, the Virasoro algebra Vir, cf. Chaps. 3 and 4.) In general, one assumes the full set $\mathscr{A}_{\text{tot}}$ of observables to form an algebra which decomposes into a direct product of algebras $\mathscr{A} \times \mathscr{A}'$ containing the Virasoro algebras $\mathsf{Vir} \subset \mathscr{A}, \overline{\mathsf{Vir}} \subset \mathscr{A}'$. The two components of the full algebra of observables are called chiral halves or holomorphic/antiholomorphic or similar.

As a consequence of the product structure, for many purposes one can restrict the investigations to one “chiral half” of the theory in such a way that only $\mathsf{Vir} \subset \mathscr{A}$ resp. $\overline{\mathsf{Vir}} \subset \mathscr{A}'$ is studied. The restrictions to one chiral half requires among other things to regard the light cone variables t_+ and t_- as completely independent variables, and, in the same way, the complex variables z and $\overline{z}$ as completely independent. The identification of $\overline{z}$ with the complex conjugate only takes place when the two chiral halves of the conformal field theory are combined.

[1] WZW = Wess–Zumino–Witten

Restricting now to one chiral half $\mathscr{A}$ and, furthermore, restricting to the subalgebra Vir we are led, first of all, to study the representations of the Virasoro algebra.

In a certain way one could claim now that conformal field theory is the representation theory of the Virasoro algebra and of certain algebras (namely chiral algebras) containing the Virasoro algebra. Therefore, in this second part of the notes we first describe the representations of the Virasoro algebra (Chap. 6) and explain as an example how the quantization of strings leads to a representation of the Virasoro algebra (Chap. 7). Next we discuss the axiomatic approach to quantum field theory according to Wightman as well as the Euclidean version according to Osterwalder–Schrader (Chap. 8) and treat the case of two-dimensional conformal field theory in a separate chapter (Chap. 9). In Chap. 10 we connect all these with the theory of vertex algebras, and in Chap. 11 we present as an example of an application of conformal field theory to complex algebraic geometry the Verlinde formula in the context of holomorphic vector bundles and moduli spaces.

Chapter 6
Representation Theory of the Virasoro Algebra

Most of the results in this chapter can be found in [Kac80]. A general treatment of the Virasoro algebra and its significance in geometry and algebra is given in [GR05*].

6.1 Unitary and Highest-Weight Representations

Let V be a vector space over $\mathbb{C}$.

Definition 6.1 (Unitary Representation). A representation $\rho : \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}} V$ (that is a Lie algebra homomorphism ρ) is called *unitary* if there is a positive semi-definite hermitian form $H : V \times V \to \mathbb{C}$, so that for all $v, w \in V$ and $n \in \mathbb{Z}$ one has

$$H(\rho(L_n)v,w) = H(v,\rho(L_{-n})w),$$
$$H(\rho(Z)v,w) = H(v,\rho(Z)w).$$

Note that this notion of a unitary representation differs from that introduced in Definition 3.7 where a unitary representation of a topological group G was defined to be a continuous homomorphism $G \to \mathrm{U}(\mathbb{H})$ into the unitary group of a Hilbert space. This is so, because we do not consider any topological structure in Vir.

One requires that $\rho(L_n)$ is formally adjoint to $\rho(L_{-n})$, to ensure that ρ maps the generators $\frac{d}{d\theta}$, $\cos(n\theta)\frac{d}{d\theta}$, $\sin(n\theta)\frac{d}{d\theta}$ (cf. Chap. 5) of the real Lie algebra $\mathrm{Vect}(\mathbb{S})$ to skew-symmetric operators. Since

$$\frac{d}{d\theta} = iL_0, \quad \cos(n\theta)\frac{d}{d\theta} = -\frac{i}{2}(L_n + L_{-n}), \quad \text{and}$$
$$\sin(n\theta)\frac{d}{d\theta} = -\frac{1}{2}(L_n - L_{-n}),$$

it follows from $H(\rho(L_n)v,w) = H(v,\rho(L_{-n})w)$ that

$$H(\rho(D)v,w) + H(v,\rho(D)w) = 0$$

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for all
$$D \in \left\{ \frac{d}{d\theta}, \cos(n\theta)\frac{d}{d\theta}, \sin(n\theta)\frac{d}{d\theta} \right\}.$$

So, in principle, these unitary representations of Vir can be integrated to projective representations $\mathrm{Diff}_+(\mathbb{S}) \to \mathrm{U}(\mathbb{P}(\mathbb{H}))$ (cf. Sect. 6.5), where $\mathbb{H}$ is the Hilbert space given by (V,H).

Definition 6.2. A vector $v \in V$ is called a *cyclic vector* for a representation $\rho : \mathsf{Vir} \to \mathrm{End}(V)$ if the set

$$\{\rho(X_1)\ldots\rho(X_m)v : X_j \in \mathsf{Vir} \quad \text{for} \quad j = 1,\ldots m, m \in \mathbb{N}\}$$

spans the vector space V.

Definition 6.3. A representation $\rho : \mathsf{Vir} \to \mathrm{End}(V)$ is called a *highest-weight representation* if there are complex numbers $h, c \in \mathbb{C}$ and a cyclic vector $v_0 \in V$, so that

$$\begin{aligned} \rho(Z)v_0 &= cv_0, \\ \rho(L_0)v_0 &= hv_0, \text{and} \\ \rho(L_n)v_0 &= 0 \quad \text{for } n \in \mathbb{Z}, n \geq 1. \end{aligned}$$

The vector v_0 is then called the *highest-weight vector* (or *vacuum vector*) and V is called a *Virasoro module* (via ρ) with *highest weight* (c,h), or simply a *Virasoro module for* (c,h).

Such a representation is also called a *positive energy representation* if $h \geq 0$. The reason of this terminology is the fact that L_0 often has the interpretation of the energy operator which is assumed to be diagonalizable with spectrum bounded from below. With this assumption any representation ρ respecting this property satisfies $\rho(L_n)v_0 = 0$ for all $n \in \mathbb{Z}, n > 0$, if v_0 is an eigenvector of $\rho(L_0)$ with lowest eigenvalue $h \in \mathbb{R}$. This follows from the fact that $w = \rho(L_n)(v_0)$ is an eigenvector of $\rho(L_0)$ with eigenvalue $h - n$ or $w = 0$ as can be seen by using the relation $L_0L_n = L_nL_0 - nL_n$:

$$\rho(L_0)(w) = \rho(L_n)\rho(L_0)v_0 - n\rho(L_n)v_0 = \rho(L_n)(hv_0) - nw = (h-n)w.$$

Now, since h is the lowest eigenvalue of $\rho(L_0)$, w has to vanish for $n > 0$.

The notation often used by physicists is $|h\rangle$ instead of v_0 and $L_n|h\rangle$ instead of $\rho(L_n)v_0$ so that, in particular, $L_0|h\rangle = h|h\rangle$.

6.2 Verma Modules

Definition 6.4. A *Verma module* for $c, h \in \mathbb{C}$ is a complex vector space $M(c,h)$ with a highest-weight representation

$$\rho : \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}}(M(c,h))$$

and a highest-weight vector $v_0 \in M(c,h)$, so that

$$\{\rho(L_{-n_1})\dots\rho(L_{-n_k})v_0 : n_1 \geq \dots \geq n_k > 0\ ,\ k \in \mathbb{N}\} \cup \{v_0\}$$

is a vector space basis of $M(c,h)$.

Every Verma module $M(c,h)$ yields a highest-weight representation with highest weight (c,h). For fixed $c,h \in \mathbb{C}$ the Verma module $M(c,h)$ is unique up to isomorphism. For every Virasoro module V with highest weight (c,h) there is a surjective homomorphism $M(c,h) \to V$, which respects the representation. This holds, since

Lemma 6.5. *For every $h,c \in \mathbb{C}$ there exists a Verma module $M(c,h)$.*

Proof. Let

$$M(c,h) := \mathbb{C}v_0 \oplus \bigoplus \mathbb{C}\{v_{n_1\dots n_k} : n_1 \geq \dots \geq n_k > 0\ ,\ k \in \mathbb{Z},\ k > 0\}$$

be the complex vector space spanned by v_0 and $v_{n_1,\dots,n_k}$, $n_1 \geq \dots \geq n_k > 0$. We define a representation

$$\rho : \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}}(M(c,h))$$

by

$$\begin{aligned}
\rho(Z) &:= c\,\mathrm{id}_{M(c,h)}, \\
\rho(L_n)v_0 &:= 0 \quad \text{for} \quad n \in \mathbb{Z}, n \geq 1, \\
\rho(L_0)v_0 &:= h v_0, \\
\rho(L_0)v_{n_1\dots n_k} &:= \left(\textstyle\sum_{j=1}^{k} n_j + h\right) v_{n_1\dots n_k}, \\
\rho(L_{-n})v_0 &:= v_n \quad \text{for} \quad n \in \mathbb{Z}, n \geq 1, \\
\rho(L_{-n})v_{n_1\dots n_k} &:= v_{n n_1\dots n_k} \quad \text{for} \quad n \geq n_1.
\end{aligned}$$

For all other $v_{n_1\dots n_k}$ with $1 \leq n < n_1$ one obtains $\rho(L_{-n})v_{n_1\dots n_k}$ by permutation, taking into account the commutation relations $[L_n, L_m] = (n-m)L_{n+m}$ for $n \neq m$, e.g., for $n_1 > n \geq n_2$:

$$\begin{aligned}
&\rho(L_{-n})v_{n_1\dots n_k} \\
&= \rho(L_{-n})\rho(L_{-n_1})v_{n_2\dots n_k} \\
&= (\rho(L_{-n_1})\rho(L_{-n}) + (-n+n_1)\rho(L_{-(n+n_1)}))v_{n_2\dots n_k} \\
&= v_{n_1 n n_2\dots n_k} + (n_1 - n)v_{(n_1+n)n_2\dots n_k}.
\end{aligned}$$

So

$$\rho(L_{-n})v_{n_1\dots n_k} := v_{n_1 n n_2\dots n_k} + (n_1 - n)v_{(n_1+n)n_2\dots n_k}.$$

Similarly one defines $\rho(L_n)v_{n_1\dots n_k}$ for $n \in \mathbb{N}$ taking into account the commutation relations, e.g.,

$$\rho(L_n)v_{n_1} := \begin{cases} 0 & \text{for} \quad n > n_1 \\ (2nh + \frac{n}{12}(n^2-1)c)v_0 & \text{for} \quad n = n_1 \\ (n+n_1)v_{n_1-n} & \text{for} \quad 0 < n < n_1. \end{cases}$$

Hence, ρ is well-defined and $\mathbb{C}$-linear. It remains to be shown that ρ is a representation, that is

$$[\rho(L_n), \rho(L_m)] = \rho([L_n, L_m]).$$

For instance, for $n \geq n_1$ we have

$$\begin{aligned} &[\rho(L_0), \rho(L_{-n})]v_{n_1\dots n_k} \\ &= \rho(L_0)v_{nn_1\dots n_k} - \rho(L_{-n})\left(\sum n_j + h\right)v_{n_1\dots n_k} \\ &= \left(\sum n_j + n + h\right)v_{nn_1\dots n_k} - \left(\sum n_j + h\right)v_{nn_1\dots n_k} \\ &= nv_{nn_1\dots n_k} \\ &= n\rho(L_{-n})v_{n_1\dots n_k} \\ &= \rho([L_0, L_{-n}])v_{n_1\dots n_k} \end{aligned}$$

and for $n \geq m \geq n_1$

$$\begin{aligned} &[\rho(L_{-m}), \rho(L_{-n})]v_{n_1\dots n_k} \\ &= \rho(L_{-m})v_{nn_1\dots n_k} - v_{nmn_1\dots n_k} \\ &= v_{nmn_1\dots n_k} + (n-m)v_{(n+m)n_1\dots n_k} - v_{nmn_1\dots n_k} \quad \text{(s.o.)} \\ &= (n-m)v_{(n+m)n_1\dots n_k} \\ &= (n-m)\rho(L_{-(m+n)})v_{n_1\dots n_k} \\ &= \rho([L_{-m}, L_{-n}])v_{n_1\dots n_k}. \end{aligned}$$

The other identities follow along the same lines from the respective definitions. $\square$

$M(c,h)$ can also be described as an induced representation, a concept which is explained in detail in Sect. 10.49. To show this, let

$$B^+ := \mathbb{C}\{L_n : n \in \mathbb{Z}, n \geq 0\} \oplus \mathbb{C}Z.$$

B^+ is a Lie subalgebra of Vir. Let $\sigma : B^+ \to \mathrm{End}_{\mathbb{C}}(\mathbb{C})$ be the one-dimensional representation with $\sigma(Z) := c$, $\sigma(L_0) := h$, and $\sigma(L_n) = 0$ for $n \geq 1$. Then the representation ρ described explicitly above is induced by σ on Vir with representation module

$$\mathrm{U}(\mathsf{Vir}) \otimes_{U(B^+)} \mathbb{C} \cong M(c,h).$$

($\mathrm{U}(\mathfrak{g})$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, see Definition 10.45.)

Remark 6.6. Let V be a Virasoro module for $c, h \in \mathbb{C}$. Then we have the direct sum decomposition $V = \bigoplus_{N \in \mathbb{N}} V_N$, where $V_0 := \mathbb{C}v_0$ and V_N for $N \in \mathbb{N}$ is, $N > 0$, the complex vector space generated by

$$\rho(L_{-n_1})\dots\rho(L_{-n_k})v_0$$
$$\text{with} \quad n_1 \geq \ldots \geq n_k > 0\,,\ \sum_{j=1}^{k} n_j = N\,,\ k \in \mathbb{N},\ k > 0.$$

The V_N are eigenspaces of $\rho(L_0)$ for the eigenvalue $(N+h)$, that is

$$\rho(L_0)\,|_{V_N} = (N+h)\mathrm{id}_{V_N}.$$

This follows from the definition of a Virasoro module and from the commutation relations of the L_m.

Lemma 6.7. *Let V be a Virasoro module for $c,h \in \mathbb{C}$ and U a submodule of V. Then*

$$U = \bigoplus_{N\in\mathbb{N}_0} (V_N \cap U).$$

A submodule of V is an *invariant linear subspace* of V, that is a complex-linear subspace U of V with $\rho(D)U \subset U$ for $D \in \mathsf{Vir}$.

Proof. Let $w = w_0 \oplus \ldots \oplus w_s \in U$, where $w_j \in V_j$ for $j \in \{1,\ldots,s\}$. Then

$$\begin{array}{rcccc} w & = & w_0 & + \ldots + & w_s, \\ \rho(L_0)w & = & hw_0 & + \ldots + & (s+h)w_s, \\ & \vdots & & & \\ \rho(L_0)^{s-1}w & = & h^{s-1}w_0 & + \ldots + & (s+h)^{s-1}w_s. \end{array}$$

This is a system of linear equations for $w_0,\ldots,w_s$ with regular coefficient matrix. Hence, the $w_0,\ldots,w_s$ are linear combinations of the $w,\ldots,\rho(L_0)^{s-1}w \in U$. So $w_j \in V_j \cap U$. □

6.3 The Kac Determinant

We are mainly interested in unitary representations of the Virasoro algebra, since the representations of Vir appearing in conformal field theory shall be unitary. To find a suitable hermitian form on a Verma module $M(c,h)$, we need to define the notion of the expectation value $\langle w\rangle$ of a vector $w \in M(c,h)$: with respect to the decomposition $M(c,h) = \bigoplus V_N$ according to Lemma 6.7, w has a unique component $w' \in V_0$. The expectation value is simply the coefficient $\langle w\rangle \in \mathbb{C}$ of this component w' for the basis $\{v_0\}$, that is $w' = \langle w\rangle v_0$. ($\langle w\rangle$ makes sense for general Virasoro modules as well.)

Let $M = M(c,h)$, $c,h \in \mathbb{R}$, be the Verma module with highest-weight representation $\rho : \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}}(M(c,h))$ and let v_0 be the respective highest-weight vector. Instead of $\rho(L_n)$ we mostly write L_n in the following. We define a hermitian form $H : M \times M \to \mathbb{C}$ on the basis $\{v_{n_1\ldots n_k}\} \cup \{v_0\}$:

$$\begin{aligned} H(v_{n_1\dots n_k}, v_{m_1\dots m_j}) &:= \langle L_{n_k} \dots L_{n_1} v_{m_1\dots m_j} \rangle \\ &= \langle L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_j} v_0 \rangle. \end{aligned}$$

In particular, this definition includes

$$H(v_0, v_0) := 1 \quad \text{and} \quad H(v_0, v_{n_1\dots n_k}) := 0 =: H(v_{n_1\dots n_k}, v_0).$$

The condition $c, h \in \mathbb{R}$ implies $H(v, v') = H(v', v)$ for all basis vectors

$$v, v' \in B := \{v_{n_1\dots n_k} : n_1 \geq \dots \geq n_k > 0\} \cup \{v_0\}.$$

The elementary but lengthy proof of this statement consists in a repeated use of the commutation relations of the L_ns. Now, the map $H : B \times B \to \mathbb{R}$ has an $\mathbb{R}$-bilinear continuation to $M \times M$, which is $\mathbb{C}$-antilinear in the first and $\mathbb{C}$-linear in the second variable:

For $w, w' \in M$ with unique representations $w = \sum \lambda_j w_j$, $w' = \sum \mu_k w'_k$ relative to basis vectors $w_j, w'_k \in B$, one defines

$$H(w, w') := \sum \sum \overline{\lambda}_j \mu_k H(w_j, w'_k).$$

$H : M \times M \to \mathbb{C}$ is a hermitian form. However, it is not positive definite or positive semi-definite in general. Just in order to decide this, the Kac determinant is used. H has the following properties:

Theorem 6.8. *Let $h, c \in \mathbb{R}$ and $M = M(c, h)$.*

1. *$H : M \times M \to \mathbb{C}$ is the unique hermitian form satisfying $H(v_0, v_0) = 1$, as well as $H(L_n v, w) = H(v, L_{-n} w)$ and $H(Zv, w) = H(v, Zw)$ for all $v, w \in M$ and $n \in \mathbb{Z}$.*
2. *$H(v, w) = 0$ for $v \in V_N$, $w \in V_M$ with $N \neq M$, that is the eigenspaces of L_0 are pairwise orthogonal.*
3. $\ker H$ *is the maximal proper submodule of M.*

Proof.

1. That the identity
$$H(L_n v, w) = H(v, L_{-n} w)$$
holds for the hermitian form introduced above can again be seen using the commutation relations. The uniqueness of such a hermitian form follows immediately from
$$H(v_{n_1\dots n_k}, v_{m_1\dots m_j}) = H(v_0, L_{n_k} \dots L_{n_1} v_{m_1\dots m_j}).$$
2. For $n_1 + \dots + n_k > m_1 + \dots + m_j$ the commutation relations of the L_n imply that $L_{n_k} \dots L_{n_1} L_{-m_1} \dots L_{-m_j} v_0$ can be written as a sum $\sum P_l v_0$, where the operator P_l begins with an L_s, $s \in \mathbb{Z}$, $s \geq 1$, that is $P_l = Q_l L_s$. Consequently, $H(v_{n_1\dots n_k}, v_{m_1\dots m_j}) = 0$.
3. $\ker H := \{v \in M : H(w, v) = 0 \ \forall w \in M\}$ is a submodule, because $v \in \ker H$ implies $L_n v \in \ker H$ since $H(w, L_n v) = H(L_{-n} w, v) = 0$. Naturally, $M \neq \ker H$ because $v_0 \notin \ker H$. Let $U \subset M$ be an arbitrary proper submodule. To show $U \subset$

$\ker H$, let $w \in U$. For $n_1 \geq \ldots \geq n_k > 0$ one has $H(v_{n_1 \ldots n_k}, w) = H(v_0, L_{n_k} \ldots L_{n_1} w)$. Assume $H(v_{n_1 \ldots n_k}, w) \neq 0$. Then $\langle L_{n_k} \ldots L_{n_1} w \rangle \neq 0$. By Lemma 6.7 this implies $v_0 \in U$ (because $L_{n_k} \ldots L_{n_1} w \in U$), and also $v_{m_1 \ldots m_j} \in U$, in contradiction to $M \neq U$. Similarly we get $H(v_0, w) = 0$, so $w \in \ker H$. □

Remark 6.9. $M(c,h) \big/ \ker H$ is a Virasoro module with a nondegenerate hermitian form H. However, H is not definite, in general.

Corollary 6.10. *If H is positive semi-definite then $c \geq 0$ and $h \geq 0$.*

Proof. For $n \in \mathbb{N}$, $n > 0$, we have

$$\begin{aligned} H(v_n, v_n) &= H(v_0, L_n L_{-n} v_0) \\ &= H(v_0, \rho([L_n, L_{-n}]) v_0) \\ &= 2nh + \frac{n}{12}(n^2 - 1)c. \end{aligned}$$

$H(v_1, v_1) \geq 0$ implies $h \geq 0$. Then, from $H(v_n, v_n) \geq 0$ we get $2nh + \frac{n}{12}(n^2-1)c \geq 0$ for all $n \in \mathbb{N}$, hence $c \geq 0$. □

Definition 6.11. Let $P(N) := \dim_{\mathbb{C}} V_N$ and $\{b_1, \ldots, b_{P(N)}\}$ be a basis of V_N. We define matrices A^N by $A^N_{ij} := H(b_i, b_j)$ for $i, j \in \{1, \ldots, P(N)\}$.

Obviously, H is positive semi-definite if all these matrices A^N are positive semi-definite. For $N = 0$ and $N = 1$ one has $A^0 = (1)$ and $A^1 = (h)$ relative to the bases $\{v_0\}$ and $\{v_1\}$, respectively. V_2 has $\{v_2, v_{1,1}\}$ ($v_2 = L_{-2} v_0$ and $v_{1,1} = L_{-1} L_{-1} v_0$) as basis. For instance,

$$\begin{aligned} H(v_2, v_2) &= \langle L_2 L_{-2} v_0 \rangle \;=\; \langle L_{-2} L_2 v_0 + 4 L_0 v_0 + \frac{2}{12} 3\, c\, v_0 \rangle \\ &= 4h + \frac{1}{2}c, \\ H(v_{1,1}, v_{1,1}) &= 8h^2 + 4h, \\ H(v_2, v_{1,1}) &= 6h. \end{aligned}$$

Hence, the matrix A^2 relative to $\{v_2, v_{1,1}\}$ is

$$A^2 = \begin{pmatrix} 4h + \frac{1}{2}c & 6h \\ 6h & 8h^2 + 4h. \end{pmatrix}$$

A^2 is (for $c \geq 0$ and $h \geq 0$) positive semi-definite if and only if

$$\det A^2 = 2h(16h^2 - 10h + 2hc + c) \geq 0.$$

This condition restricts the choice of $h \geq 0$ and $c \geq 0$ even more if H has to be positive semi-definite. In the case $c = \frac{1}{2}$, for instance, h must be outside the interval $]\frac{1}{16}, \frac{1}{2}[$. (Taking into account the other A^N, h can only have the values 0, $\frac{1}{16}$, $\frac{1}{2}$; for these values H is in fact unitary, see below.)

Theorem 6.12. [Kac80] *The* Kac determinant $\det A^N$ *depends on* (c,h) *as follows:*

$$\det A^N(c,h) = K_N \prod_{\substack{p,q\in\mathbb{N}\\ pq\le N}} (h - h_{p,q}(c))^{P(N-pq)},$$

where $K_N \ge 0$ *is a constant which does not depend on* (c,h), *the* $P(M)$ *is an in Definition 6.11, and*

$$\begin{aligned} h_{p,q}(c) := \frac{1}{48}&((13-c)(p^2+q^2)+\sqrt{(c-1)(c-25)}(p^2-q^2)\\ &-24pq-2+2c). \end{aligned}$$

A proof can be found in [KR87] or [CdG94], for example.

To derive $\det A^N(c,h) > 0$ for all $c > 1$ and $h > 0$ from Theorem 6.12, it makes sense to define

$$\begin{aligned} \varphi_{q,q} &:= h - h_{q,q}(c),\\ \varphi_{p,q} &:= (h-h_{p,q}(c))(h-h_{q,p}(c)), \quad p \ne q. \end{aligned}$$

Then by Theorem 6.12 we have

$$\det A^N(c,h) = K_N \prod_{\substack{p,q\in\mathbb{N}\\ pq\le N, p\le q}} (\varphi_{p,q})^{P(N-pq)}.$$

For $1 \le p,q \le N$ and $c > 1$, $h > 0$ one has

$$\begin{aligned} \varphi_{q,q}(c) &= h + \frac{1}{24}(c-1)(q^2-1) \; > \; 0,\\ \varphi_{p,q}(c) &= \left(h - \left(\frac{p-q}{2}\right)^2\right)^2 + \frac{1}{24}h(p^2+q^2-2)(c-1)\\ &\quad + \frac{1}{576}(p^2-1)(q^2-1)(c-1)^2\\ &\quad + \frac{1}{48}(c-1)(p-q)^2(pq+1) \; > \; 0. \end{aligned}$$

Hence, $\det A^N(c,h) > 0$ for all $c > 1$, $h > 0$.

So the hermitian form H is positive definite for the entire region $c > 1$, $h > 0$ if there is just one example $M(c,h)$ with $c > 1$, $h > 0$, such that H is positive definite. We will find such an example in the context of string theory (cf. Theorem 7.11).

The investigation of the region $0 \le c < 1$, $h \ge 0$ is much more difficult. The following theorem contains a complete description:

Theorem 6.13. *Let* $c,h \in \mathbb{R}$.

1. $M(c,h)$ *is unitary (positive definite) for* $c > 1, h > 0$.
1a. $M(c,h)$ *is unitary (positive semi-definite) for* $c \ge 1, h \ge 0$.

2. *$M(c,h)$ is unitary for $0 \leq c < 1$, $h > 0$ if and only if there exists some $m \in \mathbb{N}$, $m > 0$, so that $c = c(m)$ and $h = h_{p,q}(m)$ for $1 \leq p \leq q < m$ with*

$$h_{p,q}(m) := \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}\ ,\ m \in \mathbb{N},$$

$$c(m) := 1 - \frac{6}{m(m+1)}\ ,\ m \in \mathbb{N} \setminus \{1\}.$$

For the proof of 2: Using the Kac determinant, Friedan, Qiu, and Shenker have shown in [FQS86] that in the region $0 \leq c < 1$ the hermitian form H can be unitary only for the values of $c = c(m)$ and $h = h_{p,q}(m)$ stated in 2. Goddard, Kent, and Olive have later proven in [GKO86], using Kac–Moody algebras, that $M(c,h)$ actually gives a unitary representation in all these cases.

If $M(c,h)$ is unitary and positive semi-definite, but not positive definite, we let

$$W(c,h) := M(c,h)/\ker H.$$

Now $W(c,h)$ is a unitary highest-weight representation (positive definite).

Remark 6.14. Up to isomorphism, for every $c,h \in \mathbb{R}$ there is at most one positive definite unitary highest-weight representation, which must be $W(c,h)$. If $\rho : \mathsf{Vir} \to \mathrm{End}_{\mathbb{C}}(V)$ is a positive definite unitary highest-weight representation with vacuum vector $v_0' \in V$ and hermitian form H', the map

$$v_0 \mapsto v_0', \quad v_{n_1 \dots n_k} \mapsto \rho(L_{-n_1} \dots L_{-n_k}) v_0,$$

defines a surjective linear homomorphism $\varphi : M(c,h) \to V$, which respects the hermitian forms H and H':

$$H'(\varphi(v), \varphi(w)) = H(v,w).$$

Therefore, H is positive semi-definite and φ factorizes over $W(c,h)$ as a homomorphism $\overline{\varphi} : W(c,h) \to V$.

6.4 Indecomposability and Irreducibility of Representations

Definition 6.15. M is *indecomposable* if there are no invariant proper subspaces V, W of M, so that $M = V \oplus W$. Otherwise M is *decomposable*.

Definition 6.16. M is called *irreducible* if there is no invariant proper subspace V of M. Otherwise M is called *reducible*.

Theorem 6.17. *For each weight (c,h) we have the following:*

1. *The Verma module $M(c,h)$ is indecomposable.*
2. *If $M(c,h)$ is reducible, then there is a maximal invariant subspace $I(c,h)$, so that $M(c,h)\big/I(c,h)$ is an irreducible highest-weight representation.*
3. *Any positive definite unitary highest-weight representation (that is W(c,h), see above) is irreducible.*

Proof.

1. Let V,W be invariant subspaces of $M=M(c,h)$, and $M=V\oplus W$. By Remark 6.7, we have the direct sum decompositions
$$V=\bigoplus(M_j\cap V) \quad \text{and} \quad W=\bigoplus(M_j\cap W).$$
Since $\dim M_0=1$, this implies $(M_0\cap V)=0$ or $(M_0\cap W)=0$. So the highest-weight vector v_0 is contained either in V or in W. From the invariance of V and W it follows that $V=M$ or $W=M$.
2. Let $I(c,h)$ be the sum of the invariant proper subspaces of M. Then $I(c,h)$ is an invariant proper subspace of M and $M(c,h)\big/I(c,h)$ is an irreducible highest-weight representation.
3. Let V be a positive definite unitary highest-weight representation and $U\subsetneq V$ be an invariant subspace. Then
$$U^{\perp}=\{v\in V : H(u,v)=0\;\forall u\in U\}$$
is an invariant subspace as well, since
$$H(u,L_nv)=H(L_{-n}u,v)=0$$
and $U\oplus U^{\perp}=V$. So 3 follows from 1. □

6.5 Projective Representations of $\mathrm{Diff}_+(\mathbb{S})$

We know the unitary representations $\rho_{c,h}:\mathsf{Vir}\to\mathrm{End}(W_{c,h})$ for $c\geq 1, h\geq 0$ or $c=c(m)$, $h=h_{p,q}(m)$ from the discrete series, where $W_{c,h}:=W(c,h)$ is the unique unitary highest-weight representation of the Virasoro algebra Vir described in the preceding section. Let $\mathbb{H}:=\widehat{W}_{c,h}$ be the completion of $W_{c,h}$ with respect to its hermitean form. It can be shown that there is a linear subspace $\widetilde{W}_{c,h}\subset\mathbb{H}$, $W_{c,h}\subset\widetilde{W}_{c,h}$, so that $\rho_{c,h}(\xi)$ has a linear continuation $\overline{\rho}_{c,h}(\xi)$ on $\widetilde{W}_{c,h}$ for all $\xi\in\mathsf{Vir}\cap(\mathrm{Vect}(\mathbb{S}))$, where $\overline{\rho}_{c,h}(\xi)$ is an essentially self-adjoint operator. The representation $\rho_{c,h}$ is integrable in the following sense:

Theorem 6.18. [GW85] *There is a projective unitary representation $U_{c,h}:\mathrm{Diff}_+(\mathbb{S})\to U(\mathbb{P}(\mathbb{H}))$, so that*

$$\widehat{\gamma}(\exp(\overline{\rho}_{c,h}(\xi))) = U_{c,h}(\exp(\xi))$$

for all $\xi \in \mathrm{Vect}(\mathbb{S})$, *that is for all real vector fields* ξ *in* $\mathbb{S}$. *Furthermore, for* $X \in \mathrm{Vect}(\mathbb{S}) \otimes \mathbb{C}$ *and* $\varphi \in \mathrm{Diff}_+(\mathbb{S})$ *one has*

$$U_{c,h}(\varphi)\rho_{c,h}(X) = (\rho_{c,h}(T\varphi X) + c\alpha(X,\varphi))U_{c,h}(\varphi)$$

with a map α *on* $\mathrm{Vect}(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$. *Here, the* $U_{c,h}(\varphi)$ *are suitable lifts to* $\mathbb{H}$ *of the original* $U_{c,h}(\varphi)$ *(cf. Chap. 3).*

Further investigations in the setting of conformal field theory lead to representations of

- "chiral" algebras $\mathscr{A} \times \overline{\mathscr{A}}$ with $\mathsf{Vir} \subset \mathscr{A}$, $\overline{\mathsf{Vir}} \subset \overline{\mathscr{A}}$ (here $\overline{\mathsf{Vir}}$ is an isomorphic copy of Vir and $\mathscr{A}$ as well as $\overline{\mathscr{A}}$ are further algebras), e.g., $\mathscr{A} = U(\widehat{\mathfrak{g}})$ (universal enveloping algebra of a Kac–Moody algebra), but also algebras, which are neither Lie algebras nor enveloping algebras of Lie algebras. (Cf., e.g., [BPZ84], [MS89], [FFK89], [Gin89], [GO89].)
- Semi-groups $\mathscr{E} \times \overline{\mathscr{E}}$ with $\mathrm{Diff}_+(\mathbb{S}) \subset \mathscr{E}$, $\mathrm{Diff}_+(\mathbb{S}) \subset \overline{\mathscr{E}}$. One discusses semi-group extensions $\mathrm{Diff}_+(\mathbb{S})$, because there is no complex Lie group with $\mathrm{Vect}^{\mathbb{C}}(\mathbb{S})$ as the associated Lie algebra (cf. 5.4). Interesting cases in this context are the semi-group of Shtan and the semi-group of Neretin which are considered, for instance, in [GR05*].

We just present a first example of such a semi-group here (for a survey cf. [Gaw89]):

Example 6.19. Let $q \in \mathbb{C}$, $\tau \in \mathbb{C}$, $q = \exp(2\pi i\tau)$, $|q| < 1$, and $\Sigma_q = \{z \in \mathbb{C} | |q| \leq |z| \leq 1\}$ be the closed annulus with outer radius 1 and inner radius $|q|$. Let $g_1, g_2 \in \mathrm{Diff}_+(\mathbb{S})$ be real analytic diffeomorphisms on the circle $\mathbb{S}$. Then one gets the following parameterizations of the boundary curves of Σ_q:

$$p_1(e^{i\theta}) := q g_1(e^{i\theta}), \qquad p_2(e^{i\theta}) := g_2(e^{i\theta}).$$

The mentioned semi-group $\mathscr{E}$ is the quotient of $\mathscr{E}_0$, where $\mathscr{E}_0$ is the set of pairs (Σ, p') of Riemann surfaces Σ with exactly two boundary curves parameterized by $p' = (p'_1, p'_2)$, for which there is a $q \in \mathbb{C}$ and a biholomorphic map $\varphi : \Sigma_q \to \Sigma$ (where p_1, p_2 is a parameterization of $\partial\Sigma_q$ as above), so that $\varphi \circ p_j = p'_j$. As a set one has $\mathscr{E} = \mathscr{E}_0 \big/ \sim$, where $\sim$ means biholomorphic equivalence preserving the parameterization. The product of two equivalence classes $[(\Sigma, p')]$, $[(\Sigma', p'')] \in \mathscr{E}$ is defined by "gluing" Σ and Σ', where we identify the outer boundary curve of Σ with the inner boundary curve of Σ' taking into account the parameterizations.
The ansatz

$$A_{c,h}([\Sigma_q, p]) := \mathit{const}\; U_{c,h}(g_2^{-1}) q \exp(\overline{\rho}_{c,h}(L_0)) U_{c,h}(g_1)$$

leads to a projective representation of $\mathscr{E}$ using Theorem 6.18.

More general semi-groups can be obtained by looking at more general Riemann surfaces, that is compact Riemann surfaces with finitely many boundary curves, which are parameterized and divided into incoming ("in") and outgoing ("out") boundary curves. The semi-groups defined in this manner have unitary representations as well (cf. [Seg91], [Seg88b], and [GW85]). Starting with these observations, Segal has suggested an interesting set of axioms to describe conformal field theory (cf. [Seg88a]).

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