Chapter 5
The Virasoro Algebra

In this chapter we describe how the Witt algebra and the Virasoro algebra as its essentially unique nontrivial central extension appear in the investigation of conformal symmetries. This result has been proven by Gelfand and Fuks in [GF68]. The last section discusses the question of whether there exists a Lie group whose Lie algebra is the Virasoro algebra.

5.1 Witt Algebra and Infinitesimal Conformal Transformations of the Minkowski Plane

The quantization of classical systems with symmetries yields representations of the classical symmetry group in $\mathrm{U}(\mathbb{P})$ (with $\mathbb{P} = \mathbb{P}(\mathbb{H})$, the projective space of a Hilbert space $\mathbb{H}$, cf. Chap. 3), that is the so-called *projective representations*. As we have explained in Corollary 2.15, the conformal group of $\mathbb{R}^{1,1}$ is isomorphic to $\mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S})$ (here and in the following $\mathbb{S} := \mathbb{S}^1$ is the unit circle). Hence, given a classical theory with this conformal group as symmetry group, one studies the group $\mathrm{Diff}_+(\mathbb{S})$ and its Lie algebra first. After quantization one is interested in the unitary representations of the central extensions of $\mathrm{Diff}_+(\mathbb{S})$ or Lie $(\mathrm{Diff}_+(\mathbb{S}))$ in order to get representations in the Hilbert space as we have explained in the preceding two sections.

The group $\mathrm{Diff}_+(\mathbb{S})$ is in a canonical way an infinite dimensional Lie group modeled on the real vector space of smooth vector fields $\mathrm{Vect}(\mathbb{S})$. (We will discuss $\mathrm{Vect}(\mathbb{S})$ in more detail below.) $\mathrm{Diff}_+(\mathbb{S})$ is equipped with the topology of uniform convergence of the smooth mappings $\varphi : \mathbb{S} \to \mathbb{S}$ and all their derivatives. This topology is metrizable. Similarly, Vect $(\mathbb{S})$ carries the topology of uniform convergence of the smooth vector fields $X : \mathbb{S} \to T\mathbb{S}$ and all their derivatives. With this topology, $\mathrm{Vect}(\mathbb{S})$ is a Fréchet space. In fact, $\mathrm{Vect}(\mathbb{S})$ is isomorphic to $C^\infty(\mathbb{S},\mathbb{R})$, as we will see shortly. The proof that $\mathrm{Diff}_+(\mathbb{S})$ in this way actually becomes a differentiable manifold modeled on $\mathrm{Vect}(\mathbb{S})$ and that the group operation and the inversion are differentiable is elementary and can be carried out for arbitrary oriented, compact (finite-dimensional) manifolds M instead of $\mathbb{S}$ (cf. [Mil84]).

Schottenloher, M.: *The Virasoro Algebra*. Lect. Notes Phys. **759**, 75–85 (2008)
DOI 10.1007/978-3-540-68628-6_6

Since $\mathrm{Diff}_+(\mathbb{S})$ is a manifold modeled on the vector space $\mathrm{Vect}(\mathbb{S})$, the tangent space $T_\varphi(\mathrm{Diff}_+(\mathbb{S}))$ at a point $\varphi \in \mathrm{Diff}_+(\mathbb{S})$ is isomorphic to the vector space $\mathrm{Vect}(\mathbb{S})$. Hence, $\mathrm{Vect}(\mathbb{S})$ is also the underlying vector space of the Lie algebra $\mathrm{Lie}(\mathrm{Diff}_+(\mathbb{S}))$. A careful investigation of the two Lie brackets on $\mathrm{Vect}(\mathbb{S})$ – one from $\mathrm{Vect}(\mathbb{S})$, the other from $\mathrm{Lie}(\mathrm{Diff}_+(\mathbb{S}))$ – shows that each Lie bracket is exactly the negative of the other (cf. [Mil84]). However, this subtle fact is not important for the representation theory of $\mathrm{Lie}(\mathrm{Diff}_+(\mathbb{S}))$. Consequently, it is usually ignored. So we set

$$\mathrm{Lie}(\mathrm{Diff}_+(\mathbb{S})) := \mathrm{Vect}(\mathbb{S}).$$

The vector space $\mathrm{Vect}(\mathbb{S})$ is – like the space $\mathrm{Vect}(M)$ of smooth vector fields on a smooth compact manifold M – an infinite dimensional Lie algebra over $\mathbb{R}$ with a natural Lie bracket: a smooth vector field X on M can be considered to be a derivation $X : C^\infty(M) \to C^\infty(M)$, that is a $\mathbb{R}$-linear map with

$$X(fg) = X(f)g + fX(g) \quad \text{for} \quad f,g \in C^\infty(M).$$

The Lie bracket of two vector fields X and Y is the *commutator*

$$[X,Y] := X \circ Y - Y \circ X,$$

which turns out to be a derivation again. Hence, $[X,Y]$ defines a smooth vector field on M. For $M = \mathbb{S}$ the space $C^\infty(\mathbb{S})$ can be described as the vector space $C^\infty_{2\pi}(\mathbb{R})$ of 2π-periodic functions $\mathbb{R} \to \mathbb{R}$. A general vector field $X \in \mathrm{Vect}(\mathbb{S})$ in this setting has the form $X = f\frac{d}{d\theta}$, where $f \in C^\infty_{2\pi}(\mathbb{R})$ and where the points z of $\mathbb{S}$ are represented as $z = e^{i\theta}$, θ being a variable in $\mathbb{R}$. For $X = f\frac{d}{d\theta}$ and $Y = g\frac{d}{d\theta}$ it is easy to see that

$$[X,Y] = (fg' - f'g)\frac{d}{d\theta} \quad \text{with} \quad g' = \frac{d}{d\theta}g \text{ and } f' = \frac{d}{d\theta}f. \tag{5.1}$$

The representation of f by a convergent Fourier series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(n\theta) + b_n\sin(n\theta))$$

leads to a natural (topological) generating system for $\mathrm{Vect}(\mathbb{S})$:

$$\frac{d}{d\theta}, \quad \cos(n\theta)\frac{d}{d\theta}, \quad \sin(n\theta)\frac{d}{d\theta}.$$

Of special interest is the complexification

$$\mathrm{Vect}^{\mathbb{C}}(\mathbb{S}) := \mathrm{Vect}(\mathbb{S}) \otimes \mathbb{C}$$

of $\mathrm{Vect}(\mathbb{S})$. To begin with, we discuss only the restricted Lie algebra $\mathrm{W} \subset \mathrm{Vect}^{\mathbb{C}}(\mathbb{S})$ of polynomial vector fields on $\mathbb{S}$. Define

$$L_n := z^{1-n}\frac{d}{dz} = -iz^{-n}\frac{d}{d\theta} = -ie^{-in\theta}\frac{d}{d\theta} \in \mathrm{Vect}^{\mathbb{C}}(\mathbb{S}),$$

for $n \in \mathbb{Z}$. $L_n : C^\infty(\mathbb{S},\mathbb{C}) \to C^\infty(\mathbb{S},\mathbb{C})$, $f \mapsto z^{1-n}f'$. The linear hull of the L_n over $\mathbb{C}$ is called the Witt algebra:

$$\mathsf{W} := \mathbb{C}\{L_n : n \in \mathbb{Z}\}.$$

It has to be shown, of course, that W with the Lie bracket in $\mathrm{Vect}^{\mathbb{C}}(\mathbb{S})$ actually becomes a Lie algebra over $\mathbb{C}$. For that, we determine the Lie bracket of the L_n, L_m, which can also be deduced from the above formula (5.1). For $n,m \in \mathbb{Z}$ and $f \in C^\infty(\mathbb{S},\mathbb{C})$,

$$\begin{aligned} L_nL_mf &= z^{1-n}\frac{d}{dz}\left(z^{1-m}\frac{d}{dz}f\right) \\ &= (1-m)z^{1-n-m}\frac{d}{dz}f - z^{1-n}z^{1-m}\frac{d^2}{dz^2}f. \end{aligned}$$

This yields

$$\begin{aligned} [L_n,L_m]f &= L_nL_mf - L_mL_nf \\ &= ((1-m)-(1-n))z^{1-n-m}\frac{d}{dz}f \\ &= (n-m)L_{n+m}f. \end{aligned}$$

In a theory with conformal symmetry, the Witt algebra W is a part of the complexified Lie algebra $\mathrm{Vect}^{\mathbb{C}}(\mathbb{S}) \times \mathrm{Vect}^{\mathbb{C}}(\mathbb{S})$ belonging to the classical conformal symmetry. Hence, as we explained in the preceding chapter, the central extensions of W by $\mathbb{C}$ become important for the quantization process.

5.2 Witt Algebra and Infinitesimal Conformal Transformations of the Euclidean Plane

Before we focus on the central extensions of the Witt algebra in Theorem 5.1, another approach to the Witt algebra shall be described. This approach is connected with the discussion in Sect. 2.4 about the conformal group for the Euclidean plane. In fact, in the development of conformal field theory in the context of statistical mechanics mostly the Euclidean signature is used. This point of view is taken, for example, in the fundamental papers on conformal field theory in two dimensions (cf., e.g., [BPZ84], [Gin89], [GO89]).

The conformal transformations in domains $U \subset \mathbb{C} \cong \mathbb{R}^{2,0}$ are the holomorphic or antiholomorphic functions with nowhere-vanishing derivative (cf. Theorem 1.11). We will treat only the holomorphic case for the beginning. If one ignores the question of how these holomorphic transformations can form a group (cf. Sect. 2.4) and investigates infinitesimal holomorphic transformations, these can be written as

$$z \mapsto z + \sum_{n\in\mathbb{Z}} a_n z^n,$$

with convergent Laurent series $\sum_{n\in\mathbb{Z}} a_n z^n$. In the sense of the general relation between $\mathrm{Diff}_+(M)$ and $\mathrm{Vect}(M)$, the vector fields representing these infinitesimal transformations can be written as

$$\sum a_n z^{n+1} \frac{d}{dz}$$

in the fictional relation between the "conformal group" (see, however, Sect. 5.4) and the vector fields. The Lie algebra of all these vector fields has the sequence $(L_n)_{n\in\mathbb{Z}}$, $L_n = z^{1-n}\frac{d}{dz}$, as a (topological) basis with the Lie bracket derived above:

$$[L_n, L_m] = (n-m)L_{n+m}.$$

Hence, for the Euclidean case there are also good reasons to introduce the Witt algebra $\mathsf{W} = \mathbb{C}\{L_n : n \in \mathbb{Z}\}$ with this Lie bracket as the conformal symmetry algebra. The Witt algebra is a dense subalgebra of the Lie algebra of holomorphic vector fields on $\mathbb{C} \setminus \{0\}$. The same is true for an annulus $\{z \in \mathbb{C} : r < |z| < R\}$, $0 \le r < R \le \infty$. However, only the vector fields L_n with $n \le 1$ can be continued holomorphically to a neighborhood of 0 in $\mathbb{C}$, the other L_n s are strictly singular at 0. As a consequence, contrary to what we have just stated the vector fields L_n, $n > 1$, cannot be considered to be infinitesimal conformal transformations on a suitable neighborhood of 0. Instead, these meromorphic vector fields correspond to proper deformations of the standard conformal structure on $\mathbb{R}^{2,0} \cong \mathbb{C}$.

Without having to speak of a specific "conformal group" one can require – as it is usually done in conformal field theory à la [BPZ84] – that the primary field operators of a conformal field theory transform infinitesimally according to the L_n (a condition which will be explained in detail in Sect. 9.3). This symmetry condition yields an infinite number of constraints. This viewpoint explains the claim of "infinite dimensionality" in the citations of Sect. 2.4.

Let us point out that there is no complex Lie group H with Lie $H = \mathrm{Vect}^{\mathbb{C}}(\mathbb{S})$ as is explained in Sect. 5.4.

The antiholomorphic transformations/vector fields yield a copy $\overline{\mathsf{W}}$ of W with basis $\overline{L}_n$, so that

$$[\overline{L}_n, \overline{L}_m] = (n-m)\overline{L}_{n+m} \quad \text{and} \quad [L_n, \overline{L}_m] = 0.$$

For the Minkowski plane one has a copy of the Witt algebra as well, which in this case originates from the second factor $\mathrm{Diff}_+(\mathbb{S})$ in the characterization

$$\mathrm{Conf}(\mathbb{R}^{1,1}) \cong \mathrm{Diff}_+(\mathbb{S}) \times \mathrm{Diff}_+(\mathbb{S}).$$

In both cases there is a natural isomorphism $t : \mathsf{W} \to \mathsf{W}$ of the Witt algebra, defined by $t(L_n) := -L_{-n}$ on the basis. t is a linear isomorphism and respects the Lie bracket:

$$[t(L_n),t(L_m)] = [L_{-n},L_{-m}] = -(n-m)L_{-(n+m)} = (n-m)t(L_{n+m}).$$

Hence, t is a Lie algebra isomorphism. Since $t^2 = \mathrm{id}_W$, t is an involution. These facts explain that in many texts on conformal field theory the basis

$$L_n^\sim = -z^{n+1}\frac{d}{dz} = t\left(z^{1-n}\frac{d}{dz}\right)$$

instead of $L_n = z^{1-n}\frac{d}{dz}$ is used. Incidentally, the involution t induced on W by the biholomorphic coordinate change $z \mapsto w = \frac{1}{z}$ of the punctured plane $\mathbb{C}\setminus\{0\}$: $dz = -w^{-2}dw$ implies

$$z^{1-n}\frac{d}{dz} = w^{n-1}(-w^2)\frac{d}{dw} = -w^{n+1}\frac{d}{dw}.$$

5.3 The Virasoro Algebra as a Central Extension of the Witt Algebra

After these two approaches to the Witt algebra W we now come to the Virasoro algebra, which is a proper central extension of W. For existence and uniqueness we need

Theorem 5.1. [GF68] $H^2(\mathsf{W},\mathbb{C}) \cong \mathbb{C}$.

Proof. In the following we show: the linear map $\omega : \mathsf{W}\times\mathsf{W} \to \mathbb{C}$ given by

$$\omega(L_n,L_m) := \delta_{n+m}\frac{n}{12}(n^2-1),\ \delta_k := \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

defines a nontrivial central extension of W by $\mathbb{C}$ and up to equivalence this is the only nontrivial extension of W by $\mathbb{C}$. In order to do this we prove

1. $\omega \in Z^2(\mathsf{W},\mathbb{C})$.
2. $\omega \notin B^2(\mathsf{W},\mathbb{C})$.
3. $\Theta \in Z^2(\mathsf{W},\mathbb{C}) \Rightarrow \exists \lambda \in \mathbb{C} : \Theta \sim \lambda\omega$.

Remark: The choice of the factor $\frac{1}{12}$ in the definition of ω is in accordance with the zeta function regularization using the Riemann zeta function, cf. [GSW87, p. 96].

1. Evidently, ω is bilinear and alternating. In order to show $\omega \in Z^2(\mathsf{W},\mathbb{C})$, that is 2° of Remark 4.3, we have to check that

$$\omega(L_k,[L_m,L_n]) + \omega(L_m,[L_n,L_k]) + \omega(L_n,[L_k,L_m]) = 0$$

for $k,m,n \in \mathbb{Z}$. This can be calculated easily:

$$\begin{aligned}
&12(\omega(L_k,[L_m,L_n]) + \omega(L_m,[L_n,L_k]) \\
&+\omega(L_n,[L_k,L_m])) \\
=\ &\delta_{k+m+n}((m-n)k(k^2-1) + (n-k)m(m^2-1) \\
&+(k-m)n(n^2-1)) \\
=\ &-(m-n)(m+n)((m+n)^2-1) \\
&+(2n+m)m(m^2-1) \\
&-(2m+n)n(n^2-1) \\
=\ &0.
\end{aligned}$$

2. Assume that there exists $\mu \in \mathrm{Hom}_{\mathbb{C}}(\mathsf{W},\mathbb{C})$ with $\omega(X,Y) = \mu([X,Y])$ for all $X,Y \in \mathsf{W}$. Then for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
\omega(L_n,L_{-n}) &= \tilde{\mu}(L_n,L_{-n}) \\
\Rightarrow \tfrac{n}{12}(n^2-1) &= \mu([L_n,L_{-n}]) \\
\Rightarrow \tfrac{n}{12}(n^2-1) &= 2n\mu(L_0) \\
\Rightarrow \quad \mu(L_0) &= \tfrac{1}{24}(n^2-1).
\end{aligned}$$

The last equation cannot hold for every $n \in \mathbb{N}$. So the assumption was wrong, which implies $\omega \notin B^2(\mathsf{W},\mathbb{C})$.

3. Let $\Theta \in Z^2(\mathsf{W},\mathbb{C})$. Then for $k,m,n \in \mathbb{Z}$ we have

$$\begin{aligned}
0 &= \Theta(L_k,[L_m,L_n]) + \Theta(L_m,[L_n,L_k]) + \Theta(L_n,[L_k,L_m]) \\
&= (m-n)\Theta(L_k,L_{m+n}) + (n-k)\Theta(L_m,L_{n+k}) \\
&\quad +(k-m)\Theta(L_n,L_{k+m}).
\end{aligned}$$

For $k=0$ we get

$$(m-n)\Theta(L_0,L_{m+n}) + n\Theta(L_m,L_n) - m\Theta(L_n,L_m) = 0.$$

Hence

$$\Theta(L_n,L_m) = \frac{m-n}{m+n}\Theta(L_0,L_{m+n}) \quad \text{for} \quad m,n \in \mathbb{Z};\ m \neq -n.$$

We define a homomorphism $\mu \in \mathrm{Hom}_{\mathbb{C}}(\mathsf{W},\mathbb{C})$ by

$$\begin{aligned}
\mu(L_n) &:= \frac{1}{n}\Theta(L_0,L_n) \quad \text{for} \quad n \in \mathbb{Z}\backslash\{0\}, \\
\mu(L_0) &:= -\frac{1}{2}\Theta(L_1,L_{-1}),
\end{aligned}$$

and let $\Theta' := \Theta + \tilde{\mu}$. Then $\Theta'(L_n,L_m) = 0 \quad \text{for} \quad m,n \in \mathbb{Z}, m \neq -n$, since

$$\begin{aligned}\Theta'(L_n,L_m) &= \Theta(L_n,L_m)+\mu([L_n,L_m])\\ &= \frac{m-n}{m+n}\Theta(L_0,L_{n+m})+\mu((n-m)L_{n+m})\\ &= \frac{m-n}{m+n}\Theta(L_0,L_{n+m})+\frac{n-m}{m+n}\Theta(L_0,L_{n+m})\\ &= 0.\end{aligned}$$

So there is a map $h:\mathbb{Z}\to\mathbb{C}$ with

$$\Theta'(L_n,L_m)=\delta_{n+m}h(n) \quad \text{for} \quad n,m\in\mathbb{Z}.$$

Since Θ' is alternating, it follows:

$$h(0)=0 \qquad \text{and} \qquad h(-k)=-h(k) \quad \text{for all} \quad k\in\mathbb{Z}.$$

By definition of μ we have

$$\begin{aligned}h(1) &= \Theta'(L_1,L_{-1})\\ &= \Theta(L_1,L_{-1})+\mu([L_1,L_{-1}])\\ &= \Theta(L_1,L_{-1})+\mu(2L_0)\\ &= \Theta(L_1,L_{-1})-\Theta(L_1,L_{-1})\\ &= 0.\end{aligned}$$

It remains to be shown that there is a $\lambda\in\mathbb{C}$ with $\Theta'=\lambda\omega$, that is

$$h(n)=\frac{\lambda}{12}n(n^2-1) \quad \text{for} \quad n\in\mathbb{N}. \tag{5.2}$$

Since $\Theta'\in Z^2(\mathsf{W},\mathbb{C})$, we have for $k,m,n\in\mathbb{N}$,

$$\begin{aligned}0 &= \Theta'(L_k,[L_m,L_n])+\Theta'(L_m,[L_n,L_k])\\ &\quad +\Theta'(L_n,[L_k,L_m])\\ &= (m-n)\Theta'(L_k,L_{m+n})+(n-k)\Theta'(L_m,L_{n+k})\\ &\quad +(k-m)\Theta'(L_n,L_{k+m}).\end{aligned}$$

For $k+m+n=0$ we get

$$\begin{aligned}0 &= (m-n)h(k)+(n-k)h(m)+(k-m)h(n)\\ &= -(m-n)h(m+n)+(2n+m)h(m)\\ &\quad -(2m+n)h(n).\end{aligned}$$

The substitution $n=1$ yields the equation

$$-(m-1)h(m+1)+(2+m)h(m)-(2m+1)h(1)=0,$$

for $m \in \mathbb{N}$. Combined with $h(1) = 0$ this implies the recursion formula

$$h(m+1) = \frac{m+2}{m-1}h(m) \quad \text{for} \quad m \in \mathbb{N}\backslash\{1\}.$$

Consequently, the map h is completely determined by $h(2) \in \mathbb{C}$. We now show by induction $n \in \mathbb{N}$ that for $\lambda := 2h(2)$ the relation (5.2) holds. The cases $n = 1$ and $n = 2$ are obvious. So let $m \in \mathbb{N}$, $n > 1$, and $h(m) = \frac{\lambda}{12}m(m^2-1)$. Then

$$\begin{aligned} h(m+1) &= \frac{m+2}{m-1}\,h(m) \\ &= \frac{m+2}{m-1}\,\frac{\lambda}{12}m(m^2-1) \\ &= \frac{\lambda}{12}m(m+1)(m+2) \\ &= \frac{\lambda}{12}(m+1)((m+1)^2-1). \end{aligned}$$ □

Definition 5.2. The *Virasoro algebra* Vir is the central extension of the Witt algebra W by $\mathbb{C}$ defined by ω, that is

$$\mathsf{Vir} = \mathsf{W} \oplus \mathbb{C}Z \quad \text{as a complex vector space,}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n+m}\frac{n}{12}(n^2-1)Z,$$

$$[L_n, Z] = 0 \quad \text{for} \quad n, m \in \mathbb{Z}.$$

5.4 Does There Exist a Complex Virasoro Group?

In Sect. 2.3 we have shown that the conformal group $\mathrm{Conf}(\mathbb{R}^{2,0})$ of the Euclidean plane is not infinite dimensional. Instead, it is isomorphic to the familiar finite-dimensional group Mb of Möbius transformations which in turn is isomorphic to the Lorentz group $\mathrm{SO}(3,1)$. Here, the conformal group is defined to be the group of global conformal transformations defined on open dense subsets $M \subset \mathbb{R}^{2,0}$.

It is, however, a fact and an essential feature that in conformal field theory the infinite dimensional Lie algebra Vir is used as the fundamental set of (infinitesimal) symmetries. Even if it is impossible to interpret these symmetries as generators of conformal transformations on open subsets of the euclidean plane (cf. Sect. 2.3) it is in principle not excluded that there exists an infinite dimensional complex Lie group $\mathcal{G}$ such that the Virasoro algebra Vir is essentially the Lie algebra of $\mathcal{G}$. Such a Lie group would be called a *Virasoro group*. Such a group would play the role of an abstract infinite dimensional conformal group related to the Euclidean plane embodying all conformal symmetries.

We are thus led to discuss the following questions:

1. **Question**: Does there exist a complex Lie group $\mathscr{G}$ with the Virasoro algebra Vir as its Lie algebra?
   Closely related to this question are the following two questions.
2. **Question**: Does there exist a complex Lie group $\mathscr{H}$ with the Witt algebra W as its Lie algebra?
3. **Question**: Does there exist a real Lie group $\mathscr{F}$ such that the Lie algebra of $\mathscr{F}$ is the central extension $\mathsf{Vir}^{\mathbb{R}}$ of the real version $\mathsf{W}^{\mathbb{R}}$ of the Witt algebra given by the same cocycle ω as in Theorem 5.1?

The questions have to be formulated in a more precise manner, but the answer to the first question in its most natural setting is no, as we report in the following.

The questions are not clearly stated in the infinite dimensional setting because answering them requires to specify a topology on Vir since there is no natural topology on an infinite dimensional complex vector space in contrast to the finite-dimensional case. Since Vir can be equipped with many different topologies compatible with its structure of a complex Lie algebra we obtain a series of questions depending on the topologies considered. The topology to be chosen should be at least a locally convex topology since there exists a reasonable theory of Lie groups and Lie algebras (cf. [Mil84]) with models in locally convex spaces. However, only for Banach Lie groups one has an exponential mapping which is a local embedding and thus gives coordinates. In fact, the nonexistence of a Virasoro group is closely related to deficiencies of the exponential mapping.

If one considers locally convex topologies on Vir, it is quite natural to require that the corresponding Lie group has its models in the completion $\widehat{\mathsf{Vir}}$ of Vir. Consequently, the questions 1–3 have to be refined by asking for Lie groups such that their Lie algebras are isomorphic as topological Lie algebras to the completions $\widehat{\mathsf{Vir}}, \widehat{\mathsf{W}}$ resp. $\widehat{\mathsf{Vir}^{\mathbb{R}}}$.

What is the right topology on Vir and on the other two related Lie algebras? Regarding the definition of Vir as the central extension of the Witt algebra W and taking into account the origin of W as a Lie algebra of complex vector fields on $\mathbb{S}$ it is natural to start with the topology on W which is induced from $\mathrm{Vect}(\mathbb{S})^{\mathbb{C}}$ where on $\mathrm{Vect}(\mathbb{S})$ the natural Fréchet topology on compact convergence of the vector fields and all its derivatives is considered. The completion $\widehat{\mathsf{W}}$ of W is $\mathrm{Vect}(\mathbb{S})^{\mathbb{C}}$, and the second question reduces to the existence of a complexification of the real Lie group $\mathrm{Diff}_+(\mathbb{S})$. By a result of Lempert [Lem97*],

Theorem 5.3. $\mathrm{Diff}_+(\mathbb{S})$ *has no complexification. In particular, there even does not exist a real Lie group* $\mathscr{H}$ *with* $\mathrm{Lie}\,\mathscr{H} = \widehat{\mathsf{W}} = \mathrm{Vect}(\mathbb{S})^{\mathbb{C}}$.

Of course, the notion of a complexification has to be made precise, in particular, since in the literature different concepts are used. A (universal) complexification of a real Lie group G is a complex Lie group $G^{\mathbb{C}}$ together with a homomorphism $j : G \to G^{\mathbb{C}}$ such that any homomorphism $\psi : G \to H$ into a complex Lie group

H factors uniquely through j, that is there exists a unique complex analytic morphism $\hat{\psi} : G^{\mathbb{C}} \to H$ with $\psi = \hat{\psi} \circ j$. Finite-dimensional Lie groups always have a complexification although the homomorphism need not be injective.

Note that Theorem 5.3 would follow from the conjecture that every homomorphism ψ into a complex Lie group H is necessarily trivial. This conjecture is stated in [PS86*] (3.2.3) using the fact that $\text{Diff}_+(\mathbb{S})$ is simple according to [Her71]. But in [PS86*] it is implicitly used that H has a reasonable exponential mapping which is not true in general.

Therefore, the proof of Theorem 5.3 in [Lem97*] is based on completely different methods and the result holds for arbitrary compact and connected manifolds M of finite dimension ≥ 1 instead of $\mathbb{S}$.

With the same arguments as in [Lem97*] it can be shown that there is no Virasoro group with respect to the natural topology on Vir induced by the embedding $\mathsf{Vir} \to \text{Vect}(\mathbb{S})^{\mathbb{C}} \oplus \mathbb{C}$ as vector spaces over $\mathbb{C}$ (cf. [Nit06*]):

Theorem 5.4. *There does not exist a complex Lie group $\mathscr{G}$ with* $\text{Lie}\,\mathscr{G} = \widehat{\mathsf{Vir}}$.

In other words, there does not exist an abstract Virasoro group. On the other hand, the third question can be answered in the affirmative. There is a real Lie group $\mathscr{F}$ whose Lie algebra is the (real) nontrivial central extension of $\text{Vect}(\mathbb{S})$. $\mathscr{F}$ is a nontrivial central extension of $\text{Diff}_+(\mathbb{S})$ by $\mathbb{S}^1$.

To construct the extension group $\mathscr{F}$ we can use the restricted unitary group $\mathrm{U}_{\text{res}}(\mathbb{H}_+)$ introduced in Definition 3.16. With a suitable choice of $\mathbb{H}_+ \subset \mathbb{H} = \mathrm{L}^2(\mathbb{S})$ (the space of functions $f \in \mathrm{L}^2(\mathbb{S})$ without negative Fourier coefficients) one obtains a natural embedding of $\text{Diff}_+(\mathbb{S})$ into $\mathrm{U}_{\text{res}}(\mathbb{H}_+)$ (cf. [PS86*]) and differentiating this sequence yields a nontrivial central extension

$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Vect}(\mathbb{S})^{\sim} \longrightarrow \text{Vect}(\mathbb{S}) \longrightarrow 0$$

of $\text{Vect}(\mathbb{S}) \cong \widehat{\mathsf{W}^{\mathbb{R}}}$.

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