Chapter 4
Central Extensions of Lie Algebras and Bargmann's Theorem

In this chapter some basic results on Lie groups and Lie algebras are assumed to be known, as presented, for instance, in [HN91] or [BR77]. For example, every finite-dimensional Lie group G has a corresponding Lie algebra Lie G determined up to isomorphism, and every differentiable homomorphism $R : G \to H$ of Lie groups induces a Lie algebra homomorphism Lie $R = \dot{R}$: Lie $G \to$ Lie H. Conversely, if G is connected and simply connected, every such Lie algebra homomorphism ρ : Lie $G \to$ Lie H determines a unique smooth Lie group homomorphism $R : G \to H$ with $\dot{R} = \rho$.

In addition, for the proof of Bargmann's Theorem we need a more involved result due to Montgomory and Zippin, namely the solution of one of Hilbert's problems: every topological group G, which is a finite-dimensional topological manifold (that is every $x \in G$ has an open neighborhood U with a topological map $\varphi : U \to \mathbb{R}^n$), is already a Lie group (cf. [MZ55]): G has a smooth structure (that is, it is a smooth manifold), such that the composition $(g,h) \to gh$ and the inversion $g \to g^{-1}$ are smooth mappings.

4.1 Central Extensions and Equivalence

A Lie algebra $\mathfrak{a}$ is called *abelian* if the Lie bracket of $\mathfrak{a}$ is trivial, that is $[X,Y] = 0$ for all $X,Y \in \mathfrak{a}$.

Definition 4.1. Let $\mathfrak{a}$ be an abelian Lie algebra over $\mathbb{K}$ and $\mathfrak{g}$ a Lie algebra over $\mathbb{K}$ (the case of $\dim \mathfrak{g} = \infty$ is not excluded). An exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

is called a *central extension* of $\mathfrak{g}$ by $\mathfrak{a}$, if $[\mathfrak{a},\mathfrak{h}] = 0$, that is $[X,Y] = 0$ for all $X \in \mathfrak{a}$ and $Y \in \mathfrak{h}$. Here we identify $\mathfrak{a}$ with the corresponding subalgebra of $\mathfrak{h}$.

For such a central extension the abelian Lie algebra $\mathfrak{a}$ is realized as an ideal in $\mathfrak{h}$ and the homomorphism $\pi : \mathfrak{h} \to \mathfrak{g}$ serves to identify $\mathfrak{g}$ with $\mathfrak{h}/\mathfrak{a}$.

Schottenloher, M.: *Central Extensions of Lie Algebras and Bargmann's Theorem*. Lect. Notes Phys. **759**, 63–73 (2008)
DOI 10.1007/978-3-540-68628-6_5

Examples:

- Let
$$1 \longrightarrow A \xrightarrow{I} E \xrightarrow{R} G \longrightarrow 1$$
be a central extension of finite-dimensional Lie groups A, E, and G with differentiable homomorphisms I and R. Then, for $\dot{I} = \text{Lie } I$ and $\dot{R} = \text{Lie } R$ the sequence
$$0 \longrightarrow \text{Lie } A \xrightarrow{\dot{I}} \text{Lie } E \xrightarrow{\dot{R}} \text{Lie } G \longrightarrow 0$$
is a central extension of Lie algebras.
- In particular, every central extension E of the Lie group G by $\mathrm{U}(1)$
$$1 \longrightarrow \mathrm{U}(1) \longrightarrow E \xrightarrow{R} G \longrightarrow 1$$
with a differentiable homomorphism R induces a central extension
$$0 \longrightarrow \mathbb{R} \longrightarrow \text{Lie } E \xrightarrow{\dot{R}} \text{Lie } G \longrightarrow 0$$
of the Lie algebra Lie G by the abelian Lie algebra $\mathbb{R} \cong i\,\mathbb{R} \cong \text{Lie } \mathrm{U}(1)$.
- This holds for infinite dimensional Banach Lie groups and their Banach Lie algebras as well. For example, when we equip the unitary group $\mathrm{U}(\mathbb{H})$ with the norm topology it becomes a Banach Lie group as a real subgroup of the complex Banach Lie group $\mathrm{GL}(\mathbb{H})$ of all bounded and complex-linear and invertible transformations $\mathbb{H} \to \mathbb{H}$. Therefore, the central extension
$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{U}(\mathbb{H}) \xrightarrow{\widehat{\gamma}} \mathrm{U}(\mathbb{P}) \longrightarrow 1$$
in Lemma 3.4 induces a central extension of Banach Lie algebras
$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{u}(\mathbb{H}) \longrightarrow \mathfrak{u}(\mathbb{P}) \longrightarrow 0,$$
where $\mathfrak{u}(\mathbb{H})$ is the real Lie algebra if bounded self-adjoint operators on $\mathbb{H}$, and $\mathfrak{u}(\mathbb{P})$ is the Lie algebra of $\mathrm{U}(\mathbb{P})$

  In the same manner we obtain a central extension
$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{u}^{\sim}_{\text{res}}(\mathbb{H}) \longrightarrow \mathfrak{u}_{\text{res}}(\mathbb{H}) \longrightarrow 0$$
by differentiating the corresponding exact sequence of Banach Lie groups (cf. Proposition 3.17).
- A basic example in the context of quantization is the *Heisenberg algebra* H which can be defined as the vector space
$$\mathsf{H} := \mathbb{C}[T, T^{-1}] \oplus \mathbb{C}Z$$
with *central element* Z and with the algebra of *Laurent polynomials* $\mathbb{C}[T, T^{-1}]$. (This algebra can be replaced with the algebra of convergent Laurent series $\mathbb{C}(T)$ or with the algebra of formal series $\mathbb{C}\left[\left[T, T^{-1}\right]\right]$ to obtain the same results as for $\mathbb{C}[T, T^{-1}]$.) H will be equipped with the Lie bracket

$$[f \oplus \lambda Z, g \oplus \mu Z] := \sum k f_k g_{-k} \, Z,$$

$f,g \in \mathbb{C}[T,T^{-1}], \lambda, \mu \in \mathbb{C}$, where $f = \sum f_n T^n, g = \sum g_n T^n$ for the Laurent polynomials $f,g \in \mathbb{C}[T,T^{-1}]$ with $f_n, g_n \in \mathbb{C}$. (All the sums are finite and therefore well-defined, since for $f = \sum f_n T^n \in \mathbb{C}[T,T^{-1}]$ only finitely many of the coefficients $f_n \in \mathbb{C}$ are different from zero.)

One can easily check that the maps

$$i : \mathbb{C} \to \mathsf{H}, \; \lambda \mapsto \lambda Z,$$

and

$$\mathrm{pr}_1 : \mathsf{H} \to \mathbb{C}[T,T^{-1}], \; f \oplus \lambda Z \mapsto f,$$

are Lie algebra homomorphisms with respect to the abelian Lie algebra structures on $\mathbb{C}$ and on $\mathbb{C}[T,T^{-1}]$. We thus have defined an exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathbb{C} \xrightarrow{\;i\;} \mathsf{H} \xrightarrow{\mathrm{pr}_1} \mathbb{C}[T,T^{-1}] \longrightarrow 0 \tag{4.1}$$

with $[\lambda Z, g] = 0$. As a consequence, the Heisenberg algebra H is a central extension of the abelian Lie algebra of Laurent polynomials $\mathbb{C}[T,T^{-1}]$ by $\mathbb{C}$.

Note that the Heisenberg algebra is not abelian although it is a central extension of an abelian Lie algebra.

The map

$$\Theta : \mathbb{C}[T,T^{-1}] \times \mathbb{C}[T,T^{-1}] \to \mathbb{C}, (f,g) \mapsto \sum k f_k g_{-k},$$

is bilinear and alternating. Θ is called a cocycle in this context (cf. Definition 4.4), and the significance of the cocycle lies in the fact that the Lie algebra structure on the central extension H is determined by Θ since $[f + \lambda Z, g + \mu Z] = \Theta(f,g)Z$. The cocycle Θ can also be described by the residue of fg' at $0 \in \mathbb{C}$:

$$\Theta(f,g) = -\operatorname{Res}_{z=0} f(z) g'(z).$$

This can be easily seen by using the expansion of the product fg':

$$fg'(T) = \sum_{n \in \mathbb{Z}} \left(\sum_{k \in \mathbb{Z}} (n-k+1) f_k g_{n-k+1} \right) T^n.$$

To describe H in a slightly different way observe that the monomials $a_n := T^n, n \in \mathbb{Z}$, form a basis of $\mathbb{C}[T,T^{-1}]$. Hence, the Lie algebra structure on the Heisenberg algebra H is completely determined by

$$[a_m, a_n] = m \delta_{m+n} Z, \; [Z, a_m] = 0.$$

Here, δ_k is used as an abbreviation of Kronecker's δ_k^0.

- Another example which will be of interest in Chap. 10 in order to obtain relevant examples of vertex algebras is the *affine Kac–Moody algebra* or *current algebra* as a non-abelian generalization of the construction of the Heisenberg algebra. We begin with a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. For any associative algebra R the Lie algebra structure on $R \otimes \mathfrak{g}$ is given by

$$[r \otimes a, s \otimes b] = rs \otimes [a,b] \text{ or } [ra, sb] = rs[a,b].$$

Two special cases are $R = \mathbb{C}[T, T^{-1}]$, the algebra of complex Laurent polynomials, and $R = \mathbb{C}(T)$, the algebra of convergent Laurent series. The following construction and its main properties are valid for both these algebras and in the same way also for the algebra of formal Laurent series of $\mathbb{C}(T)$, which is used in Chap. 10 on vertex algebras. Here, we treat the case $R = \mathbb{C}[T, T^{-1}]$ with the Lie algebra $\mathfrak{g}[T, T^{-1}] = \mathbb{C}[T, T^{-1}] \otimes \mathfrak{g}$ which is sometimes called the *loop algebra* of $\mathfrak{g}$.

We fix an invariant symmetric bilinear form on $\mathfrak{g}$, that is a symmetric bilinear

$$(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C},\ a, b \mapsto (a,b),$$

on $\mathfrak{g}$ satisfying

$$([a,b],c) = (a,[b,c]).$$

The *affinization* of $\mathfrak{g}$ is the vector space

$$\hat{\mathfrak{g}} := \mathfrak{g}[T, T^{-1}] \oplus \mathbb{C}Z$$

endowed with the following Lie bracket

$$[T^m \otimes a, T^n \otimes b] := T^{m+n} \otimes [a,b] + m(a,b)\delta_{m+n}Z,$$
$$[T^m \otimes a, Z] := 0,$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$. Using the abbreviations

$$a_m := T^m a,\ b_n := T^n b,$$

this definition takes the form

$$[a_m, b_n] = [a,b]_{m+n} + m(a,b)\delta_{m+n}Z.$$

It is easy to check that this defines a Lie algebra structure on $\hat{\mathfrak{g}}$ and that the two natural maps

$$i : \mathbb{C} \to \hat{\mathfrak{g}},\ \lambda \mapsto \lambda Z,$$
$$\mathrm{pr}_1 : \hat{\mathfrak{g}} \to \mathfrak{g}[T, T^{-1}],\ f \otimes a + \mu Z \mapsto f \otimes a,$$

are Lie algebra homomorphisms. We have defined an exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{C} \xrightarrow{i} \hat{\mathfrak{g}} \xrightarrow{\mathrm{pr}_1} \mathfrak{g}[T, T^{-1}] \longrightarrow 0. \tag{4.2}$$

This exact sequence provides another example of a central extension, namely the affinization $\hat{\mathfrak{g}}$ of $\mathfrak{g}$ as a central extension of the loop algebra $\mathfrak{g}[T,T^{-1}]$.

In the case of the abelian Lie algebra $\mathfrak{g} = \mathbb{C}$ we are back in the preceding example of the Heisenberg algebra. As in that example there is a characterizing cocycle on the loop algebra

$$\Theta : \mathfrak{g}[T,T^{-1}] \times \mathfrak{g}[T,T^{-1}] \to \mathbb{C},$$
$$(T^m a, T^n b) \mapsto m(a,b)\delta_{n+m} Z,$$

determining the Lie algebra structure on $\hat{\mathfrak{g}}$.

In the particular case of a simple Lie algebra $\mathfrak{g}$ there exists only one nonvanishing invariant symmetric bilinear form on $\mathfrak{g}$ (up to scalar multiplication), the Killing form. In that case the uniquely defined central extension $\hat{\mathfrak{g}}$ of the loop algebra $\mathfrak{g}[T,T^{-1}]$ is called the *affine Kac–Moody algebra of* $\mathfrak{g}$.

- In a similar way the Virasoro algebra can be defined as a central extension of the Witt algebra (cf. Chap. 5).

Definition 4.2. An exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

splits if there is a Lie algebra homomorphism $\beta : \mathfrak{g} \to \mathfrak{h}$ with $\pi \circ \beta = \mathrm{id}_{\mathfrak{g}}$. The homomorphism β is called a *splitting map*. A central extension which splits is called a *trivial* extension, since it is equivalent to the exact sequence of Lie algebra homomorphisms

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{a} \oplus \mathfrak{g} \longrightarrow \mathfrak{g} \longrightarrow 0.$$

(Equivalence is defined in analogy to the group case, cf. Definition 3.18.)

If, in the preceding examples of central extensions of Lie groups, the exact sequence of Lie groups splits in the sense of Definition 3.19 with a differentiable homomorphism $S : G \to E$ as splitting map, then the corresponding sequence of Lie algebra homomorphisms also splits in the sense of Definition 4.2 with splitting map $\dot{S}$. In general, the reverse implication holds for connected and simply connected Lie groups G only. In this case, the sequence of Lie groups splits if and only if the associated sequence of Lie algebras splits. All this follows immediately from the properties stated at the beginning of this chapter.

Remark 4.3. For every central extension of Lie algebras

$$0 \longrightarrow \mathfrak{a} \longrightarrow \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0,$$

there is a linear map $\beta : \mathfrak{g} \to \mathfrak{h}$ with $\pi \circ \beta = \mathrm{id}_{\mathfrak{g}}$ (β is in general not a Lie algebra homomorphism). Let

$$\Theta(X,Y) := [\beta(X), \beta(Y)] - \beta([X,Y]) \quad for \quad X,Y \in \mathfrak{g}.$$

Then β is a splitting map if and only if $\Theta = 0$.

It can easily be checked that the map $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ (depending on β) always has the following properties:

1° $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is bilinear and alternating.
2° $\Theta(X,[Y,Z]) + \Theta(Y,[Z,X]) + \Theta(Z,[X,Y]) = 0$.

Moreover, $\mathfrak{h} \cong \mathfrak{g} \oplus \mathfrak{a}$ as vector spaces by the linear isomorphism

$$\psi : \mathfrak{g} \times \mathfrak{a} \to \mathfrak{h}, \quad X \oplus Y = (X,Y) \mapsto \beta(X) + Y.$$

Finally, with the Lie bracket on $\mathfrak{g} \oplus \mathfrak{a}$ given by

$$[X \oplus Z, Y \oplus Z']_{\mathfrak{h}} := [X,Y]_{\mathfrak{g}} + \Theta(X,Y)$$

for $X,Y \in \mathfrak{g}$ and $Z,Z' \in \mathfrak{a}$ the map ψ is a Lie algebra isomorphism.

The Lie bracket on $\mathfrak{h}$ can also be written as

$$[\beta(X) + Z, \beta(Y) + Z'] = \beta([X,Y]) + \Theta(X,Y).$$

Here, we treat $\mathfrak{a}$ as a subalgebra of $\mathfrak{h}$ again.

Definition 4.4. A map $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ with the properties 1° and 2° of Remark 4.3 will be called a *2-cocycle* on $\mathfrak{g}$ with values in $\mathfrak{a}$ or simply a *cocycle*.

The discussion in Remark 4.3 leads to the following classification.

Lemma 4.5. *With the notations just introduced we have*

1. *Every central extension $\mathfrak{h}$ of $\mathfrak{g}$ by $\mathfrak{a}$ comes from a cocycle $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ as in 4.3.*
2. *Every cocycle $\Theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ generates a central extension $\mathfrak{h}$ of $\mathfrak{g}$ by $\mathfrak{a}$ as in 4.3.*
3. *Such a central extension splits (and this implies that it is trivial) if and only if there is a $\mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a})$ with*

$$\Theta(X,Y) = \mu([X,Y])$$

for all $X,Y \in \mathfrak{g}$.

Proof.

1. is obvious from the preceding remark.
2. Let $\mathfrak{h}$ be the vector space $\mathfrak{h} := \mathfrak{g} \oplus \mathfrak{a}$. The bracket

$$[X \oplus Z, Y \oplus Z']_{\mathfrak{h}} := [X,Y]_{\mathfrak{g}} \oplus \Theta(X,Y)$$

for $X,Y \in \mathfrak{g}$ and $Z,Z' \in \mathfrak{a}$ is a Lie bracket if and only if Θ is a cocycle. Hence, $\mathfrak{h}$ with this Lie bracket defines a central extension of $\mathfrak{g}$ by $\mathfrak{a}$.

3. Let $\sigma : \mathfrak{g} \to \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ a splitting map, that is a Lie algebra homomorphism with $\pi \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then σ has to be of the form $\sigma(X) = X + \mu(X)$, $X \in \mathfrak{g}$, with a suitable $\mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a})$. From the definition of the bracket on $\mathfrak{h}$, $[\sigma(X),\sigma(Y)] = [X,Y] + \Theta(X,Y)$ for $X,Y \in \mathfrak{g}$. Furthermore, since σ is a Lie algebra homomorphism, $[\sigma(X),\sigma(Y)] = \sigma([X,Y]) = [X,Y] + \mu([X,Y])$. It follows that $\Theta(X,Y) = \mu([X,Y])$. Conversely, if Θ has this form, it clearly satisfies 1°

and 2°. The linear map $\sigma : \mathfrak{g} \to \mathfrak{h} = \mathfrak{g} \oplus \mathfrak{a}$ defined by $\sigma(X) := X + \mu(X)$, $X \in \mathfrak{g}$, turns out to be a Lie algebra homomorphism:

$$\begin{aligned} \sigma([X,Y]) &= [X,Y]_{\mathfrak{g}} + \mu([X,Y]) \\ &= [X,Y]_{\mathfrak{g}} + \Theta(X,Y) \\ &= [X+\mu(X), Y+\mu(Y)]_{\mathfrak{h}} \\ &= [\sigma(X), \sigma(Y)]_{\mathfrak{h}}. \end{aligned}$$

Hence, σ is a splitting map.

Examples of Lie algebras given by a suitable cocycle are the Heisenberg algebra and the Kac–Moody algebras, see above, and the Virasoro algebra, cf. Chap. 5.

As in the case of groups, the collection of all equivalence classes of central extensions for a Lie algebra is a cohomology group.

Definition 4.6.

$$\begin{aligned} \mathrm{Alt}^2(\mathfrak{g},\mathfrak{a}) &:= \{\Theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a} \,|\, \Theta \text{ satisfies condition } \mathit{1}^\circ\}. \\ Z^2(\mathfrak{g},\mathfrak{a}) &:= \{\Theta \in \mathrm{Alt}^2(\mathfrak{g},\mathfrak{a}) \,|\, \Theta \text{ satisfies condition } \mathit{2}^\circ\}. \\ B^2(\mathfrak{g},\mathfrak{a}) &:= \{\Theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{a} \,|\, \exists \mu \in \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g},\mathfrak{a}) : \Theta = \tilde{\mu}\}. \\ H^2(\mathfrak{g},\mathfrak{a}) &:= Z^2(\mathfrak{g},\mathfrak{a})/B^2(\mathfrak{g},\mathfrak{a}). \end{aligned}$$

Here, $\tilde{\mu}$ is given by $\tilde{\mu}(X,Y) := \mu([X,Y])$ for $X,Y \in \mathfrak{g}$.

Z^2 and B^2 are linear subspaces of Alt^2 with $B^2 \subset Z^2$. The above vector spaces are, in particular, abelian groups. Z^2 is the space of 2-cocycles and $H^2(\mathfrak{g},\mathfrak{a})$ is called the *second cohomology group* of $\mathfrak{g}$ with values in $\mathfrak{a}$. We have proven the following classification of central extensions of Lie algebras.

Remark 4.7. The cohomology group $H^2(\mathfrak{g},\mathfrak{a})$ is in one-to-one correspondence with the set of equivalence classes of central extensions of $\mathfrak{g}$ by $\mathfrak{a}$.

Cf. Remark 3.27 for the case of group extensions.

4.2 Bargmann's Theorem

We now come back to the question of whether a projective representation can be lifted to a unitary representation.

Theorem 4.8 (Bargmann [Bar54]). *Let G be a connected and simply connected, finite-dimensional Lie group with*

$$H^2(\mathrm{Lie}\, G, \mathbb{R}) = 0.$$

Then every projective representation $T : G \to \mathrm{U}(\mathbb{P})$ has a lift *as a unitary representation $S : G \to \mathrm{U}(\mathbb{H})$, that is for every continuous homomorphism $T : G \to \mathrm{U}(\mathbb{P})$ there is a continuous homomorphism $S : G \to \mathrm{U}(\mathbb{H})$ with $T = \widehat{\gamma} \circ S$.*

Proof. By Theorem 3.10, there is a central extension E of G and a homomorphism $\widehat{T} : E \to \mathrm{U}(\mathbb{H})$, such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{U}(1) \arrow[r] \arrow[d, "\mathrm{id}"'] & E \arrow[r, "\pi"] \arrow[d, "\widehat{T}"'] & G \arrow[r] \arrow[d, "T"] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r] & \mathrm{U}(\mathbb{H}) \arrow[r, "\widehat{\gamma}"'] & \mathrm{U}(\mathbb{P}) \arrow[r] & 1
\end{tikzcd}$$

Here, $E = \{(U,g) \in \mathrm{U}(\mathbb{H}) \times G | \widehat{\gamma}(U) = Tg\}$, $\pi = \mathrm{pr}_2$, and $\widehat{T} = \mathrm{pr}_1$. E is a topological group as a subgroup of the topological group $\mathrm{U}(\mathbb{H}) \times G$ (cf. Proposition 3.11) and $\widehat{T}$ and π are continuous homomorphisms. The lower exact sequence has local continuous sections, as we will prove in Lemma 4.9: For every $A \in \mathrm{U}(\mathbb{P})$ there is an open neighborhood $W \subset \mathrm{U}(\mathbb{P})$ and a continuous map $\nu : W \to \mathrm{U}(\mathbb{H})$ with $\widehat{\gamma} \circ \nu = \mathrm{id}_W$. Let now $V := T^{-1}(W)$. Then $\mu(g) := (\nu \circ T(g), g)$, $g \in V$, defines a local continuous section $\mu : V \to E$ of the upper sequence because $\widehat{\gamma}(\nu \circ T(g)) = Tg$, that is $(\nu \circ T(g), g) \in E$ for $g \in V$. μ is continuous because ν and T are continuous. This implies that

$$\psi : \mathrm{U}(1) \times V \to \pi^{-1}(V) \subset E, \quad (\lambda, g) \mapsto (\lambda \nu \circ T(g), g),$$

is a bijective map with a continuous inverse map

$$\psi^{-1}(U,g) = (\lambda(U), g),$$

where $\lambda(U) \in \mathrm{U}(1)$ for $U \in \widehat{\gamma}^{-1}(W)$ is given by the equation $U = \lambda(U)\nu \circ \widehat{\gamma}(U)$. Hence, the continuity of ψ^{-1} is a consequence of the continuity of the multiplication

$$\mathrm{U}(1) \times \mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{H}), \quad (\lambda, U) \mapsto \lambda U.$$

We have shown that the open subset $\pi^{-1}(V) = (T \circ \pi)^{-1}(W) \subset E$ is homeomorphic to $\mathrm{U}(1) \times V$. Consequently, E is a topological manifold of dimension $1 + \dim G$. By using the theorem of Montgomory and Zippin mentioned above, the topological group E is even a $(1 + \dim G)$-dimensional Lie group and the upper sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow E \longrightarrow G \longrightarrow 1$$

is a sequence of differentiable homomorphisms.

Now, according to Remark 4.7 the corresponding exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Lie}\,\mathrm{U}(1) \longrightarrow \mathrm{Lie}\,E \longrightarrow \mathrm{Lie}\,G \longrightarrow 0$$

splits because of the condition $H^2(\text{Lie } G, \mathbb{R}) = 0$. Since G is connected and simply connected, the sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow E \longrightarrow G \longrightarrow 1$$

splits with a differentiable homomorphism $\sigma : G \to E$ as splitting map: $\pi \circ \sigma = \mathrm{id}_G$. Finally, $S := \widehat{T} \circ \sigma$ is the postulated lift. S is a continuous homomorphism and $\widehat{\gamma} \circ \widehat{T} = T \circ \pi$ implies $\widehat{\gamma} \circ S = \widehat{\gamma} \circ \widehat{T} \circ \sigma = T \circ \pi \circ \sigma = T \circ \mathrm{id}_G = T$:

$$\begin{tikzcd}
E \arrow[d, "\widehat{T}"'] & G \arrow[l, "\sigma"'] \arrow[d, "T"] \arrow[dl, "S"'] \\
\mathrm{U}(\mathbb{H}) \arrow[r, "\widehat{\gamma}"'] & \mathrm{U}(\mathbb{P}).
\end{tikzcd}$$

□

Lemma 4.9. $\widehat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$ *has local continuous sections and therefore can be regarded as a principal fiber bundle with structure group* $\mathrm{U}(1)$.

Proof. (cf. [Sim68, p. 10]) For $f \in \mathbb{H}$ let

$$V_f := \{U \in \mathrm{U}(\mathbb{H}) : \langle Uf, f \rangle \neq 0\}.$$

Then V_f is open in $\mathrm{U}(\mathbb{H})$, since $U \mapsto Uf$ is continuous in the strong topology. Hence, $U \mapsto \langle Uf, f \rangle$ is continuous as well. (For the strong topology all maps $U \mapsto Uf$ are continuous by definition.) The set

$$W_f := \widehat{\gamma}(V_f) = \{T \in \mathrm{U}(\mathbb{P}) : \delta(T\varphi, \varphi) \neq 0\},\ \varphi = \widehat{\gamma}(f),$$

is open in $\mathrm{U}(\mathbb{P})$ since $\widehat{\gamma}^{-1}(W_f) = V_f$ is open. (The open subsets in $\mathrm{U}(\mathbb{P})$ are, by Definition 3.6, precisely the subsets $W \subset \mathrm{U}(\mathbb{P})$, such that $\widehat{\gamma}^{-1}(W) \subset \mathrm{U}(\mathbb{H})$ is open.) $(W_f)_{f \in \mathbb{H}}$ is, of course, an open cover of $\mathrm{U}(\mathbb{P})$. Let

$$\beta_f : V_f \to \mathrm{U}(1), \quad U \mapsto \frac{|\langle Uf, f \rangle|}{\langle Uf, f \rangle}.$$

β_f is continuous, since $U \mapsto \langle Uf, f \rangle$ is continuous. Furthermore, $\beta_f(e^{i\theta}U) = e^{-i\theta}\beta_f(U)$ for $U \in V_f$ and $\theta \in \mathbb{R}$, as one can see directly. One obtains a continuous section of $\widehat{\gamma}$ over W_f by

$$\nu_f : W_f \to \mathrm{U}(\mathbb{H}), \quad \widehat{\gamma}(U) \mapsto \beta_f(U)U.$$

ν_f is well-defined, since $U' \in V_f$ with $\widehat{\gamma}(U') = \widehat{\gamma}(U)$, that is $U' = e^{i\theta}U$, implies

$$\beta_f(U')U' = \beta_f(e^{i\theta}U)e^{i\theta}U = \beta_f(U)U.$$

Now $\widehat{\gamma} \circ \nu_f = \mathrm{id}_{W_f}$, since

$$\widehat{\gamma} \circ \nu_f(\widehat{\gamma}(U)) = \widehat{\gamma}(\beta_f(U)U) = \widehat{\gamma}(U) \quad \text{for} \quad U \in V_f.$$

Eventually, ν_f is continuous: let $V_1 \in W_f$ and $U_1 = \nu_f(V_1) \in \nu_f(W_f)$. Then $\beta_f(U_1) = 1$. Every open neighborhood of U_1 contains an open subset

$$B = \{U \in V_f : \|Ug_j - U_1 g_j\| < \varepsilon \text{ for } j = 1, \ldots, m\}$$

with $\varepsilon > 0$ and $g_j \in \mathbb{H}$, $j = 1, \ldots, m$. The continuity of β_f on W_f implies that there are further $g_{m+1}, \ldots, g_n \in \mathbb{H}$, $\|g_j\| = 1$, so that $|\beta_f(U) - 1| < \frac{\varepsilon}{2}$ for

$$U \in B' := \{U \in V_f : \|Ug_j - U_1 g_j\| < \frac{\varepsilon}{2} \text{ for } j = 1, \ldots, m, \ldots, n\}.$$

The image $D := \widehat{\gamma}(B')$ is open, since

$$\widehat{\gamma}^{-1}(D) = \bigcup_{\lambda \in \mathrm{U}(1)} \{U \in V_f : \|Ug_j - \lambda U_1 g_j\| < \frac{\varepsilon}{2} \text{ for } j = 1, \ldots, n\}$$

is open. (We have shown that the map $\widehat{\gamma} : \mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$ is open.) Hence, D is an open neighborhood of V_1. ν_f is continuous since $\nu_f(D) \subset B$: for $P \in D$ there is a $U \in B'$ with $P = \widehat{\gamma}(U)$, that is $\nu_f(P) = \beta_f(U)U$. This implies

$$\begin{aligned} \|\nu_f(P)g_j - U_1 g_j\| &\leq \|\beta_f(U)Ug_j - \beta_f(U)U_1 g_j\| \\ &\quad + \|(\beta_f(U) - 1)U_1 g_j\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

for $j = 1, \ldots, m$, that is $\nu_f(P) \in B$. Hence, the image $\nu_f(D)$ of the neighborhood D of V_1 is contained in B.

In spite of this nice result no reasonable differentiable structure seems to be known on the unitary group $\mathrm{U}(\mathbb{H})$ and its quotient $\mathrm{U}(\mathbb{P})$ with respect to the strong topology in order to prove a result which would state that $\mathrm{U}(\mathbb{H}) \to \mathrm{U}(\mathbb{P})$ is a differentiable principal fiber bundle. The difficulty in defining a Lie group structure on the unitary group lies in the fact that the corresponding Lie algebra should contain the (bounded and unbounded) self-adjoint operators on $\mathbb{H}$. In contrast to this situation, with respect to the operator norm topology the unitary group is a Lie group.

E is by construction the fiber product of $\widehat{\gamma}$ and T. Since $\widehat{\gamma}$ is locally trivial by Lemma 4.9 with general fiber $\mathrm{U}(1)$, this must also hold for $E \to G$. Exactly this was needed in the proof of Theorem 4.8, to show that E actually is a Lie group.

Remark 4.10. For every finite-dimensional semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ one can show $H^2(\mathfrak{g}, \mathbb{K}) = 0$ (cf. [HN91]). As a consequence of the above discussion we thus have the following result which can be applied to the quantization of certain important symmetries: if G is a connected and simply connected finite-dimensional

Lie group with semi-simple Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$, then every continuous representation $T : G \to \mathrm{U}(\mathbb{P})$ has a lift to a unitary representation. In particular, to every continuous representation $T : \mathrm{SU}(N) \to \mathrm{U}(\mathbb{P})$ (resp. $T : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(\mathbb{P})$) there corresponds a unitary representation $S : \mathrm{SU}(N) \to \mathrm{U}(\mathbb{H})$ (resp. $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{U}(\mathbb{H})$) with $\widehat{\gamma} \circ S = T$.

Note that $\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of the proper Lorentz group $\mathrm{SO}(3,1)$ and $\mathrm{SU}(2)$ is the universal covering group of the rotation group $\mathrm{SO}(3)$.

References

[Bar54] V. Bargmann. On unitary ray representations of continuous groups. *Ann. Math.* **59** (1954), 1–46.

[BR77] A.O. Barut and R. Raczka. *Theory of Group Representations and Applications*. PWN – Polish Scientific Publishers, Warsaw, 1977.

[HN91] J. Hilgert and K.-H. Neeb. *Lie Gruppen und Lie Algebren*. Vieweg, Braunschweig, 1991.

[MZ55] D. Montgomory and L. Zippin. *Topological Transformation Groups*. Interscience, New York, 1955.

[Sim68] D. Simms. *Lie Groups and Quantum Mechanics*. Lecture Notes in Mathematics **52**, Springer Verlag, Berlin, 1968.