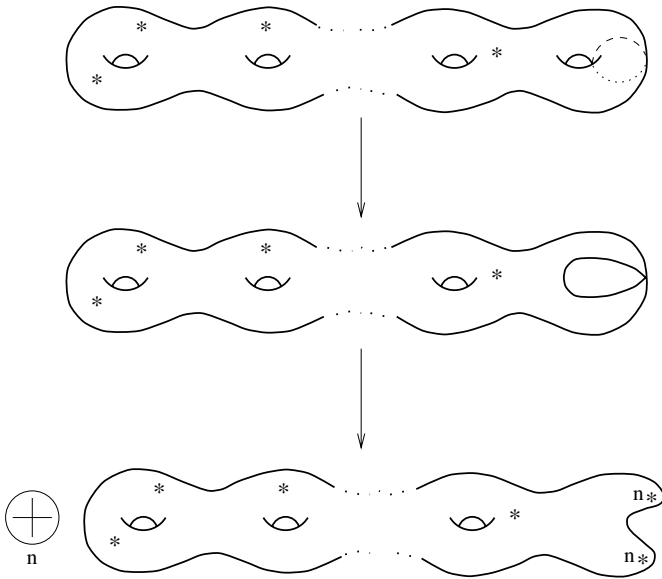


# Lecture Notes in Physics

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M. Schottenloher

# A Mathematical Introduction to Conformal Field Theory

Second Edition

 Springer

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*To Barbara*



# Preface to the Second Edition

The second edition of these notes has been completely rewritten and substantially expanded with the intention not only to improve the use of the book as an introductory text to conformal field theory, but also to get in contact with some recent developments. In this way we take a number of remarks and contributions by readers of the first edition into consideration who appreciated the rather detailed and self-contained exposition in the first part of the notes but asked for more details for the second part. The enlarged edition also reflects experiences made in seminars on the subject.

The interest in conformal field theory has grown during the last 10 years and several texts and monographs reflecting different aspects of the field have been published as, e.g., the detailed physics-oriented introduction of Di Francesco, Mathieu, and Sénéchal [DMS96\*],<sup>1</sup> the treatment of conformal field theories as vertex algebras by Kac [Kac98\*], the development of conformal field theory in the context of algebraic geometry as in Frenkel and Ben-Zvi [BF01\*] and more general by Beilinson and Drinfeld [BD04\*]. There is also the comprehensive collection of articles by Deligne, Freed, Witten, and others in [Del99\*] aiming to give an introduction to strings and quantum field theory for mathematicians where conformal field theory is one of the main parts of the text. The present expanded notes complement these publications by giving an elementary and comparatively short mathematics-oriented introduction focusing on some main principles.

The notes consist of 11 chapters organized as before in two parts. The main changes are two new chapters, Chap. 8 on Wightman's axioms for quantum field theory and Chap. 10 on vertex algebras, as well as the incorporation of several new statements, examples, and remarks throughout the text. The volume of the text of the new edition has doubled. Half of this expansion is due to the two new chapters.

We have included an exposition of Wightman's axioms into the notes because the axioms demonstrate in a convincing manner how a consistent quantum field theory in principle should be formulated even regarding the fact that no four-dimensional model with properly interacting fields satisfying the axioms is known to date. We investigate in Chap. 8 the axioms in their different appearances as postulates on operator-valued distributions in the relativistic case as well as postulates on the

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<sup>1</sup> The “\*” indicates that the respective reference has been added to the References in the second edition of these notes.

corresponding correlation functions on Minkowski and on Euclidean spaces. The presentation of the axioms serves as a preparation and motivation for Chap. 9 as well as for Chap. 10.

Chapter 9 deals with an axiomatic approach to two-dimensional conformal field theory. In comparison to the first edition we have added the conformal Ward identities, the state field correspondence, and some changes with respect to the presentation of the operator product expansion. The concepts and methods in this chapter were quite isolated in the first edition, and they can now be understood in the context of Wightman's axioms in its various forms and they also can be linked to the theory of vertex algebras.

Vertex algebras have turned out to be extremely useful in many areas of mathematics and physics, and they have become the main language of two-dimensional conformal field theory in the meantime. Therefore, the new Chap. 10 in these notes provides a presentation of basic concepts and methods of vertex algebras together with some examples. In this way, a number of manipulations in Chap. 9 are explained again, and the whole presentation of vertex algebras in these notes can be understood as a kind of formal and algebraic continuation of the axiomatic treatment of conformal field theory.

Furthermore, many new examples have been included which appear at several places in these notes and may serve as a link between the different viewpoints (for instance, the Heisenberg algebra  $H$  as an example of a central extension of Lie algebras in Chap. 4, as a symmetry algebra in the context of quantization of strings in Chap. 7, and as a first main example of a vertex algebra in Chap. 10). Similarly, Kac–Moody algebras are introduced, as well as the free bosonic field and the restricted unitary group in the context of quantum electrodynamics. Several of the elementary but important statements of the first edition have been explained in greater detail, for instance, the fact that the conformal groups of the Euclidean spaces are finite dimensional, even in the two-dimensional case, the fact that there does not exist a complex Virasoro group and that the unitary group  $U(\mathbb{H})$  of an infinite-dimensional Hilbert space  $\mathbb{H}$  is a topological group in the strong topology.

Moreover, several new statements have been included, for instance, about a detailed description of some classical groups, about the quantization of the harmonic oscillator and about general principles used throughout the notes as, for instance, the construction of representations of Lie algebras as induced representations or the use of semidirect products.

The general concept of presenting a rather brief and at the same time rigorous introduction to conformal field theory is maintained in this second edition as well as the division of the notes in two parts of a different nature: The first is quite elementary and detailed, whereas the second part requires more mathematical prerequisites, in particular, from functional analysis, complex analysis, and complex algebraic geometry.

Due to the complexity of the treatment of Wightman's axioms in the second part of the notes not all results are proven, but there are many more proofs in the second part than in the original edition. In particular, the chapter on vertex algebras is self-contained.



The final chapter on the Verlinde formula in the context of algebraic geometry, which is now Chap. 11, has nearly not been changed except for a comment on fusion rings and on the connection of the Verlinde algebra with twisted  $K$ -theory recently discovered by Freed, Hopkins, and Teleman [FHT03\*].

In a brief appendix we mention further developments with respect to boundary conformal field theory, to stochastic Loewner evolution, and to modularity together with some references.

München, March 2008

*Martin Schottenloher*

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# Preface to the First Edition

The present notes consist of two parts of approximately equal length. The first part gives an elementary, detailed, and self-contained mathematical exposition of classical conformal symmetry in  $n$  dimensions and its quantization in two-dimensions. Central extensions of Lie groups and Lie algebras are studied in order to explain the appearance of the Virasoro algebra in the quantization of two-dimensional conformal symmetry. The second part surveys some topics related to conformal field theory: the representation theory of the Virasoro algebra, some aspects of conformal symmetry in string theory, a set of axioms for a two-dimensional conformally invariant quantum field theory, and a mathematical interpretation of the Verlinde formula in the context of semi-stable holomorphic vector bundles on a Riemann surface. In contrast to the first part only few proofs are provided in this less elementary second part of the notes.

These notes constitute – except for corrections and supplements – a translation of the prepublication “Eine mathematische Einführung in die konforme Feldtheorie” in the preprint series *Hamburger Beiträge zur Mathematik*, Volume 38 (1995). The notes are based on a series of lectures I gave during November/December of 1994 while holding a *Gastdozentur* at the *Mathematisches Seminar der Universität Hamburg* and on similar lectures I gave at the *Université de Nice* during March/April 1995.

It is a pleasure to thank H. Brunke, R. Dick, A. Jochens, and P. Slodowy for various helpful comments and suggestions for corrections. Moreover, I want to thank A. Jochens for writing a first version of these notes and for carefully preparing the  $\text{\LaTeX}$  file of an expanded English version. Finally, I would like to thank the Springer production team for their support.

Munich, September 1996

*Martin Schottenloher*



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