Weak Cauchy Sequences in $L_\infty(\mu, X)$.

Georg Schlüchtermann, Mathematisches Institut der Ludwig-Maximilians-Universität, Theresienstrasse 39, 80333 München, Germany

Abstract

For a finite and positive measure space (Ω, Σ, μ) characterizations of weak Cauchy sequences in $L_\infty(\mu, X)$, the space of $\mu-$essentially bounded vector valued functions $f : \Omega \longrightarrow X$, are presented. The fine distinction between Asplund and conditionally weakly compact subsets of $L_\infty(\mu, X)$ is discussed.

AMS(MOS) classification 46E10,46G10

1 Introduction and Preliminaries

M.Talagrand gave in his celebrated paper [Ta, Th.1] a parametric Rosenthal ℓ_1-dichotomy. With the help of this result conditionally weakly compact subsets of $L_p(\mu, X), 1 \leq p < \infty$, the space of Bochner integrable functions, can be characterized. A characterization for $p = \infty$ has not been found yet. The treatment for the relatively weakly compact subsets of $L_\infty(\mu, X)$ was done, in special cases by K.T.Andrews and J.J.Uhl [AU]) and in general by the author [S3]. A basic tool in both papers is the celebrated factorization lemma of Davis, Figiel, Johnson and Pelczynski.

Here, in modified version, this method will be applied to give a complete (i.e. for all Banach spaces X) characterization of conditionally weakly compact subsets of $L_\infty(\mu, X)$ resp. weak Cauchy sequences. It is mainly based on a result to parametrizing operators $T : X \longrightarrow L_1(\mu, Y)^*$ (see the definition below). In section 3 a fine distinction between Asplund sets and conditionally weakly compact sets is sketched for $L_\infty(\mu, X)$. In the survey article of L.H.Riddle and J.J.Uhl, this was given for arbitrary Banach spaces by means of topology, vector measures and geometry. Here, this will be illustrated in the particular situation of $L_\infty(\mu, X)$.

First we fix some notations and definitions which are used in the paper. X and Y denote Banach Spaces, where $B(X)$ resp. $S(X)$ is the unit ball resp. the unit sphere of the Banach space X. Consider, if not indicated otherwise, a positive and finite measure space, which will be denoted by (Ω, Σ, μ). $L_p(\mu, X) := L_p(\Omega, \Sigma, \mu, X)$ for $1 \leq p \leq \infty$ is the usual Bochner space. $L_\infty(\mu, X^*, X)$ is the equivalence class of w^*- measurable and essentially bounded functions $f : \Omega \longrightarrow X^*$, where a function $f : \Omega \longrightarrow Y^*$ is called w^*-measurable, if for all $y \in Y$: $\langle y, f(\cdot)\rangle$ is measurable. If $U : \Omega \longrightarrow L(X, Y)$ is such that for all $x \in X : U(\cdot)(x) \in L_\infty(\mu, Y)$ (resp. $U(\cdot)(x) \in L_\infty(\mu, Y^*, Y)$), then U is called *strongly measurable* (resp. w^*-*measurable*). Define $\overline{U} : X \longrightarrow L_\infty(\mu, Y)$ (resp. $\overline{U} : X \longrightarrow L_\infty(\mu, Y^*, Y)$) by $\overline{U}(x) = U(\cdot)(x), x \in X$. U is a *parametric version* for a bounded linear operator $T : X \longrightarrow L_\infty(\mu, Y)$ (resp. for a $T : X \longrightarrow L_\infty(\mu, Y^*, Y)$) if $\overline{U} = T$.

A bounded set in a Banach space is called conditionally weakly compact, if every sequence admits a weak Cauchy subsequence. Rosenthals dichotomy theorem says that is equivalent to the fact that there is no sequence in the set equivalent to the ℓ_1-basis.

A bounded and linear operator $T : X \longrightarrow Y$ is called conditionally weakly compact, if $T(B(X))$ is conditionally weakly compact.

A convex and closed subset D of a Banach space X has the complete continuity property CCP if all bounded linear operator $V : L_1([0,1]) \longrightarrow X$, such that $V\big(\{\frac{\chi_A}{\lambda(A)}; \lambda(A) > 0\}\big) \subset D$ are Dunford-Pettis-operator, i.e. V maps weak Cauchy sequences into norm ones.

Here are some facts about conditionally weakly compact operators, which are used in the sequel:
(1.1) Let $T : X \longrightarrow Y$ be linear and bounded. Then the following conditions are equivalent:

a) T is conditionally weakly compact.

b) $T^*(B(Y^*))$ has the CCP.

c) For all bounded and linear $V: Y \longrightarrow L_\infty([0,1])$ the composition $V \circ T$ maps bounded sequences into almost everywhere convergent subsequences.

d) T factors through a space not containing a copy of ℓ_1.

a)$\Leftrightarrow$b) is done in [Bo, p.309, Th. 7.4.12], a)$\Leftrightarrow$c) is quoted from[RSU, p.528,Th. 1] and a)$\Leftrightarrow$d) is in [Di, p.237, 2 iii)].

A bounded set $K \subset X$ is an Asplund set, if for each countable subset $A \subset K$ the dual $(\overline{\text{span}} A)^*$ is norm separable.
A bounded and linear operator $T: X \longrightarrow Y$ is called Asplund operator, if $T(B(X))$ is an Asplund set.

Next we formulate some facts about Asplund operators, which are needed:
(1.2) Let $T: X \longrightarrow Y$ be linear and bounded. Then the following conditions are equivalent:

a) T is an Asplund operator.

b) $T^*(B(Y^*))$ is a set with the RNP.

c) For all bounded and linear $V: Y \longrightarrow L_\infty([0,1])$ and all $\epsilon > 0$ there is a set $E \subset [0,1], \lambda([0,1] \setminus E) < \epsilon$ such that $(V \circ T)(B(X))\chi_E$ is relatively compact in $L_\infty([0,1])$.

d) T factors through an Asplund space Z (i.e where $B(Z)$ is an Asplund set).

A reference for a)$\Leftrightarrow$b)$\Leftrightarrow$d) is [Bo, p.135, Th.5.3.5, Th.5.3.7] and [RU, p.150, Th.A] for a) $\Leftrightarrow$c).

2 Weak Cauchy Sequences in $L_\infty(\mu, X)$

We start with a dual version for the representation of operators $S: L_1(\mu, Y) \longrightarrow X^*$, which is stated e.g. in [Din, p.279,Th. 8] and which also occurs for special cases of X in [AU, p.908,Lemma 3]. As a general assumption in this section all measure spaces are assumed to be separable. This is not an essential restriction, since the result can be generalized by the usual technique of a long sequence of expectation operators (see e.g. [S2]). The proof of the following lemma is omitted. A proof can be found in [S3].

Lemma 2.1 *Let $T : X \longrightarrow L_1(\mu, Y)^*$ be a bounded linear operator. Then there exists a w^*-measurable parametric version $U : \Omega \longrightarrow L(X, Y^*)$ of T.*
In particular: If $T : X \longrightarrow L_\infty(\mu, Y)$, then there exists a strongly measurable parametric version $U : \Omega \longrightarrow L(X, Y)$ of T.

The following proposition is a parametric version of the factorization of conditionally weakly compact operators (see 1.1.b,d; [S3, Prop. 2.2.])

Proposition 2.2 *: Let $T : X \longrightarrow L_1(\mu, Y)^*$ be bounded and linear. Then T is conditionally weakly compact if and only if there exist a w^*-measurable parametric version $U : \Omega \longrightarrow L(X, Y^*)$ of T and an absolutely convex, w^*-closed $K \subset X^*$ with the CCP, such that $U(\omega)^*(B(Y^{**})) \subset K$ for a.a. $\omega \in \Omega$.*

Proof: " $\Rightarrow$ ": By 1.1.d), T factors through a space R, which does not contain a copy of ℓ_1, i.e. there exist bounded linear operators $T_1 : X \longrightarrow R$ and $T_2 : R \longrightarrow L_1(\mu, Y)^*$, such that $T = T_2 \circ T_1$. Consider $T_2 : R \longrightarrow L_1(\mu, Y)^*$. By 2.1. there exists an operator-valued map $\tilde{U} : \Omega \longrightarrow L(R, Y^*)$, such that $\overline{\tilde{U}} = T_2$. Define for $\omega \in \Omega :\ U(\omega) := \tilde{U}(\omega) \circ T_1$. Then $U : \Omega \longrightarrow L(X, Y^*)$ has the property that $U(\cdot)(x)$ is w^*-measurable for all $x \in X$, since $\tilde{U}$ enjoys this property and T_1 is a bounded operator. Evidently $\overline{U} = T$ and $U(\omega)^* = T_1^* \circ \tilde{U}(\omega)^*$ for $\omega \in \Omega$. Assume now w.l.o.g. that $\|T_2\| \leq 1$, then $\|\tilde{U}(\omega)^*\| \leq 1$ for a.a. $\omega \in \Omega$. Define $K := T_1^*(B(R^*))$, then for a.a.$\omega \in \Omega$:

$$U(\omega)^*(B(Y^{**})) \subset K$$

and K is absolutely convex, w$^*-$closed and has the CCP according to 1.1 b).
" $\Leftarrow$ ":

$$\begin{aligned} \text{Define } \tilde{T} \ &: \ L_1(\mu, Y) \longrightarrow X^* \\ & \quad f \longmapsto \int_\Omega U(\omega)^*(f(\omega))d\mu(\omega). \end{aligned}$$

Let K be as presumed. For all $y \in B(Y)$ the map $\Omega \ni \omega \longmapsto U(\omega)^*(y)$ is w^*-measurable, since U is w^*-measurable. Hence, $\omega \longmapsto U(\omega)^*(f(\omega))$ is measurable for all $f \in L_1(\mu, Y)$. Let $f \in B(L_1(\mu, Y))$, $f = \sum_{i=1}^{n} y_i \chi_{A_i}$.

$$\text{Then} \quad \tilde{T}(f) = \underbrace{\sum_{i=1}^{n} \underbrace{\int_{A_i} \underbrace{U(\omega)^*(y_i)}_{\in \|y_i\| K} \, d\mu(\omega)}_{\in \|y_i\| \mu(A_i) K}}_{\in \sum_{i=1}^{n} \|y_i\| \mu(A_i) K \subset K}.$$

It is easy to see that $T^*|_{L_1(\mu,Y)} = \tilde{T}$.
Hence, by the Goldstine argument (see [Di, p.13]) and the $w^* - w^*$-continuity of $\tilde{T}|_{B(L_1(\mu,Y))}$, $T^*(B(L_1(\mu, Y)^{**})) \subset K$ is absolutely convex, w^*-compact and has the CCP. By 1.1.b), T is conditionally weakly compact. □

The $L_\infty(\mu, Y)$–version is an easy consequence of the preceding proposition.

Corollary 2.3: Let $T : X \longrightarrow L_\infty(\mu, Y)$ be bounded and linear. Then T is conditionally weakly compact if and only if there exist a strongly measurable version $U : \Omega \longrightarrow L(X, Y)$ of T and an absolutely convex, w*-closed subset $K \subset X^*$ with the CCP, such that $U(\omega)^*(B(Y^*)) \subset K$ for a.a. $\omega \in \Omega$.
Proof: " $\Rightarrow$ " : Transform the proof of the preceding proposition literally, exchange Y^* resp. Y^{**} by Y resp. Y^*, and observe that the operator valued map $\tilde{U} : \Omega \longrightarrow L(X, Y)$, used in the proof, is strongly measurable according to 2.1.
" $\Leftarrow$ " : Embed $L_\infty(\mu, Y)$ into $L_1(\mu, Y^*)^*$ and Y into Y^{**} isometrically. Then this direction follows directly from the above proposition.
□

The next theorem will give a characterization of weak Cauchy sequences in $L_1(\mu, Y)^*$ respectively $L_\infty(\mu, Y)$. It should be seen as an analogue to the characterization of weak zero sequences given in [S3]. For this set

$$c := \{(x_n) \subset \mathbb{R}; \lim_{n\to\infty} x_n \text{ exists}\}.$$

Theorem 2.4 *:*

a) *Let* $(f_n)_{n\in\mathbb{N}} \subset L_1(\mu, Y)^*$ *be a bounded sequence. Then* (f_n) *is weak Cauchy if and only if there is a closed and absolutely convex set* $K \subset c$ *with the CCP and a set* $N \in \Sigma, \mu(N) = 0$*, such that* $(|\langle f_n(\omega), y^{**}\rangle|)_{n\in\mathbb{N}} \in K$ *for all* $\omega \in \Omega \setminus N, y^{**} \in B(Y^{**})$ *(see [S3, Cor. 2.3]).*

b) *Let* $(f_n)_{n\in\mathbb{N}} \subset L_\infty(\mu, Y)$ *be a bounded sequence. Then* (f_n) *is weak Cauchy if and only if there is a closed and absolutely convex set* $K \subset c$ *with the CCP and a set* $N \in \Sigma, \mu(N) = 0$*, such that* $(|\langle f_n(\omega), y^*\rangle|)_{n\in\mathbb{N}} \in K$ *for all* $\omega \in \Omega \setminus N, y^* \in B(Y^*)$.

*Proof:*a) Define $T : \ell_1 \longrightarrow L_1(\mu, Y)^*, T(e_n) = f_n$.
Then by 1.1.b) (f_n) is weak Cauchy if and only if $T^* : L_1(\mu, Y)^{**} \longrightarrow c$ and $T^*(B(L_1(\mu, Y)^*))$ has the CCP. By 2.2 this is equivalent to the existence of an operator-valued map $U : \Omega \longrightarrow L(\ell_1, Y^*)$, such that $\overline{U} = T$ and an absolutely convex, w^*-closed set $K \subset \ell_\infty$ with the CCP such that in turn for a.a. $\omega \in \Omega :\ U(\omega)^*(B(Y^{**})) \subset K$

$$\begin{aligned}
&\iff (|\langle e_n, U(\omega)^*(y^{**})\rangle|)_{n\in\mathbb{N}} \in K \text{ f.a.a. } \omega \in \Omega \text{ and } y^{**} \in B(Y^{**}) \\
&\iff (|\langle U(\omega)(e_n), y^{**}\rangle|)_{n\in\mathbb{N}} \in K \text{ f.a.a. } \omega \in \Omega \text{ and } y^{**} \in B(Y^{**}) \\
&\iff (|\langle f_n(\omega), y^{**}\rangle|)_{n\in\mathbb{N}} \in K \text{ f.a.a. } \omega \in \Omega \text{ and } y^{**} \in B(Y^{**}).
\end{aligned}$$

Since $(\langle f_n(\omega), y^{**}\rangle) \in c$ for all $y^{**} \in B(Y^{**})$ and a.a. $\omega \in \Omega$ the set K can be chosen in c.

b) As in part "a)" but use 2.3 instead of 2.2. □

For a further characterization of weak Cauchy sequences the following characterization of weak zero sequences in $L_\infty(\mu, X)$ is useful. It is an immediate consequence of [S3, Th.2.7].

Proposition 2.5 *Let* $(f_n) \subset L_\infty(\mu, X)$ *be bounded. Then the following conditions are equivalent:*

a) (f_n) *is a weak zero sequence.*

b) *There is a set* $N \subset \Omega, \mu(N) = 0$*, such that:*

   i) *For all* $\omega \in \Omega \setminus N : (f_n(\omega))$ *is weak zero.*

ii) For all sequences $(x_j^*) \subset B(X^*), (\omega_j) \subset \Omega \backslash N$ *there are subsequences* $(x_{j_k}^*), (\omega_{j_k})$, *such that* $\lim_{n\to\infty} \lim_{k\to\infty} \langle f_n(\omega_{j_k}), x_{j_k}^* \rangle = 0$.

The following corollary follows from the previous theorem.

Corollary 2.6: Let $(f_n) \subset L_\infty(\mu, X)$ be bounded. Then the following conditions are equivalent:

a) (f_n) is weak Cauchy.

b) For all subsequences (f_{n_m}) there is a set $N \in \Sigma, \mu(N) = 0$, such that all sequences $(x_j^*) \subset B(X^*), (\omega_j) \subset \Omega \setminus N$ admit subsequences $(x_{j_k}^*), (\omega_{j_k})$, such that $\lim_{m\to\infty} \lim_{k\to\infty} \langle (f_{n_{m+1}} - f_{n_m})(\omega_{j_k}), x_{j_k}^* \rangle = 0$.

The following theorem gives the final characterization of weak Cauchy sequences.

Theorem 2.7 *Let* $(f_n) \subset L_\infty(\mu, X)$ *be bounded. Then the following conditions are equivalent:*

a) (f_n) *is weak Cauchy.*

b) There is a set $N \in \Sigma, \mu(N) = 0$, *such that for all sequences* $(x_j^*) \subset B(X^*), (\omega_j) \subset \Omega \backslash N$ *there exist subsequences* $(x_{j_k}^*), (\omega_{j_k})$, *such that* $\lim_{m,n\to\infty} \lim_{k\to\infty} \langle (f_m - f_n)(\omega_{j_k}), x_{j_k}^* \rangle = 0$.

Proof: a) $\Rightarrow$ *b)*: Define $T : \ell_1 \longrightarrow L_\infty(\mu, X),\ T(e_n) := f_n$. Let $U : \Omega \longrightarrow L(\ell_1, X)$ be an operator-valued version of T with $\sup_{\omega\in\Omega} \|U(\omega)\| = \sup_{n\in\mathbb{N}} \|f_n\|$. For $n \in \mathbb{N}$ define $\tilde{f_n} : \Omega \longrightarrow X$ by $\tilde{f_n}(\omega) := U(\omega)(e_n)$. Then for all choices of representatives $(\overline{f_n})$ of (f_n) it holds:

$$\exists\ N \in \Sigma,\ \mu(N) = 0\ \forall\ \omega \in \Omega \setminus N :\ \tilde{f_n}(\omega) = \overline{f_n}(\omega),\ (n \in \mathbb{N}).$$

Let $(\omega_j) \subset \Omega \setminus N, (x_j^*) \subset B(X^*)$ be given. Since $(\overline{f_n})$ are uniformly bounded, there exists a subsequence $(\omega_{j_k}), (x_{j_k}^*)$, such that for all $n \in \mathbb{N}$: $\lim_{k\to\infty} \langle \overline{f_n}(\omega_{j_k}), x_{j_k}^* \rangle$ exists. Define a functional

$F : \mathrm{span}(f_n) \longrightarrow \mathbb{R}$ by $F(f) := \lim_{k\to\infty}\langle \overline{f}(\omega_{j_k}), x^*_{j_k}\rangle$. Since the sequence $(x^*_{j_k})$ is bounded and the limit exists by the above, F can be seen as an element of $(\overline{\mathrm{span}}(f_n))^*$. But this implies "b)".
$b) \Rightarrow a)$ follows by the corollary. $\square$

Remark 2.8: Let $K \subset L_\infty(\mu, X)$ be bounded and separable. Select a dense sequence $(f_n) \subset K$ and choose a corresponding sequence of representatives $(\overline{f_n}), \overline{f_n} : \Omega \longrightarrow X$. Define now $\mathcal{C}_K := \{(\overline{\omega}, \overline{x^*}) = (\omega_k, x^*_k) \in \Omega^{\mathbb{N}} \times B(X^*)^{\mathbb{N}};\ \lim_{k\to\infty}\langle \overline{f_n}(\omega_k), x^*_k\rangle$ exists for all $n \in \mathbb{N}\}$. Then according to the proof above for all $(\overline{\omega}, \overline{x^*})$: $F_{(\overline{\omega},\overline{x^*})}(f) := \lim_{k\to\infty}\langle \overline{f}(\omega_k), x^*_k\rangle$ defines an element of $(\overline{\mathrm{span}}K)^*$. By 2.5 and 2.7 a sequence $(f_n) \subset K$ is convergent to $f \in L_\infty(\mu, X)$ resp. weak Cauchy if and only if $\lim_{n\to\infty} F_{(\overline{\omega},\overline{x^*})}(f - f_n) = 0$ resp. $\lim_{n,m\to\infty} F_{(\overline{\omega},\overline{x^*})}(f_n - f_m) = 0$.

One might suspect that weak Cauchy in $L_\infty(\mu, X)$ may imply a strong condition on the collective image set of the sequence such as the following one:

> Let $(f_n)_{n\in\mathbb{N}} \subset L_\infty(\mu, X)$ be weak Cauchy. Then:
> For all $\epsilon > 0$ there are sets $\Omega_\epsilon \subset \Omega, \mu(\Omega \setminus \Omega_\epsilon) < \epsilon$, $L_\epsilon \subset X$ conditionally weakly compact, such that $\bigcup\limits_{\omega\in\Omega_\epsilon} (f_n(\omega))_{n\in\mathbb{N}} \subset L_\epsilon$.

But this assumption fails, as example 2.6 in [S3] demonstrates. Note that the Banach space X in this particular example is weak sequentially complete. Thus the notions of weak Cauchy and weak convergence resp. conditionally weakly compact and weakly compact coincide.

In the following, some special conditionally weakly compact sets will be given. But first a simple (and well-known) example will demonstrate that a similiar result as for weak convergent sequences, namely that $(x_n) \subset X$ weak zero and $(f_n) \subset L_\infty(\mu)$ bounded imply $(f_n x_n)$ weak zero in $L_\infty(\mu, X)$, does not hold for weak Cauchy sequences.

Example 2.9: Let $X := \mathbb{R}$, $f_n := r_n$ the Rademacher functions and $x_n := 1$. Then (r_n) is bounded and (x_n) is Cauchy but $(f_n x_n) = (r_n)$ is not conditionally weakly compact according to 2.6 ((r_n) also spans a copy of ℓ_1 in $L_\infty([0, 1])$ [Di, p.223]).

Nevertheless a certain subclass of conditionally weakly compact sets can be characterised.

Corollary 2.10:

a) Let $(f_n) \subset L_\infty(\mu)$ and $(x_n) \subset X$ be weak Cauchy. Then $(f_n x_n) \subset L_\infty(\mu, X)$ is weak Cauchy.

b) Let $C \subset L_\infty(\mu)$ and $L \subset X$ be conditionally weakly compact, then $K := C \otimes L := \{fx; f \in C, x \in L\} \subset L_\infty(\mu, X)$ is conditionally weakly compact.

Proof: We use 2.7. Since $(f_n) \subset L_\infty(\mu)$ is weak Cauchy, it is bounded (assume w.l.o.g. $(f_n) \subset B(L_\infty(\mu))$), and there exists $N \in \Sigma, \mu(N) = 0$, such that

For all sequences $(\omega_j) \subset \Omega \setminus N$ there is a subsequence (ω_{j_k}) :

$$(1) \qquad \lim_{n,m\to\infty} \lim_{k\to\infty} |f_n(\omega_{j_k}) - f_m(\omega_{j_k})| = 0.$$

Let $\epsilon > 0$, $(\omega_j) \subset \Omega \setminus N$, $(x_j^*) \subset B(X^*)$ be given. Then there are subsequences $(\omega_{j_k}), (x_{j_k}^*)$, such that (1) is satisfied and there is a $x^* \in B(X^*)$ with

$$(2) \qquad \forall\ n \in \mathbb{N} : \lim_{k\to\infty} \langle x_n, x_{j_k}^* - x^* \rangle = 0.$$

Assume further w.l.o.g. that $(x_n) \subset B(X)$, since it is a weak Cauchy sequence. According to (1) and since (x_n) is weak Cauchy, there exists $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0$:

$$\lim_{k\to\infty} |f_n(\omega_{j_k}) - f_m(\omega_{j_k})| < \frac{\epsilon}{4} \text{ and } |\langle x_n - x_m, x^* \rangle| < \frac{\epsilon}{2}.$$

Then for $n, m \geq n_0$:

$$\begin{aligned}
&\lim\nolimits_{k\to\infty} && |\langle f_n(\omega_{j_k})x_n - f_m(\omega_{j_k})x_m, x_{j_k}^* \rangle| \leq \\
&\lim\nolimits_{k\to\infty} && |\langle f_n(\omega_{j_k})x_n - f_n(\omega_{j_k})x_m, x_{j_k}^* \rangle| + \frac{\epsilon}{2} = \\
&\lim\nolimits_{k\to\infty} && |\langle x_n - x_m, x_{j_k}^* \rangle||f_n(\omega_{j_k})| + \frac{\epsilon}{2} \leq \\
&\lim\nolimits_{k\to\infty} && |\langle x_n - x_m, x_{j_k}^* - x^* + x^* \rangle| + \frac{\epsilon}{2} \leq_{(2)} \\
&\lim\nolimits_{k\to\infty} && |\langle x_n - x_m, x^* \rangle| + \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

by the above .

Part b) follows directly from a). □

3 Asplund Sets and Conditionally Weakly Compact Sets

In the article "Martingales and the fine line between Asplund spaces and spaces not containing a copy of ℓ_1," L.H. Riddle and J.J.Uhl discussed the difference between Asplund sets and conditionally weakly compact sets (see [RU]). Here a parametric version is given. Namely it will be shown, how the characterization of conditionally weakly compact subsets, derived in part 2 and a corresponding one for Asplund sets is separated. But first the operator results in section 2 will be transferred to the Asplund property. The proofs will be omitted, since they run parallely to 2.2 resp. 2.3, only replacing "conditionally weakly compact" by "Asplund", "CCP" by "RNP" and applying 1.2 instead of 1.1.

Proposition 3.1 *Let $T : X \longrightarrow L_1(\mu, Y)^*$ be bounded and linear. Then T is an Asplund operator if and only if there exist a w^*-measurable version $U : \Omega \longrightarrow L(X, Y^*)$ of T and an absolutely convex and w^*-closed subset $K \subset X^*$ with the RNP, such that $U(\omega)^*(B(Y^{**})) \subset K$ for a.a. $\omega \in \Omega$.*

Corollary 3.2: Let $T : X \longrightarrow L_\infty(\mu, Y)$ be bounded and linear. Then T an Asplund operator if and only if there exists a strongly measurable parametric version $U : \Omega \longrightarrow L(X, Y)$ of T and there an absolutely convex, w^*- closed set $K \subset X^*$ with the RNP, such that $U(\omega)^*(B(Y^*)) \subset K$ for a.a. $\omega \in \Omega$.

The characterization in Corollary 2.3 of conditionally weakly compact sets is not very useful in practice. So, an equivalent version of 2.3 is presented, which gives the analogue to the characterization of the relatively weakly compact sets in [S3] resp. relatively weakly compact sets in [DRS]. The result follows directly from 2.7.

Proposition 3.3 *For a bounded subset $K \subset L_\infty(\mu, X)$ the following conditions are equivalent:*

a) K is conditionally weakly compact.

b) For all sequences $(f_n)_{n\in\mathbb{N}} \subset K$ there are a subsequence $(f_{n_m})_{m\in\mathbb{N}})$ and a set $N \subset \Omega, \mu(N) = 0$, such that:

i) For all $\omega \in \Omega \setminus N : (f_{n_m}(\omega))$ is weak Cauchy.

ii) For all sequences $(x_j^*) \subset B(X^*), (\omega_j) \subset \Omega \setminus N$ there are subsequences $(x_{j_k}^*), (\omega_{j_k})$ with $\lim\limits_{l,m \to \infty} \lim\limits_{k \to \infty} \langle (f_{n_m} - f_{n_l})(\omega_{j_k}), x_{j_k}^* \rangle = 0$.

Remark 2.8 exhibits certain functionals, which are characteristic for the determination of weak sequential convergence. Let $\mathcal{C}_K$ be defined as in 2.8. The next two theorem will demonstrate the "parametric" fine line between Asplund sets and conditionally weakly compact sets. But first an auxiliary result is needed.

Lemma 3.4 *Let $K \subset L_\infty(\mu, X)$ be separable and $L_K := \overline{span} K$.*

a) $\mathcal{F}_K := \{F_{(\overline{\omega}, \overline{x^*})}; (\overline{\omega}, \overline{x^*}) \in \mathcal{C}_K\}$ is w^*-compact in $B(L_K^*)$.

b) There exists a finite Borel measure ν on $(\mathcal{F}_K, w^*)$, such that $L_\infty[0,1]$ and $L_\infty(\nu)$ are w^*-homeomorphic and isometrically.

Proof: a): Let $(f_n) \subset K$ be dense. For simplicity let (f_n) denote the sequence of representatives. Further, let $(F^k) \subset \mathcal{F}_K$ be w*- converging to $F \in B(L_K^*)$. Select $(\overline{\omega}^k, \overline{x^*}^k) = (w_j^k, x_j^{*k}) \in \mathcal{C}_K$, such that $F_{(\overline{\omega}^k, \overline{x^*}^k)} = F^k$. By definition:

$$\forall\, k \in \mathbb{N}\ \forall\, n \in \mathbb{N}\ \exists\, j_{k,n} \in \mathbb{N}\ \forall\, j \geq j_{k,n} : |F^k(f_n) - \langle f_n(\omega_j^k), x_j^{*k} \rangle| < \frac{1}{k}.$$

One may assume:

$$\forall\ k \in \mathbb{N}\ \forall\ n \in \mathbb{N} :\ j_{k,n} \leq j_{k,n+1}.$$

Define $(\overline{\omega}, \overline{x^*}) := (\omega_l, x_l^*)$, where $\omega_l := \omega_{j_{l,l}}^l$ and $x_l^* := x_{j_{l,l}}^{*l}$.
Claim: $\forall\ n \in \mathbb{N} : \lim_{l \to \infty} \langle f_n(\omega_l), x_l^* \rangle = F(f_n)$:
To prove this, let $\epsilon > 0$ and $n \in \mathbb{N}$ be given. Choose $k_0 \in \mathbb{N}$, such that $\frac{1}{k_0} < \frac{\epsilon}{2}$ and :

$$(1) \qquad \forall\ k \geq k_0 :\ |F(f_n) - F^k(f_n)| < \frac{\epsilon}{2}.$$

For $k \geq k_0 : \exists\ j_{k,n} \in \mathbb{N}\ \forall\ j \geq j_{k,n}$:

$$(2) \qquad |F^k(f_n) - \langle f_n(\omega_j^k), x_j^{*k} \rangle| < \frac{1}{k}.$$

Let $k \geq n$, then for $l = j_{k,k} \geq j_{k,n}$ by (2):

$$|F^k(f_n) - \langle f_n(\omega_l), x_l^* \rangle| < \frac{1}{k} < \frac{\epsilon}{2}.$$

Application of (1) gives the claim.

b) According to a), $\mathcal{F}_K$ is w^*-compact in $B(L_K^*)$. Hence, $(\mathcal{F}_K, w^*)$ is a compact metrizable space, since L_K is separable. Thus, by a result of C.Kuratowski's [Ku, p.227] there is a regular Borel measure ν defined on $(\mathcal{F}_K, w^*)$, such that $([0,1], \mathcal{L}, \lambda)$ is measure isomorphic to $(\mathcal{F}_K, \mathcal{B}(w^*), \nu)$. Hence there is a result of Caratheodory an isometry between $L_1[0,1]$ and $L_1(\nu)$ (see e.g. [La, p.128,Cor]). But this gives b). □

In the following ν will denote the above regular Borel measure on $\mathcal{F}_K$.

Theorem 3.5 *For a bounded and separable subset $K \subset L_\infty(\mu, X)$ the following conditions are equivalent:*

a) K is an Asplund set.

b) There is an increasing sequence $(\mathcal{F}_l)$ of w^*- compact subsets of $\mathcal{F}_K$, such that $\lim_{l\to\infty} \nu(\mathcal{F}_l) = 1$ and for all sequences $(f_n)_{n\in\mathbb{N}} \subset K$ there exists a subsequence $(f_{n_m})_{m\in\mathbb{N}}$, such that

$$\lim_{k,m\to\infty} \sup_{F\in\mathcal{F}_l} |F(f_{n_m} - f_{n_k})| = 0 \text{ for all } l \in \mathbb{N}.$$

Proof: According to 1.2.c), K is an Asplund set if and only if for all $T : L_K \longrightarrow L_\infty([0,1])$ and for all $\epsilon > 0$ there is a set $E \subset [0,1]$, $\lambda([0,1] \setminus E) < \epsilon$, so that $\chi_E \circ T$ is compact.

a)⇒b) Let K be a separable Asplund set. Consider

$$\begin{aligned} T: \quad L_K &\longrightarrow L_\infty(\nu) \ (\cong L_\infty([0,1])), \\ f &\longmapsto (F \longmapsto F(f)) \end{aligned}$$

By 1.2.c) and since ν is a regular Borel measue on $(\mathcal{F}_K, w^*)$ we have

$$\forall\ l \in \mathbb{N}\ \exists\ \mathcal{F}_l \subset \mathcal{F}_K,\ \nu(\mathcal{F}_l) \geq 1 - \frac{1}{l} :\ \chi_{\mathcal{F}_l} \circ T \text{ is compact on } K.$$

Since $(\mathcal{F}_l)$ may be chosen to be increasing, b) follows immediately.
b)$\Rightarrow$a) Let $T : L_K \longrightarrow L_\infty[0,1]$ be given. Then, according to 3.4 b) let $I : L_\infty[0,1] \longrightarrow L_\infty(\nu)$ be the isometry, which is a w*-w*-homeomorphism. By b) conclude:

$$\exists\ (\mathcal{F}_l) \text{ increasing },\ \lim_{l\to\infty} \nu(\mathcal{F}_l) = 1 :\ \forall\ l \in \mathbb{N} :$$

$$\chi_{\mathcal{F}_l} \circ I \circ T \text{ is compact on } K.$$

The property of I in combination with 1.2.c) shows immediately that K is an Asplund set. □

Theorem 3.6 *For a bounded and separable subset $K \subset L_\infty(\mu, X)$ the following conditions are equivalent:*

a) K is conditionally weakly compact.

b) For all sequence $(f_n)_{n\in\mathbb{N}} \subset K$ there exist a subsequence $(f_{n_m})_{m\in\mathbb{N}}$ an increasing sequence $(\mathcal{F}_l)$ of w^*- compact subsets of $\mathcal{F}_K$ with $\lim_{l\to\infty} \nu(\mathcal{F}_l) = 1$ such that

$$\lim_{k,m\to\infty} \sup_{F\in\mathcal{F}_l} |F(f_{n_m} - f_{n_k})| = 0 \textit{ for all } l \in \mathbb{N}.$$

Proof: a)$\Rightarrow$b): Let K be conditionally weakly compact. Let T be defined as in 3.5 a$\Rightarrow$b). Then, by 1.1.c) for all sequences $(f_n) \subset K$, there is a subsequence (f_{n_m}), so that $T(f_{n_m})$ converges pointwise. Using Egorov's theorem and the regularity of ν, there is an $l \in \mathbb{N}$ with a w*-compact set $\mathcal{F}_l, \nu(\mathcal{F}_l) \geq 1 - \frac{1}{l}$, so that $(\chi_{\mathcal{F}_l} \circ T)(f_{n_m})$ converges uniformly. Since $(\mathcal{F}_l)$ may be chosen to be increasing, b) follows.
b)$\Rightarrow$a) Use 1.1.c). For this purpose suppose $T : L_K \longrightarrow L_\infty([0,1])$. Let $(f_n) \subset K$ be a sequence and I be the isometry between $L_\infty[0,1]$ and $L_\infty(\nu)$ (see the proof of 3.5 b)$\Rightarrow$a)). Then according to b) there is a subsequence (f_{n_m}), such that $(I \circ T)(f_{n_m})$ converges a.e.. The property of I and the regularity of ν implies the a.e. convergence of $T(f_{n_m})$ and a) is proved. □

Remark 3.7:

a) It should be mentioned that 2.6, 2.7 and the results in section 3 can be formulated for $L_1(\mu, X)^*$, where the functions $f \in L_1(\mu, X^*, X)$ depends heavily on the parametrization formulated e.g. in 2.1.

b) In addition all the results can be extended to some measure spaces, which are not necessarily finite. The key theorem 2.1, which stated the parametrization, is true as long as (Ω, Σ, μ) is localizable (see e.g [Din] for the definition of *localizable* and [S2] for the technique to extend the result).

References

[AU] Andrews,K.T.,Uhl,J.J.: *On weak compactness in $L_\infty(\mu, X)$*, Indiana Univ. Math. J. **30,6** (1981), p.907-915

[BH] Batt,J.,Hiermeyer,W.: *On compactness in $L_p(\mu, X)$ in the weak topology and in the topology $\sigma(L_p(\mu, X), L_q(\mu, X^*))$*, Math.Z. **182** (1983),p.409-432

[Bo] Bourgin,R.: *Geometric aspects of convex sets with the Radon-Nikodým Property,* Lecture Notes in Mathematics **993** (1983)

[Di] Diestel,J.: *Sequences and Series in Banach Spaces,* Springer Berlin/New York (1984)

[Din] Dinculeanu,N.: *Vector Measures,* VEB Deutscher Verlag der Wissenschaften, Berlin (1966)

[DRS] Diestel,J.,Ruess,W.,Schachermayer,W.: *On weak compactness in $L_1(\mu, X)$,* Proc. Amer. Math. Soc. **118** (1993), p.447-453

[DU] Diestel,J.,Uhl,,J.: *Vector Measures,* AMS Math.Survey **15**, (1977)

[Ku] Kuratowski,C.: *Topologie I,* Monografje Matematyczne Vol.3, Warsawa (1933)

[La] Lacey,H.E.: *The isometric theory of classical Banach spaces,* Springer, Berlin (1974)

[RSU] Riddle,L.H.,Saab,E.,Uhl,J.J.: Sets with the weak Radon-Nikodým property in dual Banach spaces, Ind.Univ.J.Math. **32,4** (1983), p.527-541

[RU] Riddle,L.H.,Uhl,J.J.: *Martingales and the fine line between Asplund spaces and spaces not containing a copy* ℓ_1, in: Martingale Theory in Harmonic Analysis and Banach spaces, Lecture Notes in Mathematics **939** (1983) p.145-156

[S1] Schlüchtermann,G.: *Renorming in the space of Bochner integrable functions* $L_1(\mu, X)$, Manus. Math. **73** (1991), p.397-409

[S2] Schlüchtermann,G.: *On weakly compact operators,* Math.Ann. **292** (1992), p.263-266

[S3] Schlüchtermann,G.: *Weak Compactness in* $L_\infty(\mu, X)$, J.Funct.Anal. **125,2**,(1994), p.379-388

[Ta] Talagrand,M.: *Weak Cauchy sequences in* $L^1(E)$, Amer. J. Math. **106** (1984), p.703-724