WEAK COMPACTNESS CRITERIA IN SYMMETRIC SPACES OF MEASURABLE OPERATORS

by

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Abstract *The principal result of the paper reduces the study of certain weakly compact sets in Banach spaces of measurable operators to that of the corresponding sets of generalised singular value functions. In particular, under natural conditions, it is shown that the orbit of a relatively weakly compact subset of the Köthe dual of a symmetric space of measurable operators affiliated with some semifinite von Neumann algebra is again relatively weakly compact.*

0. Introduction The study of weak compactness in Banach spaces of measurable functions is a classical theme. Central results in the structure and characterisation of weakly compact sets in spaces of type $L^1(\mu)$ go back to the seminal work of Grothendieck [Gr] and Dunford-Pettis [DS]. A central role is played here by the order structure, in particular, through varios characterisations in terms of the behaviour of disjoint sequences. Refinements of methods based on disjoint sequence techniques lead naturally to various characterisations of weakly compact sets in the more general Banach lattice setting [BVL], [MN]. On the other hand, the classical Banach spaces of measurable functions generally admit not only a lattice structure, but are rearrangement-invariant. The study of the structure of weakly compact sets in Köthe spaces of measurable functions from the point of view of rearrangement-invariance seems to have been initiated by Fremlin [Fr], for spaces which are exact interpolation spaces for some pair $(L^1(\mu), L^!\infty(\mu))$. A central consequence ([Fr], Section 28) of this study, is that, under natural conditions, the orbit (see Proposition 2.1 below) of a relatively weakly compact set in the Köthe dual of a rearrangement invariant space of measurable functions is itself relatively weakly compact. The results of [Fr] were subsequently refined and extended to a very wide class of rearrangement-invariant spaces by Chilin, Sedaev and Sukochev [CSS]. In this study, the essential underlying duality is expressed in terms of the natural pairing of Lorentz and Marcinkiewicz spaces, rather than the classical (L^1, L^∞) pairing. In this setting, a fundamental role is played by the theory of submajorisation in the sense of Hardy, Littlewood and Polya [BS], [KPS].

A precursor to the work of Fremlin is the study of weak compactness in symmetric sequence spaces by Garling [Ga1]. This work led to the further study of weakly compact sets [Ga2] in ideals of operators in Hilbert space, with a general setting that included the familiar Schatten ideals. The principal result of [Ga2] reduces the study of weakly compact subsets of certain ideals of compact operators to the study of the corresponding subsets of singular value sequences in symmetric sequence spaces. The settings considered by Garling and Fremlin may be viewed as special cases of a general approach given by the theory of non-commutative symmetric spaces of measurable operators in which the classical duality for spaces of type $L^\infty(\mu), L^1(\mu)$ is replaced by the duality between a semi-finite von Neumann algebra and its predual. This approach is based on the classical theory of rearrangement-invariant Banach spaces of measurable functions, the theory of non-comm! utative integration introduced by Segal (see [Ne]), and the interplay of these themes in the systematic development of the theory of Banach spaces of measurable operators. While many authors have

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contributed to this development, we shall adopt here the approach given in [DDP1,4] where the necessary basic ideas concerning duality in the sense of Köthe are established in the non-commutative setting.

The principal result of the the present paper is a simultaneous extension of the work of [Fr] and [Ga2] to the setting of symmetric Banach spaces of measurable operators affiliated with a semifinite von Neumann algebra. In particular, under natural conditions, we show (Theorem 2.13) that the orbit of a relatively weakly compact subset of the Köthe dual of some symmetric Banach space of measurable operators is again relatively weakly compact, and that the study of weakly compact subsets in the non-commutative setting can be reduced to the study of the corresponding sets of generalised singular value functions (in the sense of Fack and Kosaki [FK]) in rearrangement-invariant spaces of measurable functions on the positive half-line. These ideas have been explored earlier [DDP2], [GS]. However, the results given in these sources either apply explicitly only to the case that the underlying trace is finite [GS] or are restricted in applicability [DDP2] by additional ass! umptions in the case of infinite trace. In particular, while the results stated in [DDP2] contain as special cases the corresponding results for trace ideals given by [Ga2], they do not recover the full generality given by Fremlin when specialised to the commutative case. Moreover, one of the principal steps given in [DDP2] Proposition 2.6 appears to contain a gap, which is removed by the approach of the present paper. The approach that we follow here is direct and is based on non-commutative disjoint sequence techniques that elaborate similar methods given in [Su1], [CDS], and which are of independent interest in their own right.

1. Preliminaries We denote by $\mathcal{M}$ a semifinite von Neumann algebra on the Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. The identity in $\mathcal{M}$ is denoted by $\mathbf{1}$, and we denote by $\mathcal{M}^p$ the set of all projections in $\mathcal{M}$. A linear operator x:dom$(x) \to \mathcal{H}$, with domain dom$(x) \subseteq \mathcal{H}$, is called *affiliated with* $\mathcal{M}$ if $ux = xu$ for all unitary u in the commutant $\mathcal{M}'$ of $\mathcal{M}$. The closed and densely defined operator x, affiliated with $\mathcal{M}$, is called *τ-measurable* if for every $\epsilon > 0$ there exists $p \in \mathcal{M}^p$ such the $p(\mathcal{H}) \subseteq$dom(x) and $\tau(\mathbf{1} - p) < \epsilon$. With the sum and product defined as the respective closures of the algebraic sum and product, $\widetilde{\mathcal{M}}$ is a *-algebra. Equipped with the measure topology (see [FK]), $\widetilde{\mathcal{M}}$ is a complete topological *-algebra. These facts and their proofs can be found in the papers [Ne] and [Te]. For standard facts concerning von Neumann algebras, we refer to [SZ],[Ta].

We recall the notion of generalized singular value function [FK]. Given a self-adjoint operator x in $\mathcal{H}$ we denote by $e^x(\cdot)$ the spectral measure of x. Now assume that $x \in \widetilde{\mathcal{M}}$. Then $e^{|x|}(B) \in \mathcal{M}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(e^{|x|}(s, \infty)) < \infty$. For $x \in \widetilde{\mathcal{M}}$ and $t \geq 0$ we define

$$\mu_t(x) = \inf\{s \geq 0 : \tau(e^{|x|}(s, \infty)) \leq t\}.$$

The function $\mu(x) : [0, \infty) \to [0, \infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of x; note that $\mu_t(x) < \infty$ for all $t > 0$. Suppose that $\alpha > 0$. If we consider $\mathcal{M} = L^\infty([0, \alpha), m)$, where m denotes Lebesgue measure on the interval $[0, \alpha)$, as an Abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2([0, \alpha), m)$, with the trace given by integration with respect to m, it is easy to see that $\widetilde{\mathcal{M}}$ consists of all measurable functions on $[0, \alpha)$ which are bounded except on a set of finite measure. Further, if $f \in \widetilde{\mathcal{M}}$, then the generalized singular value function $\mu(f)$ is precisely the classical non-increasing rearrangement of the function $|f|$. On the other hand, if $(\mathcal{M}, \tau)$ is the space of all bounded linear operators in some Hilbert space equipped with the canonical trace tr, then $\widetilde{\mathcal{M}} = \mathcal{M}$ and, if $x \in \mathcal{M}$ is compact, then the generalised singular value function $\mu(x)$ may be identified in a natural manner with the sequence $\{\mu_n(x)\}_{n=0}^\infty$ of singular values of $|x| = \sqrt{x^*x}$, repeated according to multiplicity and arranged in non-increasing order.

By $L^0([0,\alpha),m)$ we denote the space of all $\mathbb{C}$-valued Lebesgue measurable functions on the interval $[0,\alpha)$ (with identification m-a.e.). A Banach space $(E,\|\cdot\|_E)$, where $E\subseteq L^0([0,\alpha),m)$ is called a *rearrangement-invariant Banach function space* on the interval $[0,\alpha)$, if it follows from $f\in E, g\in L^0([0,\alpha),m)$ and $\mu(g)\le\mu(f)$ that $g\in E$ and $\|g\|_E\le\|f\|_E$. If $(E,\|\cdot\|_E)$ is a rearrangement-invariant Banach function space on $[0,\alpha)$, then E is said to be *symmetric* if $f,g\in E$ and $g\prec\prec f$ imply that $\|g\|_E\le\|f\|_E$. Here $g\prec\prec f$ denotes submajorization in the sense of Hardy-Littlewood-Polya :

$$\int_0^t \mu_s(g)ds \le \int_0^t \mu_s(f)ds, \quad \text{for all } t>0.$$

If $(E,\|\cdot\|_E)$ is a symmetric Banach function space on $[0,\alpha)$, then E will be called *fully symmetric* if $f\in E, g\in L^0([0,\alpha),m)$ and $g\prec\prec f$ implies $g\in E$ and $\|g\|_E\le\|f\|_E$. The general theory of rearrangement-invariant spaces may be found in [KPS], BS] and [LT].

Given a semifinite von Neumann algebra $(\mathcal{M},\tau)$ and a rearrangement-invariant Banach function space $(E,\|\cdot\|_E)$ on $([0,\tau(\mathbf{1}),m)$, we define the non-commutative space $E(\mathcal{M},\tau)$ by setting

$$E(\mathcal{M},\tau):=\{x\in\widetilde{\mathcal{M}}:\mu(x)\in E\},\quad \|x\|_{E(\mathcal{M},\tau)}:=\|\mu(x)\|_E.$$

If E is symmetric, or if $\mathcal{M}$ is non-atomic, then the space $(E(\mathcal{M},\tau),\|\cdot\|_{E(\mathcal{M},\tau)}$ is a Banach space and is called the symmetric (respectively, rearrangement-invariant) operator space associated with $(\mathcal{M},\tau)$ corresponding to $(E,\|\cdot\|_E)$. The Köthe dual $E(\mathcal{M},\tau)^\times$ is defined to be the set of all $x\in\widetilde{\mathcal{M}}$ such that $xy\in L^1(\mathcal{M},\tau)$ for all $y\in E(\mathcal{M},\tau)$. With the norm defined by setting

$$\|x\|_{E(\mathcal{M},\tau)^\times}:=\sup\{\tau(|xy|)\ :\ y\in E(\mathcal{M},\tau),\ \|y\|_{E(\mathcal{M},\tau)}\le 1\},\quad x\in E(\mathcal{M},\tau)^\times,$$

the Köthe dual $E(\mathcal{M},\tau)^\times$ is a Banach space. Basic properties of Köthe duality, in the commutative setting may be found in [KPS], [BS] (where the Köthe dual is called the *associate* space). In the non-commutative setting, the reader is referred to [DDP4], where it is shown in particular that if E is symmetric, then the space $(E(\mathcal{M},\tau)^\times,\|\cdot\|_{(E(\mathcal{M},\tau)^\times})$ may be identified with the space $(E^\times(\mathcal{M},\tau),\|\cdot\|_{E^\times(\mathcal{M},\tau)})$.

We shall need the following notation. We set

$$\widetilde{\mathcal{M}}_0:=\{x\in\widetilde{\mathcal{M}}\ :\ \mu_t(x)\to 0 \text{ as } t\to\infty\},\quad L_0(\mathcal{M},\tau):=\widetilde{\mathcal{M}}_0\cap\left(L^1(\mathcal{M},\tau)+\mathcal{M}\right),$$

and make the obvious remark that $\widetilde{\mathcal{M}}_0$ coincides with $\widetilde{\mathcal{M}}$ if $\tau(\mathbf{1})<\infty$. For brevity, we denote the space $L_0(L^\infty([0,\infty),m)$ by $L_0[0,\infty)$. By [DDP4] Proposition 2.7, the closure of the space $L^1(\mathcal{M},\tau)\cap\mathcal{M}$ in the space $L^1(\mathcal{M},\tau)+\mathcal{M}$ coincides with $L_0(\mathcal{M},\tau)$. Moreover, for any rearrangement-invariant Banach function space E on $[0,\tau(\mathbf{1}))$, the inclusions

$$L^1(\mathcal{M},\tau)\cap\mathcal{M}\subseteq E(\mathcal{M},\tau),\ \ E^\times(\mathcal{M},\tau)\subseteq L^1(\mathcal{M},\tau)+\mathcal{M}.$$

hold with continuous embeddings. It is clear that the embedding $E\subseteq L_0[0,\infty)$ implies the embedding $E(\mathcal{M},\tau)\subseteq\widetilde{\mathcal{M}}_0$.

2. Weak compactness criteria

The following characterisation of weak compactness in rearrangement-invariant Banach function spaces extends a similar criterion due to Fremlin [Fr] in a more restricted setting. We recall first [KPS] that if E is a rearrangement-invariant Banach function space on $[0,\alpha)$, the the *fundamental function* φ_E is defined by setting

$$\varphi_E(t) := \|\chi_{[0,t)}\|_E \quad t \in [0,\alpha).$$

Proposition 2.1 *Let E be a rearrangement-invariant Banach function space on $[0,\alpha)$, $0 < \alpha \leq \infty$. Let $K \subset E^\times$ be bounded. If either $\alpha < \infty$, or $\alpha = \infty$ and $\varphi_E(\infty) = \infty$, then the following properties are equivalent:*

(i) *K is relatively $\sigma(E^\times, E)-$compact;*

(ii) *For each $f \in E$ and sequence $\{Q_n\}_{n\in\mathbb{N}} \subseteq [0,\alpha)$ of measurable subsets with $Q_n \downarrow \emptyset$,*

$$\sup_{g\in K} |\int_{Q_n} fgdm| \to 0.$$

(iii) *$\mu(K) := \{\mu(g); g \in K\}$ is relatively $\sigma(E^\times, E)$-compact;*

(iv) *For every $f \in E$ and sequence $\{f_n\} \subseteq E$ with $f_n \prec\prec f$, $n \in \mathbb{N}$ and $f_n \to 0$ in measure*

$$\sup\left\{\int_{[0,\alpha)} \mu_t(f_n)\mu_t(g)dt : g \in K\right\} \to_n 0.$$

(v) *$Orb(K) := \{h \in L^1 + L_\infty;\ h \prec\prec g$ for some $g \in K\}$ is relatively $\sigma(E^\times, E)$-compact.*

Proof The equivalence of all statements except that given in (iv) is proved in [CSS], Theorem 2. It suffices to show the equivalence of each with the assertion of (iv). To show that (iii)$\Rightarrow$ (iv), we follow an argument of AA Sedaev. Let $0 \leq f \in E$ and let $\epsilon > 0$ be given. Assume first that $\alpha = \infty$ and that $\varphi_E(\infty) = \infty$. Using the equivalence (ii)$\Leftrightarrow$ (iii) applied to $\mu(K)$, there exists $r > 0$ such that

$$\sup_{g\in K} \int_{[0,\alpha)} \mu_t(g)\mu_t(f)\chi_{[0,r)\cup[\frac{1}{r},\infty)}(t)dt < \epsilon.$$

Since $\mu_t(f_n) \to 0$ as $n \to \infty$, there exists $N(r) \in \mathbb{N}$ such that, for all $n \geq N(r)$,

$$\mu_r(f_n) < \epsilon/C(r),$$

where

$$C(r) = \left(\sup_{g\in K} \|g\|_{E^\times}\right) \varphi_E\left(\frac{1}{r}\right).$$

We obtain

$$\begin{aligned}\sup_{g\in K} \int_{[0,\alpha)} \mu_t(f_n)\mu_t(g)dt &\leq \sup_{g\in K} \int_{[0,r)\cup[\frac{1}{r},\infty)} \mu_t(f_n)\mu_t(g)dt + \sup_{g\in K} \int_{[r,\frac{1}{r})} \mu_t(f_n)\mu_t(g)dt \\ &\leq \epsilon + \mu_r(f_n)\|\chi_{[r,\frac{1}{r})}\|_E \sup_{g\in K} \|g\|_{E^\times} \\ &\leq 2\epsilon\end{aligned}$$

for all $n \geq N(r)$. This suffices to establish the implication in the case that $\alpha = \infty$ and the proof in the case that $\alpha < \infty$ is almost identical.

The chain of equivalences will be complete by showing the implication (iv)$\Rightarrow$ (ii). Assume then that $f \in E$ and that $\{Q_n\}_{n\in\mathbb{N}}$ is a sequence of measurable subsets of $[0,\alpha)$ for which $Q_n \downarrow \emptyset$. Note first, that if $\alpha = \infty$ and $\varphi_E(\infty) = \infty$, then $\chi_{[0,\infty)} \notin E$ and so $E \subseteq L_0[0,\infty)$. If $\alpha < \infty$, then also $E \subseteq L_0[0,\infty)$. In both cases, it follows that $f\chi_{Q_n} \to 0$ in measure. Since $f\chi_{Q_n} \prec\prec f$ for $n \in \mathbb{N}$, the assertion of (ii) now follows from the inequality

$$\sup_{g\in K} \Big| \int_{[0,\alpha)} f(t)g(t)\chi_{Q_n}(t)dt \Big| \leq \sup_{g\in K} \int_{[0,\alpha)} \mu_t(g)\mu_t(f\chi_{Q_n})dt,$$

and this suffices to complete the proof. □

2.2. Remark: It is worth noting that the implication (iii)$\Rightarrow$ (i) may fail if the condition $\varphi_E(\infty) = \infty$ is omitted in the case that $\alpha = \infty$. In fact, one need only take E to be $L^\infty[0,\infty)$ and the set K to be the sequence $\{\chi_{[n,n+1]}; n \in \mathbb{N}\} \subseteq E^\times = L^1[0,\infty)$.

The equivalence of (i) and (ii) is well known for Köthe function spaces [BVL]. Consequently the real depth of [CSS] Theorem 2 lies in the implication $(i) \Rightarrow (v)$. For fully symmetric spaces, this implication is due to DH Fremlin [Fr]. However, as noted in [CSS], it follows from a result of Sedaev [Se] that there exist rearrangement invariant Banach function spaces that do not admit any equivalent fully symmetric norms. These examples show that Proposition 2.1 yields a proper and non-trivial extension of Fremlin's result. The implication $(i) \Rightarrow (iv)$ appears in [DDP2], Proposition 2.6. However, the proof given there contains a gap.

The main purpose of this paper is to extend Proposition 2.1 to the non-commutative setting. We begin with some auxiliary results, which may be of independent interest.

Proposition 2.3 *Let E be a symmetric Banach function space on $[0,\tau(\mathbf{1}))$, and let $K \subseteq E^\times(\mathcal{M},\tau)$ be bounded. If $K \subseteq E^\times(\mathcal{M},\tau)$ is relatively $\sigma(E^\times(\mathcal{M},\tau), E(\mathcal{M},\tau))$-compact, then*

$$\lim_{n\to\infty} \sup_{y\in K} |\tau(ye_n x)| \to 0$$

for every sequence $\{e_n\}_{n=1}^\infty$ of mutually orthogonal projections in $\mathcal{M}^p$ and every $x \in E(\mathcal{M},\tau)$.

Proof The Proposition follows immediately from the fact that xK is relatively $\sigma(L^1(\mathcal{M},\tau),\mathcal{M})$-compact in $L^1(\mathcal{M},\tau)$ and [SZ, Theorem 5.14]. □

We recall, that if $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, then $\mathcal{N}$ will be called **proper** if the restriction $\tau_{\mathcal{N}}$ of τ to $\mathcal{N}$ is again semi-finite. We shall repeatedly use the following proposition which follows from an inspection of the proof of [DDP3], Theorem 3.5.

Proposition 2.4 *Let $\mathcal{M}$ be non-atomic, and let $0 \leq x \in \widetilde{\mathcal{M}}$. There exists a proper non-atomic commutative von Neumann subalgebra $\mathcal{N}$ in $\mathcal{M}$ with $x \in \widetilde{\mathcal{N}}$ and there exists a trace preserving $*$-isomorphism ϕ between $L^\infty[0,\tau(\mathbf{1}))$ and $\mathcal{N}$ such that, if $\tilde{\phi}$ denotes the natural extension of ϕ to $(L^1 + L_\infty)[0,\tau(\mathbf{1}))$, then $\tilde{\phi}(\mu(x)) = x$ and for every $0 < s \leq \tau(\mathbf{1})$ there exists a projection $p \in \mathcal{M}$ such that*

$$p \leq r(x), \quad px = xp \quad \text{and} \quad \mu(px) = \mu(\mu(x)\chi_{[0,s)}),$$

where $r(x)$ is the right support projection of x.

The following assertion coincides with [CDS, Lemma 2.4]. Its proof follows from the fact that the projections $f - f \wedge (\mathbf{1} - e), f \vee (\mathbf{1} - e) - (\mathbf{1} - e)$ are equivalent, together with the equality

$$f \vee (\mathbf{1} - e) - (\mathbf{1} - e) = (f \vee (\mathbf{1} - e)) \wedge e.$$

Lemma 2.5 *If $e, f \in \mathcal{M}^p$, and if $\tau(f) < \infty$, then*

$$\tau(f \wedge (\mathbf{1} - e)) \geq \tau(f) - \tau(e).$$

The next lemma follows from [GI, Lemma 1.9 (iii)].

Lemma 2.6 *If $(\mathcal{M}, \tau)$ is a finite von Neumann algebra, if $0 \leq x \in L^1(\mathcal{M}, \tau)$ and if $p \in \mathcal{M}^p$, then*

$$\mu(x)\chi_{[\tau(1)-\tau(p),\tau(1))} \prec\prec pxp.$$

Theorem 2.7 *Let E be a symmetric Banach function space on $[0, \tau(\mathbf{1}))$ with $\tau(\mathbf{1}) < \infty$, and let $K \subseteq E^\times(\mathcal{M}, \tau)$ be bounded. If $(\mathcal{M}, \tau)$ is non-atomic, then the following statements are equivalent.*

(i) *K is relatively $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$-compact.*

(ii) *$\mu(K) := \{\mu(x) : x \in K\} \subseteq E^\times$ is relatively $\sigma(E^\times, E)$-compact.*

Proof Suppose that K is relatively $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$- compact. If $\mu(K)$ is not relatively $\sigma(E^\times, E)$-compact, then it follows from [BVL] (see also Proposition 2.1) that there exist $\epsilon > 0$, a sequence $\{x_n\}_{n=1}^\infty \subseteq K$, a sequence $\{e_n\}_{n=1}^\infty$ of mutually disjoint subsets of $[0, \tau(\mathbf{1}))$ and $g \in E$ such that

$$\epsilon < |\int_{[0,\tau(\mathbf{1})} \mu_t(x_n) g(t) \chi_{e_n}(t) dt| \leq \int_{[0,\tau(\mathbf{1}))} \mu_t(\mu(x_n)\chi_{e_n}) \mu_t(g\chi_{e_n}) dt, \tag{2.0}$$

It may be clearly assumed that $g \in E$ is non-negative. From Proposition 2.4(ii), we may assume that there exists a sequence of projections $\{p_n'\}_{n=1}^\infty$ such that

$$|x_n| p_n' = p_n' |x_n|, \quad p_n' \leq r(|x_n|), \quad \tau(p_n') = m(e_n), \quad n \geq 1 \tag{2.1}$$

and

$$\mu(|x_n| p_n') = \mu\left(\mu(x_n)\chi_{e_n}\right), \quad n \geq 1. \tag{2.2}$$

Further, for each $n \geq 1$, there exists $y_n \in L^1(p_n' \mathcal{M} p_n', \tau(p_n' \cdot p_n'))$ such that

$$y_n \geq 0, \quad y_n |x_n| = |x_n| y_n, \quad \mu(y_n) = \mu(g\chi_{e_n}), \quad \mu(p_n' |x_n| p_n' y_n) = \mu(\mu(x_n) g \chi_{e_n}). \tag{2.3}$$

Since $\tau(p'_n) \to 0$, it follows that $y_n p'_n |x_n| p'_n = y_n p'_n u_n^* x_n \to 0$ for the measure topology. Here, $x_n = u_n |x_n|$, $n = 1, 2, \dots$ is the usual polar decomposition. Via [Su1] Proposition 2.2 and passing to a subsequence and relabelling if necessary, there exist two sequences of mutually orthogonal projections $\{p_n\}_{n=1}^\infty$, $\{q_n\}_{n=1}^\infty$ such that

$$\|y_n p'_n u_n^* x_n - q_n y_n p'_n u_n^* x_n p_n\|_{L_1(p\mathcal{M}p, \tau(p\cdot p))} \to 0 \tag{2.4}$$

where $p := \bigvee_{n\ge 1} p'_n$. It follows from (2.4) that

$$\tau(y_n p'_n u_n^* x_n) - \tau(q_n y_n p'_n u_n^* x_n p_n) \to 0$$

or, equivalently, that

$$\tau(p'_n |x_n| p'_n y_n) - \tau(x_n p_n q_n y_n p'_n u_n^*) \to 0. \tag{2.5}$$

¿From (2.3), it follows that

$$\tau(p'_n |x_n| p'_n y_n) = \int_{[0,\tau(\mathbf{1}))} \mu_t(x_n) g \chi_{e_n}(t) dt,$$

and so from (2.0) and (2.5), it follows that

$$\tau(x_n p_n q_n y_n p'_n u_n^*) \not\to 0. \tag{2.6}$$

For brevity, set $z_n := q_n y_n p'_n u_n^*$, $n \ge 1$, and observe that $x_n p_n z_n \to 0$ for the measure topology. Again applying [Su1] Proposition 2.2, and passing to a subsequence and relabelling if necessary, we may select two sequences of mutually orthogonal projections $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ such that

$$\|x_n p_n z_n - a_n x_n p_n z_n b_n\|_{L_1(p\mathcal{M}p, \tau(p\cdot p))} \to 0 \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\tau(x_n p_n z_n b_n a_n) = \tau(a_n x_n p_n z_n b_n) \not\to 0. \tag{2.8}$$

We now let

$$Z_n := p_n z_n b_n a_n, \ n \ge 1 \quad \text{and} \quad Z := \sum_{n=1}^\infty Z_n$$

where the latter series converges in $L_1(\mathcal{M}, \tau)$. Indeed, from (2.3) we obtain that

$$\mu(Z_n) = \mu(p_n z_n b_n a_n) \le \mu(z_n) = \mu(q_n y_n p'_n u_n^*) \le \mu(y_n) = \mu(g\chi_{e_n}). \tag{2.9}$$

Since the sequence $\{Z_n\}_{n=1}^\infty = \{p_n z_n b_n a_n\}_{n=1}^\infty$ consists of left and right disjointly supported elements, it follows that

$$|\sum_{n=N}^M Z_n| = \sum_{n=N}^M |Z_n|$$

for all $1 \le N < M$. Using (2.9), let us now observe that

$$\begin{aligned}
\tau\left(e^{\sum_{n=N}^M |Z_n|}(s,\infty)\right) &= \sum_{n=N}^M \tau\left(e^{|Z_i|}(s,\infty)\right) \\
&= \sum_{n=N}^M m(\mu(Z_i) > s) \\
&\le \sum_{n=N}^M m\left(\mu(g\chi_{e_n}) > s\right) \\
&\le \sum_{n=N}^M m\left(g\chi_{e_n} > s\right) \\
&= m\left(g\chi_{\cup_{n=N}^M e_n} > s\right).
\end{aligned}$$

¿From this and from the definition of the generalised singular value function, it follows that

$$\mu(\sum_{n=N}^{M} Z_n) = \mu(\sum_{n=N}^{M} |Z_n|) \leq \mu(\sum_{n=N}^{M} g\chi_{\cup_{n=N}^{M} e_n}),$$

whenever $1 \leq N < M$. Since $\tau(\mathbf{1}) < \infty$, it follows that $g \in L^1([0, \tau(\mathbf{1}))$, and so

$$\|\sum_{n=N}^{M} Z_n\|_{L_1(\mathcal{M},\tau)} \leq \|\sum_{n=N}^{M} g\chi_{\cup_{n=N}^{M} e_n}\|_{L_1([0,\tau(\mathbf{1}))} \to 0, \quad \text{as} \quad N, M \to \infty.$$

Consequently,

$$\mu_t(Z) = \lim_{N\to\infty} \mu_t(\sum_{n=1}^{N} Z_n) \leq \mu_t(g), \quad \forall t > 0,$$

whence $Z \in E(\mathcal{M}, \tau)$. It follows from (2.8) that

$$\tau(x_n p_n Z) = \tau(x_n p_n Z_n) = \tau(x_n p_n z_n b_n) \nrightarrow 0.$$

Via Proposition 2.3, this yields a contradiction to the assumption that K is relatively $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$-compact.

Conversely, if K is a bounded subset of $E^\times(\mathcal{M}, \tau)$ which is not relatively $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$- compact, then it follows from [DDP2] Proposition 2.3 that there exists a sequence $\{e_n\}_{n=1}^\infty$ of mutually orthogonal projections in $\mathcal{M}^p, 0 \leq x \in E(\mathcal{M}, \tau)$ and a sequence of elements $\{y_n\}_{n=1}^\infty \subseteq K$ and $\epsilon > 0$ such that

$$\max\{|\tau(e_n x y_n)|, |\tau(y_n x e_n)|\} > \epsilon. \tag{2.10}$$

It follows from (2.10) , [CS] Proposition 1.1 and [KPS] II.2.36 that

$$\int_{[0,\tau(\mathbf{1}))} \chi_{[0,\tau(e_n))}\mu_t(x)\mu_t(y_n)dt > \epsilon, \quad n \geq 1.$$

Since $\tau(e_n) \to 0$, it follows from Proposition 2.1 (ii), that $\mu(K)$ is not relatively $\sigma(E^\times, E)$ compact, and this suffices to complete the proof of the Theorem. $\square$

The removal of the assumption that $\tau(\mathbf{1}) < \infty$ in Theorem 2.7 preceding is based on the construction of a sequence of mutually orthogonal projections. As the construction is technical and of independent interest, we present the details in the following Lemma.

Lemma 2.8 *Let $\{x_n\}_{n\geq 1} \subseteq \widetilde{\mathcal{M}}$ and let $\{p'_n\}_{n\geq 1} \subseteq \mathcal{M}$ be any sequence of projections such that*

$$\tau(p'_n) \geq 2\sum_{i=1}^{n-1} \tau(p'_i), \quad |x_n|p'_n = p'_n|x_n|, \quad p'_n \leq r(|x_n|), \quad n \geq 1.$$

If $x_n = u_n|x_n|$ is the polar decomposition, then there exists a sequence $\{p_n\}_{n\geq 1}$ of mutually orthogonal projections such that

$$p_n \leq p'_n, \quad \tau(p_n) \geq \tau(p'_n) - 2\sum_{i=1}^{n-1} \tau(p'_i), \quad r(p_n u_n^*) \perp r(p_m u_m^*), \quad n, m \geq 1, n \neq m,$$

Proof For convenience, we set $\alpha_n = \tau(p'_n), n \geq 1$. Observing that u_n^* is a partial isometry with initial space the left support $l(x_n)$ of x_n and with final space the right support $r(x_n)$ of x_n, it follows that, for any subprojection $p' \leq p'_n$, the operator $p' u_n^*$ is a partial isometry with left support p'. In particular, for every $n \geq 1$, $p'_n u_n^*$ is a partial isometry with $l(p'_n u_n^*) = p'_n$. We set

$$q_n = \bigvee_{i=1}^{n-1} p'_i, \quad n \geq 1.$$

We define the sequence $\{p_n\}_{n\geq 1}$ inductively as follows. Let $p_1 = p'_1$ and suppose that projections $p_1, p_2, \dots p_{n-1}$ have been constructed. We set

$$\overline{p}_n = p'_n \wedge (\mathbf{1} - q_n),$$

and note that according to Lemma 2.5

$$\tau(\overline{p}_n) = \tau(p'_n \wedge (\mathbf{1} - q_n)) \geq \tau(p'_n) - \tau(q_n) \geq \alpha_n - \sum_{i=1}^{n-1} \tau(p'_i) = \alpha_n - \sum_{i=1}^{n-1} \alpha_i. \tag{2.11}$$

We further let

$$v_n := r(p'_n u_n^*) \wedge (\mathbf{1} - \bigvee_{i=1}^{n-1} r(p_i u_i^*)), \quad n = 1, 2, \dots .$$

We have

$$v_n \perp \bigvee_{i=1}^{n-1} r(p_i u_i^*),$$

and, using Lemma 2.5,

$$\begin{aligned} \tau(v_n) = \tau\left(r(p'_n u_n^*) \wedge (\mathbf{1} - \bigvee_{i=1}^{n-1} r(p_i u_i^*))\right) &\geq \tau(r(p'_n u_n^*)) - \tau(\bigvee_{i=1}^{n-1} r(p_i u_i^*)) \\ &\geq \tau(p'_n) - \sum_{i=1}^{n-1} \tau(p_i) \geq \alpha_n - \sum_{i=1}^{n-1} \alpha_i. \end{aligned} \tag{2.12}$$

We have used here the fact that

$$\tau(r(p'_n u_n^*)) = \tau(l(p'_n u_n^*)) = \tau(p'_n).$$

We now set

$$\overline{\overline{p}}_n := l(u_n^* v_n), \quad p_n := \overline{p}_n \wedge \overline{\overline{p}}_n, \quad n \geq 1.$$

Since

$$l(u_n^* v_n) = u_n^* v_n u_n \leq u_n^* r(p'_n u_n^*) u_n = u_n^*(u_n p'_n u_n^*) u_n = r(x_n) p'_n r(x_n) = p'_n, \quad n \geq 1$$

we see that

$$p_n \leq \overline{\overline{p}}_n \leq p'_n, \quad n \geq 1.$$

Since $p_n \leq \overline{p}_n$, it follows

$$p_n \leq p'_n \wedge (\mathbf{1} - q_n).$$

Thus

$$p_n \perp q_n$$

and since $p'_i \leq q_n$ for all $i = 1, 2, \ldots, n-1$ it follows that $p_n \perp p'_i,\ i = 1, 2, \ldots, n-1$ and further that

$$p_n \perp p_i,\ i = 1, 2, \ldots, n-1.$$

Hence, since

$$r(p_n u_n^*) = u_n p_n u_n^* \leq u_n(l(u_n^* v_n))u_n^* = u_n(u_n^* v_n u_n)u_n^* = l(x_n) v_n l(x_n), \quad n \geq 1$$

and since $v_n \leq r(p'_n u_n^*) = u_n p'_n u_n^* \leq l(x_n),\ n \geq 1$ we infer that

$$r(p_n u_n^*) \leq v_n, \quad n \geq 1$$

and therefore

$$r(p_n u_n^*) \perp \bigvee_{i=1}^{n-1} r(p_i u_i^*).$$

Further, (see (2.11) and (2.12))

$$\overline{p}_n, \overline{\overline{p}}_n \leq p'_n, \quad \tau(\overline{p}_n), \tau(\overline{\overline{p}}_n) \geq \alpha_n - \sum_{i=1}^{n-1} \alpha_i.$$

We shall now show that

$$\tau(p_n) \geq \alpha_n - 2\sum_{i=1}^{n-1} \alpha_i.$$

Without loss of generality, we may assume that

$$p'_n = \mathbf{1}, \quad \tau(\mathbf{1}) = \alpha_n.$$

Since

$$\tau(\overline{p}_n), \tau(\overline{\overline{p}}_n) \geq \alpha_n - \sum_{i=1}^{n-1} \alpha_i$$

we have

$$\begin{aligned}
\tau(\overline{p}_n \wedge \overline{\overline{p}}_n) &= \tau(\overline{p}_n \wedge (\mathbf{1} - (\mathbf{1} - \overline{\overline{p}}_n)) \\
&\geq \tau(\overline{p}_n) - \tau(\mathbf{1} - \overline{\overline{p}}_n) \\
&\geq \left(\alpha_n - \sum_{i=1}^{n-1} \alpha_i\right) - \sum_{i=1}^{n-1} \alpha_i \\
&= \alpha_n - 2\sum_{i=1}^{n-1} \alpha_i.
\end{aligned}$$

This completes the construction of the sequence $\{p_n\}_{n=1}^{\infty}$ and the proof of the Lemma. $\square$

We will now extend the result of Theorem 2.7 to the case that $\tau(\mathbf{1}) = \infty$. Before proceeding, let us note that $L_0[0,\infty)$ is a separable subspace of $(L^1 + L^\infty)[0,\infty)$. This follows from [KPS] II.4.5, Theorem 4.8. Consequently, the norm on $L_0[0,\infty)$ is absolutely continuous [KPS] II 1.3.

Theorem 2.9 *Let E be a rearrangement-invariant, symmetric Banach function space on $[0,\tau(\mathbf{1}))$. If $\mathcal{M}$ is non-atomic and if $K \subseteq E^{\times}(\mathcal{M},\tau)$ is relatively $\sigma(E^{\times}(\mathcal{M},\tau),E(\mathcal{M},\tau))$-compact, then*

$$\mu(K) := \{\mu(x) : x \in K\} \subseteq E^{\times}$$

is relatively $\sigma(E^{\times},E)$-compact.

Proof We assume first that $E \subseteq L_0[0,\infty)$. If $\mu(K)$ is not relatively $\sigma(E^{\times},E)$-compact, then it follows as in the proof of Theorem 2.7 that there exist $\epsilon > 0$, a sequence $\{x_n\}_{n=1}^{\infty} \subseteq K$, a sequence $\{e_n\}_{n=1}^{\infty}$ of mutually disjoint subsets of $[0,\infty)$ and $g \in E$ such that

$$\epsilon < |\int_{[0,\infty)} \mu_t(x_n)g(t)\chi_{e_n}(t)dt| \leq \int_{[0,\infty)} \mu_t(\mu(x_n)\chi_{e_n})\mu_t(g\chi_{e_n})dt, \tag{2.13}$$

for every $n \geq 1$. It may be assumed further that $g \geq 0$, that $m(e_n) < \infty$ for all $n \geq 1$ and that there exists a sequence $\{t_n\}_{n=1}^{\infty}$ of real numbers such that

$$e_n \subseteq [0,t_n), \quad n \geq 1.$$

Since

$$\sum_{j=1}^{\infty} m(e_j \cap [0,t_n)) < \infty, \quad \forall n \geq 1$$

it follows that $\lim_{j\to\infty} m(e_j \cap [0,t_n)) = 0, n \geq 1$. Passing to a subsequence we may therefore assume that

$$\sum_{n=2}^{\infty} m(e_n \cap [0,t_{n-1})) < \infty \tag{2.14}$$

Suppose that $m(e_n) \to_n 0$. By passing to a subsequence and relabelling if neccessary, we may assume that

$$\sum_{n\geq 1} m(e_n) < \infty.$$

As in the proof of the first part of Theorem 2.7, there exist sequences $\{p'_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ which satisfy (2.1),(2.2),(2.3). If we set $q := \bigvee_{n\geq 1} p'_n$, then $\tau(q) < \infty$ and the proof of the implication (i)$\Rightarrow$ (ii) of Theorem 2.7 applies verbatim to the von Neumann algebra $(q\mathcal{M}q, \tau(q\cdot q))$. We may therefore assume that $m(e_n) \not\to 0$ and that there exists $\delta > 0$ such that

$$m(e_n) > \delta, \quad \lim_{j\to\infty} \int_{e_j \cap [0,t_n)} \mu_t(x_j)g(t)dt \to 0, \quad n \geq 1 \tag{2.15}$$

We may further assume that $e_n \subseteq [t_{n-1},t_n), n \geq 2$, and since $m(e_n) > \delta, n = 1,2,\ldots$, it follows that $t_n \to \infty$.

We show now that $m(e_n) \to \infty$. If this is not the case, then there exists a positive number Δ such that

$$\delta < m(e_n) < \Delta, \quad n \geq 1.$$

Observe that

$$\int_{e_n} \mu_t(x_n)g(t)dt \leq \int_{[t_{n-1},t_{n-1}+\Delta)} \mu_t(x_n)\mu_{t-t_{n-1}}(g\chi_{e_n})\,dt = \int_{[t_{n-1},t_{n-1}+\Delta)} \mu_t(x_n)\tilde{g}(t)dt,$$

where

$$\tilde{g} = \sum_{n=1}^{\infty} \mu_{t-t_{n-1}}(g\chi_{e_n}).$$

Note that $\tilde{g}$ is equimeasurable with $g\chi_e$, where $e = \cup_{n=1}^{\infty} e_n$. Since $K \subseteq E^{\times}(\mathcal{M},\tau)$ is bounded, it also follows that K is bounded in $L^1(\mathcal{M},\tau) + \mathcal{M}$; thus, there exists a constant $M > 0$ such that

$$\mu_t(x_n) \leq \int_{[0,1)} \mu_s(x_n) ds \leq M,$$

for all $t \geq 1$ and for all $n \geq 1$. Consequently,

$$\int_{[t_{n-1},t_{n-1}+\Delta)} \mu_t(x_n)\tilde{g}(t)dt \leq M \int_{[t_{n-1},t_{n-1}+\Delta)} \tilde{g}(t)dt \leq M \int_{[0,\Delta)} \mu_t\left(\tilde{g}\chi_{[t_{n-1},\infty)}\right) dt,$$

for all $t_n \geq 1$. Since the norms

$$\int_{[0,\Delta)} \mu_t(\cdot)dt, \quad \|\cdot\|_{(L^1+L^\infty)[0,\infty)} = \int_{[0,1)} \mu_t(\cdot)dt$$

are equivalent, it follows from the absolute continuity of the norm on $L_0[0,\infty)$ that

$$\int_{[0,\Delta)} \mu_t\left(\tilde{g}\chi_{[t_{n-1},\infty)}\right) dt \to_n 0$$

since $t_n \to \infty$. It now follows that

$$\int_{e_n} \mu_t(x_n)g(t)dt \to_n 0.$$

This contradicts (2.13), and so $m(e_n) \to \infty$.

Combining (2.13), (2.15) and passing to a subsequence and relabelling if necessary, we may assume that there exists a sequence of disjoint intervals $\{[s_n,t_n)\}_{n=1}^{\infty}$ with $t_{n-1} \leq s_n < t_n \uparrow_n \infty$ such that

$$\mu_{t_n}(x_n) > 0, \quad \int_{[s_n,t_n)} \mu_t(x_n)g(t)dt > \epsilon, \quad \int_{[s_n,s_n+2\sum_{i=1}^{n-1}\alpha_i)} \mu_t(x_n)g(t)dt < \epsilon/2, \quad n \geq 1, \tag{2.16}$$

where $\alpha_n = t_n - s_n$, $\quad n \geq 1$. By Proposition 2.4(ii), there exist projections $\{p'_n\}_{n=1}^{\infty}$ such that

$$|x_n|p'_n = p'_n|x_n|, \quad \tau(p'_n) = t_n - s_n \ (= \alpha_n), \quad p'_n \leq r(|x_n|)$$

where $r(|x_n|)$ is the right support projection of $|x_n|$, and such that

$$\mu(|x_n|p'_n) = \mu\left(\mu(x_n)\chi_{[s_n,t_n)}\right), \quad n \geq 1. \tag{2.17}$$

Since $\alpha_n \to \infty$, it is clear that we may assume that $\alpha_n > 2\sum_{i=1}^{n-1}\alpha_i$. ¿From Lemma 2.8, there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of mutually orthogonal projections such that

$$p_n \leq p'_n, \quad n \geq 1,$$

such that

$$r(p_n u_n^*) \perp r(p_m u_m^*), \quad m,n \geq 1, \quad m \neq n,$$

and such that

$$\tau(p_n) \geq \alpha_n - 2\sum_{i=1}^{n-1} \alpha_i, \quad n \geq 1.$$

Since

$$(t_n - s_n) - \tau(p_n) \leq 2\sum_{i=1}^{n-1} \alpha_i \quad \text{and} \quad \mu(|x_n|p_n') = \mu\left(\mu(x_n)\chi_{[s_n,t_n)}\right)$$

we have via Lemma 2.6

$$\mu(x_n)\chi_{[s_n+2\sum_{i=1}^{n-1}\alpha_i,t_n)} \prec\prec \mu(p_n(|x_n|p_n')p_n) = \mu(p_n p_n'|x_n|p_n' p_n) = \mu(p_n|x_n|p_n). \tag{2.18}$$

Via Proposition 2.4(i), for each $n \geq 1$ there exists $y_n \in L^1(p_n\mathcal{M}p_n, \tau(p_n \cdot p_n))$ such that

$$\mu(y_n) = \mu\left(g\chi_{[s_n+2\sum_{i=1}^{n-1}\alpha_i,t_n)}\right) \tag{2.19}$$

and such that

$$\tau(p_n(|x_n|p_n')p_n \cdot p_n y_n p_n) = \int_{[0,\infty)} \mu_t(p_n|x_n|p_n)\mu_t(g\chi_{[s_n+2\sum_{i=1}^{n-1}\alpha_i,t_n)})dt. \tag{2.20}$$

Via well-known properties of rearrangements (see [KPS] p.72) and (2.13), (2.16)-(2.20), it now follows that

$$\begin{aligned} \tau(p_n|x_n|p_n \cdot p_n y_n p_n) &\geq \int_{[0,\infty)} \mu_t(x_n)\chi_{[s_n+2\sum_{i=1}^{n-1}\alpha_i,t_n)} g\chi_{[s_n+2\sum_{i=1}^{n-1}\alpha_i,t_n)}(t)dt \\ &= \int_{[s_n,t_n)} \mu_t(x_n)g(t)dt - \int_{[s_n,s_n+2\sum_{i=1}^{n-1}\alpha_i)} \mu_t(x_n)g(t)dt \\ &\geq \epsilon/2. \end{aligned} \tag{2.21}$$

We now set

$$Y_n := \sum_{n=1}^{N} p_n y_n p_n u_n^*, \quad N \geq 1.$$

Since the sequence $\{p_n y_n p_n u_n^*\}_{n\geq 1}$ is both left and right disjointly supported, and since (2.19) implies that

$$\mu(|p_n y_n p_n u_n^*|) \leq \mu\left(g\chi_{[s_n+2\sum_{i=1}^{n-1}\alpha_i,t_n)}\right), \quad n \geq 1, \tag{2.22}$$

it follows as in the proof of Theorem 2.7 that

$$\mu(\sum_{n=N}^{M} Y_n) \leq \mu\left(g\chi_{[s_N,t_M)}\right), \quad 1 \leq N < M < \infty.$$

Since $\lim_{t\to\infty} \mu_t(g) = 0$, it follows that

$$\|g\chi_{[s_N,\infty)}\|_{L_1+L_\infty} \to 0.$$

Consequently,

$$\|\sum_{n=N}^{M} Y_n\|_{(L_1+L_\infty)(\mathcal{M},\tau)} \to 0, \quad \text{as} \quad N, M \to \infty,$$

and so the series $\sum_{n=1}^{\infty} Y_n$ converges in $(L^1 + L^\infty)(\mathcal{M}, \tau)$, to some element $Y \in (L^1 + L^\infty)(\mathcal{M}, \tau)$. Further, since

$$\mu_t(Y) = \lim_{N\to\infty} \mu_t(\sum_{n=1}^{N} Y_n) \leq \mu_t(g),$$

for all $t > 0$, it follows that $Y \in E(\mathcal{M}, \tau)$. Via (2.20), we finally obtain

$$\begin{aligned}
\tau(x_n p_n Y) &= \tau(x_n p_n \cdot p_n y_n p_n u_n^*) \\
&= \tau(u_n^* x_n p_n \cdot p_n y_n p_n) \\
&= \tau(|x_n| p_n \cdot p_n y_n p_n) \\
&= \tau(p_n |x_n| p_n \cdot p_n y_n p_n) \\
&\geq \epsilon/2.
\end{aligned}$$

This contradicts Proposition 2.3, and completes the proof of the Theorem under the assumption that $E \subseteq L_0[0, \infty)$.

Suppose now that $E \not\subseteq L_0[0, \infty)$. It follows that $\chi_{[0,\infty)} \in E$, and hence, $L^\infty[0, \infty) \subseteq E$. Consequently, $E^\times \subseteq L^1[0, \infty)$. Moreover, since it is clear that$E \cap L_0[0, \infty) \subseteq E$, it follows that $E^\times \subseteq (E \cap L_0[0, \infty))^\times$. It follows that K is relatively $\sigma(L^1(\mathcal{M}, \tau), \mathcal{M})$-compact and relatively $\sigma(E^\times(\mathcal{M}, \tau), (E \cap L_0)(\mathcal{M}, \tau))$-compact. If we observe that

$$E = L^\infty[0, \infty) + (E \cap L_0[0, \infty),$$

it follows that in order to show that K is relatively $\sigma(E^\times, E)$ compact, it suffices to consider only the case that $E(\mathcal{M}, \tau)) = L_\infty(\mathcal{M}, \tau) = \mathcal{M}$.

We suppose then that $E = L^\infty[0, \infty)$ and that $K \subseteq L^1(\mathcal{M}, \tau)$ is relatively $\sigma(L^1(\mathcal{M}, \tau), \mathcal{M})$-compact. The proof that the set $\mu(K)$ is relatively $\sigma(L^1([0, \infty), L_\infty[0, \infty))$ compact, follows exactly the same lines as the first part of the proof, but there are some technical modifications that need to be made, which we now indicate.

We assume then that $\epsilon > 0, \{x_n\}_{n=1}^\infty \subseteq K, \{e_n\}_{n=1}^\infty, g \in L^\infty[0, \infty)$ satisfy (2.13). As in the first part of the proof, it may be assumed that $m(e_n) \not\to 0$ and that (2.15) holds. To show now that $m(e_n) \to \infty$, we again assume that this is not the case and that there exists a positive number Δ such that $\delta < m(e_n) < \Delta$ for all $n \geq 1$. For any $N \in \mathbb{N}$, for any n such that $t_{n-1} > N\Delta$, and for any $g \in L^\infty[0, \infty)$, observe that

$$\begin{aligned}
N \int_{[t_{n-1}, t_{n-1}+\Delta)} \mu_t(x_n) g(t) dt &\leq N \|g\|_\infty \int_{[t_{n-1}, t_{n-1}+\Delta)} \mu_t(x_n) dt \\
&\leq \|g\|_\infty \int_{[0, t_{n-1})} \mu_t(x_n) dt \leq \|g\|_\infty \sup_{n\geq 1} \|x_n\|_{L^1(\mathcal{M},\tau)},
\end{aligned}$$

using the fact that $\mu(x_n)$ is non-increasing. This implies that

$$\int_{[t_{n-1}, t_{n-1}+\Delta)} \mu_t(x_n) g(t) dt \to_n 0 \tag{2.23}$$

for all $g \in L^\infty[0, \infty)$. Again setting

$$\tilde{g} = \sum_{n=1}^{\infty} \mu_{t-t_{n-1}}(g\chi_{e_n})$$

and noting that

$$\int_{e_n} \mu_t(x_n) g(t) dt \leq \int_{[t_{n-1}, t_{n-1}+\Delta)} \mu_t(x_n) \tilde{g}(t) dt,$$

the assertion of (2.23) yields a contradiction to (2.13). It again follows that $m(e_n) \to_n \infty$. The sequences $\{p_n\}_{n\geq 1}, \{y_n\}_{n\geq 1}$ are now defined exactly as before and we again set

$$Y_n := \sum_{n=1}^{N} p_n y_n p_n u_n^*, \quad N \geq 1,$$

and let

$$Y_N = w_N |Y_N|$$

be the polar decomposition for all $N \geq 1$. Via the inequality (2.23), we obtain that

$$\| \, |Y_N| \, \|_{\mathcal{M}} = \| \sum_{n=1}^{N} |p_n y_n p_n u_n^*| \, \|_{\mathcal{M}} \leq \|g\|_\infty$$

for all $N \geq 1$. This implies that the sequence $\{|Y_N|\}_{N\geq 1}$ converges for the strong operator topology to some element $Z \in \mathcal{M}$. Further, let

$$w := \lim_{N\to\infty} w_N,$$

where the limit is taken with respect to the strong operator topology (it is clear that each w_N is given by a sum of left and right disjointly supported partial isometries). We set finally

$$Y := wZ \in \mathcal{M}.$$

Since, for any positive integer n, we still have

$$p_n Y = p_n y_n p_n u_n^*$$

we may repeat the final arguments for the case that $E \subseteq L_0[0, \infty)$, and this suffices to complete the proof of the theorem. $\square$

We will now remove the assumption that $\mathcal{M}$ be non-atomic in the preceding Theorem 2.9 in the setting of fully symmetric spaces. To this end, we need a slightly strengthened version of [CDS], Lemma 1.3.

If $\mathcal{M} \subseteq \mathcal{N}$ is a proper von Neumann subalgebra, then the conditional expectation

$$\mathcal{E}_{\mathcal{M}} : L^1(\mathcal{N}, \tau_{\mathcal{N}}) + \mathcal{N} \to L^1(\mathcal{M}, \tau) + \mathcal{M}$$

is defined as in the commutative setting via the equality

$$\tau(x\mathcal{E}_{\mathcal{M}}(y)) = \tau(xy), \quad x \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}, \quad y \in L^1(\mathcal{N}, \tau_{\mathcal{N}}) + \mathcal{N},$$

and an appeal to the fact that the spaces $L^1(\mathcal{M}, \tau) + \mathcal{M}$, $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ are dual in the sense of Köthe. See, for example, [DDP3, Theorem 5.6].

Lemma 2.10 *Let $\mathcal{M} \subseteq \mathcal{N}$ be a proper von Neumann subalgebra. If E is a fully symmetric Banach function space on $[0, \tau(\mathbf{1}))$, then $\mathcal{E}_{\mathcal{M}}(y) \in E(\mathcal{M}, \tau)$ for all $y \in E(\mathcal{N}, \tau_{\mathcal{N}})$ and*

$$\tau(x\mathcal{E}_{\mathcal{M}}(y)) = \tau_{\mathcal{N}}(xy), \quad x \in E(\mathcal{M}, \tau)^\times, \quad y \in E(\mathcal{N}, \tau_{\mathcal{N}}).$$

Proof We observe that for each $y \in E(\mathcal{N}, \tau_{\mathcal{N}})$, the mapping $x \to \tau_{\mathcal{N}}(xy)$, $x \in E(\mathcal{M}, \tau)^{\times}$, is a normal linear mapping in the sense of [DDP4, Definition 5.8]. Consequently, by [DDP4, Theorem 5.11], there exists a unique element $\mathcal{E}(y) \in E^{\times\times}(\mathcal{M}, \tau)$ such that

$$\tau(x\mathcal{E}(y)) = \tau_{\mathcal{N}}(xy), \quad x \in E(\mathcal{M}, \tau)^{\times}, \quad y \in E(\mathcal{N}, \tau_{\mathcal{N}}).$$

It follows immediately that $\mathcal{E}$ coincides with the restriction of $\mathcal{E}_{\mathcal{M}}$ to $E(\mathcal{N}, \tau_{\mathcal{N}})$. Since E is fully symmetric, and $\mathcal{E}_{\mathcal{M}}(y) \prec\prec \mu(y) \in E$ for all $y \in E(\mathcal{N}, \tau_{\mathcal{N}})$, it follows that $\mathcal{E}(y) = \mathcal{E}_{\mathcal{M}}(y) \in E(\mathcal{M}, \tau)$ and this suffices to conclude the proof of the lemma. $\square$

Theorem 2.11 *If E is a fully symmetric Banach function space on $[0, \tau(\mathbf{1}))$, and if $K \subset E(\mathcal{M}, \tau)^{\times}$ is relatively $\sigma(E(\mathcal{M}, \tau)^{\times}, E(\mathcal{M}, \tau))$- compact, then $\mu(K) \subseteq E^{\times}$ is relatively $\sigma(E^{\times}, E)-$compact.*

Proof Let $K \subset E(\mathcal{M}, \tau)^{\times}$ be relatively $\sigma(E(\mathcal{M}, \tau)^{\times}, E(\mathcal{M}, \tau))-$ compact. Let

$$\mathcal{N} := \mathcal{M} \otimes L_{\infty}(0, 1), \quad \tau_{\mathcal{N}} := \tau \otimes m.$$

Clearly $\mathcal{M} \otimes \chi_{[0,1]}$ is a proper von Neumann subalgebra of $(\mathcal{N}, \tau_{\mathcal{N}})$. Clearly we may identify the former subalgebra with $\mathcal{M}$ by considering the mapping

$$\phi : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}, \quad \phi(x) := x \otimes \chi_{[0,1]}.$$

For the necessary details, we refer to [FK],[CS]. If $z \in E(\mathcal{N}, \tau)$, then there exists $y \in E(\mathcal{N}, \tau)$ such that

$$\mathcal{E}_{\mathcal{M}\otimes\chi_{[0,1)}} z = y \otimes \chi_{[0,1)}.$$

Consequently, for all $x \in E(\mathcal{M}, \tau)$,

$$(\tau \otimes m)((x \otimes \chi_{[0,1)})z) = (\tau \otimes m)((x \otimes \chi_{[0,1)})\mathcal{E}_{\mathcal{M}\otimes\chi_{[0,1)}} z) = \tau(xy).$$

from this it follows that the restriction of ϕ to $E(\mathcal{M}, \tau)^{\times}$ is

$$\sigma(E(\mathcal{M}, \tau)^{\times}, E(\mathcal{M}, \tau)) \text{ to } \quad \sigma(E(\mathcal{N}, \tau_{\mathcal{N}})^{\times}, E(\mathcal{N}, \tau_{\mathcal{N}}))$$

continuous. This implies that $\phi(K) \subseteq E(\mathcal{N}, \tau_{\mathcal{N}})^{\times}$ is relatively $\sigma(E(\mathcal{N}, \tau_{\mathcal{N}})^{\times}, E(\mathcal{N}, \tau_{\mathcal{N}}))-$compact. Since $\mathcal{N}$ is non-atomic, Theorem 2.9 implies that $\mu(\phi(K))$ is relatively $\sigma(E^{\times}, E)-$compact. Since $\mu(K) = \mu(\phi(K))$,the assertion of the theorem follows. $\square$

We will now show that the converse to Theorem 2.11 holds, provided that $E \subseteq L_0[0, \infty)$. As shown by Remark 2.2 (a), the converse to Theorem 2.11 fails if no restriction is placed on E.

Theorem 2.12 *Let $E \subseteq L_0$ be a symmetric Banach function space on $[0, \tau(\mathbf{1}))$. If K is a bounded subset of $E^{\times}(\mathcal{M}, \tau)$ and if $\mu(K)$ is relatively $\sigma(E^{\times}, E)$-compact, then K is relatively $\sigma(E^{\times}(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$-compact.*

Proof We assume first that $\mathcal{M}$ is non-atomic. If K is not relatively $\sigma(E^{\times}(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$ compact, then it follows from the argument given in the second part of the proof to Theorem 2.7, that we may assume there exists a sequence $\{e_n\}_{n=1}^{\infty}$ of mutually orthogonal projections in $\mathcal{M}^p, 0 \le x \in E(\mathcal{M},\tau)$ and a sequence of elements $\{y_n\}_{n=1}^{\infty} \subseteq K$ and $\epsilon, \delta > 0$ such that

$$\tau(e_n) > \delta, \quad \max\{|\tau(e_n x y_n)|, |\tau(y_n x e_n)|\} > \epsilon, \quad n \ge 1$$

For each $n \ge 1$, set $q_n = \bigvee_{i \ge n} e_i$, observe that $e_n \le q_n \downarrow 0$, and that

$$\max\{|\tau(e_n q_n x y_n q_n e_n)|, |\tau(e_n q_n y_n x q_n e_n)|\} > \epsilon. \tag{2.24}$$

Let $N > 0$ be a fixed natural number and, via Proposition 2.4, choose $p_N \in \mathcal{M}^p$ such that $\tau(p_N) = N$ and

$$x p_N = p_N x \quad \text{and} \quad \mu(p_N x) = \mu(\mu(x)\chi_{[0,N)}).$$

¿From (2.24), it follows that

$$\max\left\{|\tau(e_n q_n p_N x y_n q_n e_n) + \tau(e_n q_n p_N^{\perp} x y_n q_n e_n)|, \quad |\tau(e_n q_n y_n x p_N q_n e_n) + \tau(e_n q_n y_n x p_N^{\perp} q_n e_n)|\right\} > \epsilon.$$

We suppose first that the inequality

$$\max\{|\tau(e_n q_n p_N x y_n q_n e_n)|, \quad |\tau(e_n q_n y_n x p_N q_n e_n)|\} > \epsilon/2 \tag{2.25}$$

holds for infinitely many natural numbers n. Relabelling if necessary, we assume that (2.25) holds for all $n \ge 1$. It follows that

$$\int_{[0,\tau(e_n))} \mu_t(q_n p_N)\mu_t(x)\mu_t(y_n)dt > \epsilon/2, \quad n \ge 1 \tag{2.26}$$

Since

$$\mu(q_n p_N) = \mu^{1/2}(|q_n p_N|^2) = \mu^{1/2}(p_N q_n p_N),$$

it follows from the fact that $q_n \downarrow_n 0$ that $p_N q_n p_N \downarrow_n 0$ and so also

$$\mu(q_n p_N) = \mu^{1/2}(p_N q_n p_N) \downarrow_n 0$$

since $\tau(p_N) < \infty$. Consequently, $\mu(q_n p_N)\mu(x) \downarrow_n 0$ and so the assumption that $\mu(K)$ is relatively $\sigma(E^{\times}, E)$ compact implies via the implication (i)$\rightarrow$ (iv) of Proposition 2.1 that

$$\int_{[0,\tau(e_n))} (\mu_t(q_n p_N)\mu_t(x))\,\mu_t(y_n)dt \rightarrow_n 0.$$

This however is a contradiction to (2.26). Consequently, for each $N = 1, 2, \ldots$, the inequality (2.25) holds for at most finitely many $n \ge 1$. Thus, for each $N = 1, 2, \ldots$, we deduce the inequality:

$$\max\left\{|\tau(e_n q_n p_N^{\perp} x y_n q_n e_n)|, \quad |\tau(e_n q_n y_n x p_N^{\perp} q_n e_n)|\right\} > \epsilon/2, \quad n \ge 1.$$

We may assume therefore that

$$\tau(|p_N^{\perp} x y_N|) > \frac{\epsilon}{2}, N = 1, 2, \ldots \tag{2.27}$$

It follows from [CS] Proposition 1.1 and (2.27) that

$$\int_0^{\infty} \mu_t(p_N^{\perp} x)\mu_t(y_N)dt > \frac{\epsilon}{2}, \quad \int_N^{\infty} \mu_t(x)\mu_{t-N}(y_N)dt > \frac{\epsilon}{2}. \tag{2.28}$$

If
$$z_N(t) = \begin{cases} 0, & t < N \\ \mu_{t-N}(y_N), & t \geq N \end{cases},$$
then (2.28) becomes
$$\int_N^\infty \mu_t(x) z_N(t)dt > \frac{\epsilon}{2}. \tag{2.29}$$
Since $\mu(z_N) = \mu(y_N) \in \mu(K)$, $N = 1, 2, \ldots,$, the inequality (2.29) yields a contradiction via Proposition 2.1 to the assumption that $\mu(K)$ is relatively $\sigma(E^\times, E)$ compact.

Finally, the assumption that $\mathcal{M}$ is non-atomic may be removed by identifying $\mathcal{M}$ as a proper subalgebra of the non-atomic non Neumann allgebra $(\mathcal{M} \otimes L^\infty[0,1), \tau \otimes m)$ and we omit the details. This suffices to complete the proof of the Theorem. $\square$

It is worth remarking that the only role played by the assumption that E is fully symmetric is to permit the application of a technique which removes the further assumption that $\mathcal{M}$ is non-atomic. As is clear from an inspection of the proof, Theorem 2.12 remains valid under the assumption that $\mathcal{M}$ is non-atomic, without further restriction on E other than the essential requirement that $E \subseteq L_0[0, \infty)$.

By synthesising Theorems 2.9, 2.11 and 2.12 and using Proposition 2.1, we may now obtain the principal result of this paper.

Theorem 2.13 *Let $E \subseteq L_0[0, \infty)$ be symmetric Banach function space on $[0, \tau(\mathbf{1})$, let $K \subseteq E^\times(\mathcal{M}, \tau)$ be bounded, and let $(\mathcal{M}, \tau)$ be an arbitrary semi-finite von Neumann algebra. If $(\mathcal{M}, \tau)$ is non-atomic or if E is fully symmetric, then the following statements are equivalent.*

(i) K *is relatively* $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$*-compact;*

(ii) $\mu(K)$ *is relatively* $\sigma(E^\times, E)$*-compact;*

(iii) Orb $(\mu(K))$ *is relatively* $\sigma(E^\times, E)$*-compact;*

(iv) $\mathrm{Orb}(K) := \{x \in L^1(\mathcal{M}, \tau) + \mathcal{M} : x \prec\prec y$ for some $y \in K\}$ *is relatively* $\sigma(E^\times(\mathcal{M}, \tau), E(\mathcal{M}, \tau))$*-compact.*

If the assumption that $E \subseteq L_0[0, \infty)$ is omitted, then the implication (i)$\Rightarrow$ (ii) continues to hold, but the implication (ii)$\Rightarrow$ (i) fails.

Finally we remark that Theorem 2.13 continues to remain valid if E is assumed to be merely rearrangement-invariant, provided $(\mathcal{M}, \tau)$ is non-atomic. This follows from an inspection of the proofs and [Su2]. We leave the details to the interested reader.

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