Perturbation of Linear Semigroups

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An extension of a result on the essential spectral radius of a perturbed semigroup obtained by L. WEIS is presented. In addition integral inequalities for the measure of non-compactness for strongly measurable operator-valued functions and linear semigroups are given.

1. Introduction

We will consider perturbation theorems for linear semigroups, where in particular we compare the essential spectral radii of the perturbed with the unperturbed semigroup. The main interest for this investigation arises from the fact that the asymtotic behaviour for $t \to \infty$ of a C_0–semigroup $(U(t))$ is connected with the essential spectral radius of the operators $U(t),\ t \geq 0$. The definition of the essential spectral radius is as follows. Let $T \in L(X)$. Then

$$\begin{aligned} r_e(T) \quad := \quad & \sup\{|\lambda|;\ \lambda \in \sigma(T) \text{ but } \lambda \text{ is not an eigenvalue of finite algebraic} \\ & \text{multiplicity }\}\ \text{[We, p.7] (see [Vi, Vo1, We] for details).} \end{aligned}$$

The operator ideal of strictly singular operators $SS(X,Y)$ (or Kato-operators) enjoys the property that for $B \in SS(X,Y)$ $r_e(T+B) = r_e(T)$ for all $T \in L(X,Y)$ [Sche, Th.2.17;Cor.3.4].

Let us suppose that G is the generator of the C_0–semigroup $(S(t))$ and let $B \in L(X)$. Then $G+B$ generates the so called perturbed semigroup, say $(T(t))$. In applications, one is interested to know if the essential spectral radii of the perturbed and unperturbed semigroup are equal, since then the asymptotic behaviour is not changed.

L.Weis gave conditions under which $r_e(T(t)) = r_e(S(t))$ [We]; we will extend his result in 3.5. I. VIDAV and J. VOIGT ensured this equality under certain compactness conditions [Vi, Vo1]. Their results were extended again by L.Weis to the spaces L_p for $1 \leq p \leq \infty$ and $C(K)$-spaces. Unfortunately his proof is based on a proposition given in the same paper, whose proof contains a gap as J.Voigt showed (see [Vo2, 1.2]).

Here we will show the invariance of the essential spectral radius for a larger class of spaces, which includes that of L. WEIS. The assumption L. WEIS used for this result was "$BS(s_1)B \ldots BS(s_n)B$ is strictly singular" for all finite sets $\{s_1, \ldots, s_n\} \subset \mathbb{R}_+$. This hypothesis can be modified to the condition that the products of the operators are compact, weakly compact resp. conditionally weakly compact, and this conditions can be verified more easily in applications, cf. [We].

Throughout the paper X, Y are Banach spaces, $B(X)$ and $S(X)$ denote the unit ball and the unit sphere of X, respectively. $L(X,Y)$ is the space of linear and bounded operators, $W(X,Y)$, $(CW(X,Y))$ is the space of weakly compact resp. conditionally weakly compact operators. (Ω, Σ, μ) stands for a finite measure space, and $U : \Omega \longrightarrow L(X,Y)$ is called strongly measurable, if $U(\cdot)(x)$ is measurable for all $x \in X$.

First we give some results on the essential spectral radius, which are used later, and we will develop the connection to the measure of non-compactness.

Remark 1.1.

a) For $T \in L(X.Y)$, the quantity

$$c(T) := \inf\{\|T|_Z\|;\ Z \text{ is a closed subspace with } \operatorname{codim} Z < \infty\}$$

is called the *measure of non-compactness*. We note that $c(T) = 0$ if and only if T is compact [LeS, p.9] The connection to the essential spectral radius is given by $r_e(T) = \lim\limits_{n\to\infty} c(T^n)^{\frac{1}{n}}$ (see e.g. [We, p.7]).

b) For a C_0-semigroup $(S(t))_{t\geq 0}$ let $\omega := \inf\{t^{-1}\ln\|S(t)\|; t > 0\}$ and $\omega_e := \inf\{\gamma \leq \omega; \exists\ M \in \mathbb{R} :\ c(S(t)) \leq Me^{\gamma t}\}$. Then

$$r_e(S(t)) = e^{\omega_e t} \text{ for all } t > 0.$$

ω (resp. ω_e) are called the type (resp. essential type) of the semigroup (see [We, Prop.2.1]).

c) Let $|||\cdot|||$ be an equivalent norm on X with

$$\forall\ x \in X : m\|x\| \leq |||x||| \leq M\|x\| \quad m, M > 0 \text{ suitable}.$$

If c_{eq} denotes the measure of non-compactness for this equivalent norm, then we have

$$\forall\ T \in L(X) : c(T) \leq \frac{M}{m} c_{eq}(T) \text{ and } c_{eq}(T) \leq \frac{M}{m} c(T).$$

Thus, for a C_0-semigroup $(S(t))$, the essential spectral radius is invariant for different equivalent norms on X, using 1.1 a).

d) $\mathcal{S} := \{Z \subset X;\ \operatorname{codim} Z < \infty\}$ will denote the set of all subspaces with finite co-dimension. We have $Z \in \mathcal{S}$ if and only if there are $x_1^*, \dots, x_n^* \in X^*$, so that $Z = \bigcap\limits_{i=1}^{n} x_i^{*-1}(\{0\})$. Hence

$$Z_1, Z_2 \in \mathcal{S} \Rightarrow Z_1 \cap Z_2 \in \mathcal{S}.$$

e) Let $U : \Omega \longrightarrow L(X,Y)$ be strongly measurable and bounded. We suppose that there is a decreasing sequence $(Z_n) \subset \mathcal{S}$, in order that $\omega \mapsto \|U(\omega)|_{Z_n}\|$ is measurable and $c(U(\omega)) = \lim_{n\to\infty} \|U(\omega)|_{Z_n}\|$ for almost all $\omega \in \Omega$.

Then we have:

$$c\Big(\int_\Omega U(\omega)d\mu(\omega)\Big) \le \int_\Omega c(U(\omega))d\mu(\omega).$$

Proof. By the definition of the measure of non-compactness one has:

$$\begin{aligned} c\Big(\int_\Omega U(\omega)d\mu(\omega)\Big) &\le \limsup_{n\to\infty} \sup_{x\in B(Z_n)} \|\int_\Omega U(\omega)(x)d\mu(\omega)\| \\ &\le \limsup_{n\to\infty} \int_\Omega \|U(\omega)|_{Z_n}\|d\mu(\omega) = \int_\Omega c(U(\omega))d\mu(\omega) \end{aligned}$$

by the Lebesgue domination theorem.

Thus e) is proved. □

2. Essential spectral radius for perturbed semigroups

Before we start with the the investigation of the spectral radius of a perturbed semigroup we will state some auxiliary results, which may be of independent interest. In the following proposition we will present a "multiplicative" version of 1.1.e). This proposition will be used in the proof of the main theorem, in which we state the equality of the essential spectral radius for the unperturbed and perturbed semigroup. The next proposition also improves [KS, Th. 3.7 a)], provided a certain assumption is imposed on the operator-valued function.

Proposition 2.1. *Let $U_1, \dots, U_k : \Omega \longrightarrow L(X)$ be bounded and strongly measurable. Let us suppose that for all $i = 1, \dots, k$ there exists a decreasing sequence $(Z_n^i) \subset \mathcal{S}$, so that for all $i = 1, \dots, k$:*

$$\omega \longmapsto \|U_i(\omega)|_{Z_n^i}\| \text{ is measurable and } c(U_i(\omega)) = \lim_{n\to\infty} \|U_i(\omega)|_{Z_n^i}\| \text{ a.e. on } \Omega.$$

Then we have:

a) If X does not contain ℓ_1, then

$$c\Big(\int_\Omega U_1(\omega)\cdots U_k(\omega)d\mu(\omega)\Big) \le \int_\Omega c(U_1(\omega))\cdots c(U_k(\omega))d\mu(\omega).$$

b) If X is separable, then

$$c\Big(\int\limits_{\Omega} U_1(\omega)\cdots U_k(\omega)d\mu(\omega)\Big) \le 2\int\limits_{\Omega} c(U_1(\omega))\cdots c(U_k(\omega))d\mu(\omega).$$

c) If X is arbitrary, then

$$c\Big(\int\limits_{\Omega} U_1(\omega)\cdots U_k(\omega)d\mu(\omega)\Big) \le 4\int\limits_{\Omega} c(U_1(\omega))\cdots c(U_k(\omega))d\mu(\omega).$$

Proof. a) We define for $n \in \mathbb{N}$:

$$Z_n := \bigcap_{i=1}^{k} Z_n^i.$$

Then with 1.1 c) we conclude that $Z_n \in \mathcal{S}$ for all $n \in \mathbb{N}$. For $\omega \in \Omega$, $n \in \mathbb{N}$ and $i = 1, \ldots, k$ we define $\phi_{n,i}(\omega) := \inf\limits_{j=1,\ldots,k} \|U_i(\omega)|_{Z_n^j}\|$. The functions $\phi_{n,i}$ are measurable for $n \in \mathbb{N}$ and $i = 1, \ldots, k$. Keeping in mind that $c(U_i(\omega)) \le \|U_i(\omega)|_{Z_n}\| \le \phi_{n,i}(\omega) \le \|U_i(\omega)|_{Z_n^i}\|$ for all $n \in \mathbb{N}$, $i = 1, \ldots, k$ and $\omega \in \Omega$, according to the assumption we get:

$$\forall\ i = 1, \ldots, k:\ c(U_i(\omega)) = \lim_{n\to\infty} \phi_{n,i}(\omega) \text{ a.e. on } \Omega.$$

Since the U_i are bounded for all $i = 1, \ldots, k$ we may assume w.l.o.g that

$$\sup_{i=1,\ldots,k}\ \sup_{\omega\in\Omega} \|U_i(\omega)\| \le 1.$$

Let $0 < \epsilon < \frac{1}{\mu(\Omega)}$. The Egorov theorem provides us with a set $\Omega_\epsilon \in \Sigma$, $\mu(\Omega\backslash\Omega_\epsilon) < \frac{\epsilon}{2}$, such that

$$\sup_{i=1,\ldots,k}\ \sup_{\omega\in\Omega_\epsilon} \big|\phi_{n,i}(\omega) - c(U_i(\omega))\big| \overset{n\to\infty}{\longrightarrow} 0.$$

For simplicity we assume that $k = 2$. The argument can easily be transferred to any $k \in \mathbb{N}$. We can find an $n_0 \in \mathbb{N}$, such that for all $n \ge n_0$:

$$\sup_{i=1,2}\ \sup_{\omega\in\Omega_\epsilon} \big|\phi_{n,i}(\omega) - c(U_i(\omega))\big| < \frac{\epsilon}{8\mu(\Omega)}.$$

Because $c(U_i(\omega)) \le \phi_{n,i}(\omega) \le 1$ for all $\omega \in \Omega$ and $i = 1, 2$, we conclude:

$$\begin{aligned}
(1) \qquad \sup_{\omega\in\Omega_\epsilon} \big|\phi_{n,1}(\omega)\cdot\phi_{n,2}(\omega) - c(U_1(\omega))\cdot c(U_2(\omega))\big| \quad &< \\
\sup_{\omega\in\Omega_\epsilon} \big|\phi_{n,1}(\omega)\cdot\phi_{n,2}(\omega) - \big(\phi_{n,1}(\omega) - \frac{\epsilon}{8\mu(\Omega)}\big)\cdot\big(\phi_{n,2}(\omega) - \frac{\epsilon}{8\mu(\Omega)}\big)\big| \quad &\le \\
\Big|\frac{\epsilon}{8\mu(\Omega)} + \frac{\epsilon}{8\mu(\Omega)} - \frac{\epsilon^2}{64\mu(\Omega)^2}\Big| \quad &< \quad \frac{\epsilon}{4\mu(\Omega)}.
\end{aligned}$$

Using (1), for $n \geq n_0$ we deduce:

$$\int_{\Omega_\epsilon} \phi_{n,1}(\omega) \cdot \phi_{n,2}(\omega) d\mu(\omega) \leq \int_{\Omega_\epsilon} c(U_1(\omega)) \cdot c(U_2(\omega)) d\mu(\omega) + \frac{\epsilon}{4}.$$

Since $\|U_i(\omega)\| \leq 1$ for all $\omega \in \Omega$ and $i = 1, 2$, for $n \geq n_0$ we have

$$(2) \quad \int_{\Omega} \phi_{n,1}(\omega) \cdot \phi_{n,2}(\omega) d\mu(\omega) \leq \int_{\Omega} c(U_1(\omega)) \cdot c(U_2(\omega)) d\mu(\omega) + \frac{\epsilon}{2}.$$

Let $P : X \longrightarrow Z_{n_0}$ denote the projection onto Z_{n_0}. Then $I - P$ is a finite rank operator and thus, $V(\omega) := (I - P) \cdot U_2(\omega)$ is compact for all $\omega \in \Omega$. Since $\ell_1 \not\subset X$, by [S1, Th. 3.2] we know that $\overline{V} : X \longrightarrow L_1(\mu, X)$ is compact, too. Therefore by 1.1 b) there is a subspace $\tilde{Z} \subset Z_{n_0}$, $\tilde{Z} \in \mathcal{S}$, such that $\|\overline{V}|_{\tilde{Z}}\| \leq \frac{\epsilon}{4}$. For $x \in B(\tilde{Z})$ we estimate:

$$
\begin{aligned}
(3) \qquad & \int_{\Omega} \|U_1(\omega) \cdot U_2(\omega)(x)\| d\mu(\omega) \\
= \; & \int_{\Omega} \|U_1(\omega) \cdot P \cdot U_2(\omega)(x) + \\
& \quad U_1(\omega) \cdot (I - P) \cdot U_2(\omega)(x)\| d\mu(\omega) \\
\overset{\|U_1(\omega)\| \leq 1}{\leq} \; & \int_{\Omega} \|U_1(\omega) \cdot P \cdot U_2(\omega)(x)\| d\mu(\omega) + \frac{\epsilon}{4} \\
\overset{\|U_1(\cdot)|_{Z_{n_0}}\| \leq \phi_{n_0,1}(\cdot)}{\leq} \; & \int_{\Omega} \phi_{n_0,1}(\omega) \cdot \|P \cdot U_2(\omega)(x)\| d\mu(\omega) + \frac{\epsilon}{4} \\
= \; & \int_{\Omega} \phi_{n_0,1}(\omega) \cdot \|\big(U_2(\omega) - (I - P) \cdot U_2(\omega)\big)(x)\| d\mu(\omega) + \frac{\epsilon}{4} \\
\overset{\tilde{Z} \subset Z_{n_0}}{\leq} \; & \int_{\Omega} \phi_{n_0,1}(\omega) \cdot \phi_{n_0,2}(\omega) d\mu(\omega) + \\
& \int_{\Omega} \phi_{n_0,1}(\omega) \cdot \|(I - P) \cdot U_2(\omega)\big)(x)\| d\mu(\omega) + \frac{\epsilon}{4} \\
\leq \; & \int_{\Omega} \phi_{n_0,1}(\omega) \cdot \phi_{n_0,2}(\omega) d\mu(\omega) + \frac{\epsilon}{2}.
\end{aligned}
$$

Using (2) and (3), we get

$$c\Big(\int_{\Omega} U_1(\omega) \cdot U_2(\omega) d\mu(\omega)\Big) \leq \sup_{x \in B(\tilde{Z})} \|\int_{\Omega} U_1(\omega) \cdot U_2(\omega)(x) d\mu(\omega)\| \leq$$

$$\sup_{x\in B(\tilde{Z})}\int\limits_{\Omega}\|U_1(\omega)\cdot U_2(\omega)(x)\|d\mu(\omega)\overset{(3)}{\leq}\int\limits_{\Omega}\phi_{n_0,1}(\omega)\cdot\phi_{n_0,2}(\omega)d\mu(\omega)+\frac{\epsilon}{2}\overset{(2)}{\leq}$$
$$\int\limits_{\Omega}c(U_1(\omega))\cdot c(U_2(\omega))d\mu(\omega)+\epsilon.$$

Since $\epsilon>0$ is arbitrary the proof to a) is done. b) and c) are a direct consequence of the submultiplicity of the measure of noncompactness c (cf. [LeS]) and of [KS, Th.3.7 b),c)]. The measurability of $c(U_i(\cdot))$, $i=1,\ldots n$, is given by the assumption. □

Remark 2.2. Up to now it is not known to us, whether the constants in the integral equations of 2.1 can be lowered or even removed. But as we will see later, for the application to perturbed semigroups this is of no significance.

The next lemma will give conditions under which it is possible to find a suitable sequence $(Z_n)\subset\mathcal{S}$, needed for 1.1 e) resp. 2.1.

Lemma 2.3. *Let $T>0$ and let $U:[0,T]\longrightarrow L(X,Y)$ be strongly measurable and bounded, such that either*

i) $[0,T]\ni t\longmapsto\|U(t)|_Z\|$ is decreasing for all $Z\in\mathcal{S}$ or

ii) $[0,T]\ni t\longmapsto\|U(t)|_Z\|$ is increasing for all $Z\in\mathcal{S}$.

Then there is a decreasing sequence $(Z_n)\subset\mathcal{S}$,, so that

$$c(U(t))=\lim_{n\to\infty}\|U(t)|_{Z_n}\|\text{ for almost all } t\in[0,T].$$

Proof. For $Z\in\mathcal{S}$ we consider the function $\|U(\cdot)|_Z\|:[0,T]\ni t\longmapsto\|U(t)|_Z\|$ as an element in the order complete Banach lattice $L_1([0,T])$ (see [LT, p.4]). Then there exists $\phi:=\inf_{Z\in\mathcal{S}}\|U(\cdot)|_Z\|$ in the Banach lattice $L_1([0,T])$ (see [LT, p.4,1.a.3]). Hence we find a sequence $(Z_n)\subset\mathcal{S}$, in order that $\phi=\inf\limits_{n\in\mathbb{N}}\|U(\cdot)|_{Z_n}\|$ in $L_1([0,T])$. According to 1.1 d), the sequence may be chosen to be decreasing.

Let us define $\psi(t):=\inf\limits_{n\in\mathbb{N}}\|U(t)|_{Z_n}\|$, $t\in[0,T]$. The definition of ψ reveals that for all $Z\in\mathcal{S}$, $\psi(t)\leq\|U(t)|_Z\|$ for almost all $t\in[0,T]$. For $Z\in\mathcal{S}$ we define $N_Z:=\{t\in[0,T];\psi(t)>\|U(t)|_Z\|\}$.
We claim: $M:=\bigcup_{Z\in\mathcal{S}}N_Z$ has measure zero.
Proof of the claim: We will give the proof in the case i), since ii) is similar. For $k\in\mathbb{N}$ we define $M_k:=\{t\in[0,T];\text{ there exists } Z\in\mathcal{S}:\|U(t)|_Z\|+\frac{1}{k}<\psi(t)\}$. Then evidently $M=\bigcup_{k\in\mathbb{N}}M_k$. We prove that $\lambda(M_k)=0$ for all $k\in\mathbb{N}$.
Let $k\in\mathbb{N}$. We suppose that $\lambda^*(M_k)>0$, where λ^* denotes the outer measure.

We define $a := \inf M_k$ and $b := \sup M_k$. Then we can find a $t_1 \in]a,b[\cap M_k$, such that

$$(1)\quad \lambda^*(]t_1,b[\cap M_k) > 0.$$

Such a t_1 exists according to the definition of a, b and since $\lambda^*(M_k) > 0$. Because $t_1 \in M_k$, the definition of M_k provides a $\tilde{Z}_1 \in \mathcal{S}$, so that

$$(2)\quad \|U(t_1)|_{\tilde{Z}_1}\| + \frac{1}{k} < \psi(t_1).$$

Since $\psi(t) \leq \|U(t)|_{\tilde{Z}_1}\|$ for almost all $t \in [0,T]$, we can find a $t_2 > t_1$, $t_2 \in]t_1,b[\cap M_k$, $\lambda^*(]t_2,b[\cap M_k) > 0$, so that $\psi(t_2) \leq \|U(t_2)|_{\tilde{Z}_1}\|$. The general assumption that $t \longmapsto \|U(t)|_{\tilde{Z}_1}\|$ is decreasing gives us finally:

$$\psi(t_2) + \frac{1}{k} \leq \|U(t_2)|_{\tilde{Z}_1}\| + \frac{1}{k} \leq \|U(t_1)|_{\tilde{Z}_1}\| + \frac{1}{k} < \psi(t_1).$$

We now proceed in identical fashion, applying (1) and (2) to t_2. We obtain a sequence of strictly increasing numbers $(t_m) \subset M_k$, such that $\psi(t_{m+1}) + \frac{1}{k} < \psi(t_m)$, which is a contradiction, since $\psi(t) \geq 0$ for all $t \in [0,T]$. Hence $\lambda(M) = 0$. This means that there is a set $N \subset [0,T]$ of Lebesgue measure zero, such that for all $t \in [0,T] \setminus N$ and all subspaces $Z \in \mathcal{S}$ of finite co-dimension

$$\psi(t) \leq \|U(t)|_Z\|.$$

Hence, the proof of the lemma is completed. □

Remark 2.4.

a) Since we are interested in the application of the results stated in 2.1 and 2.5, we will select a special situation, where especially the assumption of 2.5 is satisfied. For this purpose we suppose that for a dense set $(x_n)_{n\in\mathbb{N}} \subset S(X)$ the map $t \longmapsto \|U(t)(x_n)\|$ is decreasing for all $n \in \mathbb{N}$. Then the assumptions of the preceding lemma are evidently fulfilled. The same is true if "decreasing" is replaced by "increasing".

b) Let $(S(t))$ be a C_0-semigroup. Then

$$\exists\ \omega \in \mathbb{R}\ \exists\ M > 0\ \forall\ t \in [0,\infty[\colon\ \|S(t)\| \leq Me^{\omega t}.$$

With the help of a classical result (see e.g. [Pa, p.19]), we find an equivalent norm $|||\cdot|||$ on X, such that $|||S(t)||| \leq e^{\omega t}$. According to 1.1 c) the essential spectral radius is not affected by changing to an equivalent norm. Hence, we will consider a C_0-semigroup $(S(t))$ in the next lemma, such that $\|S(t)\| \leq e^{\omega t}$.

The preceding results are used to deduce an important inequality for operators of a C_0-semigroup.

Lemma 2.5. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X with $\|S(t)\| \leq e^{\omega t}(t \in [0,\infty[)$ and suitable $\omega \in \mathbb{R}$. Then:*

$$\exists\ M > 0\ \forall\ t > 0\ \forall\ m \in \mathbb{N}\ \forall\ B \in L(X):$$

$$c\Big(\int_{\Omega_m} S(s_1)BS(s_2)\ldots BS(s_m)BS(t-s_1-\ldots-s_m)ds_1\ldots ds_m\Big) \leq$$

$$c(B)^m M \int_{\Omega_m} c(S(s_1))\ldots c(S(s_m))c(S(t-s_1-\ldots-s_m))ds_1\ldots ds_m,$$

where $\Omega_m := \{(s_1,\ldots,s_m) \in [0,t]^m; \sum\limits_{j=1}^{m} s_j \leq t\}$.
If $\ell_1 \not\subset X$, then $M = 1$, if X is separable, then $M = 2$, and we have $M = 4$ otherwise.

Proof. We define

$$\hat{S}(s) := e^{-\omega t}S(s).$$

Then obviously $(\hat{S}(s))_{s\geq 0}$ is a C_0-semigroup with $\sup\limits_{s\geq 0}\|\hat{S}(s)\| \leq 1$.

For $i = 1,\ldots,m$ let us define operator-valued maps

$$\begin{aligned} U_i\quad &:\quad \Omega_m \longrightarrow L(X)\\ &\omega \longmapsto U_i(\omega) := \hat{S}_{s_i} \circ B,\ \omega = (s_1,\ldots,s_m) \in \Omega_m. \end{aligned}$$

We need the existence of a decreasing sequence $(Z_n^i) \subset \mathcal{S}$, with the property that for $i = 1,\ldots,m$:

$$\omega \longmapsto \|U_i(\omega)|_{Z_n^i}\| \text{ is measurable and}$$

$$c(U_i(\omega)) = \lim_{n\to\infty} \|U_i(\omega)|_{Z_n^i}\| \text{ a.e. on } \Omega_m$$

to be able to conclude the inequality of 2.1. To this end, we apply 2.3 to $[0,t] \ni s \longmapsto \hat{S}(s) \circ B$. The assumptions are satisfied, because for all subspaces Z and all $s, h > 0$:

$$\|\hat{S}(s+h)|_Z\| \leq \|(\hat{S}(h)\circ\hat{S}(s))|_Z\| \leq \|\hat{S}(h)\|\cdot\|\hat{S}(s)|_Z\| \leq \|\hat{S}(s)|_Z\|,$$

hence in particular $s \longmapsto \|\hat{S}(s)\circ B|_Z\|$ is also decreasing for all $Z \in \mathcal{S}$.
As a consequence there is a decreasing sequence $(Z_n) \subset \mathcal{S}$, such that for $i = 1,\ldots,m$:

$$c(U_i(\omega)) = c(\hat{S}(s_i)\circ B) = \lim_{n\to\infty}\|\hat{S}(s_i)\circ B|_{Z_n}\| = \lim_{n\to\infty}\|U_i(\omega)|_{Z_n}\| \text{ f.a.a. } \omega \in \Omega_m.$$

Furthermore $s \longmapsto \|\hat{S}(t-s)|_Z\|$ is increasing. Hence, according to 2.3, there is a decreasing sequence $(Z_n^{m+1}) \subset \mathcal{S}$, so that

$$c(\hat{S}(t-s)) = \lim_{n\to\infty} \|\hat{S}(t-s)|_{Z_n^{m+1}}\| \text{ f.a.a. } s \in [0,t].$$

Thus we can deduce f.a.a. $\omega = (s_1, \ldots, s_m) \in \Omega$:

$$c(\hat{S}(t-(s_1+\ldots+s_m))) = \lim_{n\to\infty} \|\hat{S}_{t-(s_1+\ldots+s_m)}|_{Z_n^{m+1}}\|.$$

We can apply 2.1, since the assumptions are verified, and we find an $M > 0$, such that:

$$c\Big(\int\limits_{\Omega_m} \hat{S}(s_1)B\hat{S}(s_2)\ldots B\hat{S}(s_m)B \cdot \hat{S}(t-s_1-\ldots-s_m)ds_1\ldots ds_m\Big) \leq$$
$$M \int\limits_{\Omega_m} c(\hat{S}(s_1)\circ B)\ldots c(\hat{S}(s_m)\circ B)\cdot c(\hat{S}(t-s_1-\ldots-s_m))ds_1\ldots ds_m \leq$$
$$c(B)^m M \int\limits_{\Omega_m} c(\hat{S}(s_1))\ldots c(\hat{S}(s_m))\cdot c(\hat{S}(t-s_1-\ldots-s_m))ds_1\ldots ds_m.$$

The last inequality is due to the fact that the measure of non-compactness is submultiplicative [LeS, p.9,Lemma 3.2.]. But this means for the original semigroup $(S(t))_{t\geq 0}$:

$$\begin{aligned}
&c\Big(\int\limits_{\Omega_m} S(s_1)BS(s_2)\ldots BS(s_m)BS(t-s_1-\ldots-s_m)ds_1\ldots ds_m\Big) &=\\
&c\Big(\int\limits_{\Omega_m} e^{\omega s_1}\hat{S}(s_1)Be^{\omega s_2}\hat{S}(s_2)\ldots Be^{\omega s_m}\hat{S}(s_m)B\cdot e^{\omega(t-s_1-\ldots-s_m)} \\
&\qquad \hat{S}(t-s_1-\ldots-s_m)ds_1\ldots ds_m\Big) &=\\
&e^{\omega t}c\Big(\int\limits_{\Omega_m} \hat{S}(s_1)B\hat{S}(s_2)\ldots B\hat{S}(s_m)B\hat{S}(t-s_1-\ldots-s_m)ds_1\ldots ds_m\Big) &\leq\\
&e^{\omega t}c(B)^m M \int\limits_{\Omega_m} c(\hat{S}(s_1))\ldots c(\hat{S}(s_m))\cdot c(\hat{S}(t-s_1-\ldots-s_m))ds_1\ldots ds_m &=\\
&c(B)^m M \int\limits_{\Omega_m} e^{\omega s_1}c(\hat{S}(s_1))\ldots e^{\omega s_m}c(\hat{S}(s_m)) \\
&\qquad e^{\omega(t-s_1-\ldots-s_m)}c(\hat{S}(t-s_1-\ldots-s_m))ds_1\ldots ds_m &=\\
&c(B)^m M \int\limits_{\Omega_m} c(S(s_1))\ldots c(S(s_m))c(S(t-s_1-\ldots-s_m))ds_1\ldots ds_m.
\end{aligned}$$

The value of M is given according to 2.1. □

Remark 2.6. If $(S(t))_{t\geq 0}$ is a C_0-semigroup, there is an $\tilde{M} > 0$ and an $\omega \in \mathbb{R}$, so that for all $t \geq 0$: $\|S(t)\| \leq \tilde{M}e^{\omega t}$ (see [Pa, p.4,Th.2.2]). According to [Pa, p.19] there is an equivalent norm $|||\cdot|||$ on X, such that

$$\forall\ x \in X : \ \|x\| \leq |||x||| \leq \tilde{M}\|x\| \text{ and } |||S(t)||| \leq e^{\omega t}.$$

Hence using 2.5 and the inequality of 1.1 c) we get for an arbitrary C_0-semigroup:

$$\forall\ t > 0\ \forall\ m \in \mathbb{N}\ \forall\ B \in L(X):$$

$$c\Big(\int\limits_{\Omega_m} S(s_1)BS(s_2)\dots BS(s_m)BS(t-s_1-\dots-s_m)ds_1\dots ds_m\Big) \leq$$

$$M\tilde{M}^{2(m+1)}c(B)^m \overline{\int\limits_{\Omega_m}} c(S(s_1))\dots c(S(s_m))c(S(t-s_1-\dots-s_m))ds_1\dots ds_m,$$

where $\Omega_m := \{(s_1,\dots,s_m) \in [0,t]^m;\ \sum\limits_{j=1}^{m} s_j \leq t\}$.

We are done with the required preliminary results and can derive the main theorem, which is concerned with the essential spectral radius of a perturbed semigroup. The proof follows from the Dyson-Phillips-expansion
(see [Kat, p.497,IX,2.11]).

Theorem 2.7. *We suppose that X has the property that either $W(X,X) \subset SS(X,X)$ or $CW(X,X) \subset SS(X,X)$. Let $B \in L(X)$. If for some $m \in \mathbb{N}$ and all $m-$ tupels $(t_1,t_2,\dots,t_m)$ of positive numbers the operator $BS(t_1)BS(t_2)\dots BS(t_m)B$ is either weakly compact or conditionally weakly compact, then*

$$\forall\ t \geq 0 : \ r_e(S(t)) = r_e(T(t)),$$

where $(T(t))$ is the perturbed semigroup.

Proof. Let us recall the representation of the perturbed semigroup $(T(t))_{t\geq 0}$ according to the Dyson-Phillips expansion (see e.g. [Kat, p.497]):
By iterating

$$T(t) = S(t) + \int\limits_0^t S(s)BT(t-s)ds$$

we get

$$T(t) = \sum_{j=0}^{m} W_j(t)\ +\ R_m(t)$$

where,

$$\begin{aligned}
W_0(t) &= S(t) \\
W_j(t) &= \int\limits_{s_1+\ldots+s_j\le t} S(s_1)BS(s_2)\ldots BS(s_j)BS(t-s_1-\ldots-s_j)ds_1\ldots ds_j, \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j=1,\ldots,m. \\
R_m(t) &= \int\limits_{s_1+\ldots+s_{m+1}\le t} S(s_1)BS(s_2)\ldots BS(s_{m+1})B \\
& \qquad\qquad \cdot T(t-s-s_1-\ldots-s_{m+1})ds_1\ldots ds_{m+1}
\end{aligned}$$

for $t \geq 0$. By [S2, Th.] $R_m(t)$ is weakly compact resp. conditionally weakly compact, hence strictly singular. For the proof of $r_e(T(t)) = r_e(S(t))$, we basically follow the line of the proof of [We, p.11,3.1.Th.].

Let ω_e and $\tilde{\omega}_e$ be the essential type of $(S(t))$ and $(T(t))$. First we show that $\tilde{\omega}_e \leq \omega_e$. Let ω', ω'' so that $\omega_e < \omega' < \omega''$ and let M be according to 1.1 b) such that

$$c(S(t)) \leq \hat{M}e^{\omega' t}.$$

Since $R_m(t)$ is strictly singular, $R_m(t)$ does not change the essential spectral radius (see [Sche, p.1067,Th.2.17;p.1069,Cor.3.4]). Thus we obtain:

$$r_e(T(t)) = r_e\Big(\sum_{j=0}^{m} W_j(t)\Big) \leq c\Big(\sum_{j=0}^{m} W_j(t)\Big).$$

By 1.1 b) and 2.6 we have

$$\begin{aligned}
r_e(T(t)) &\leq \hat{M}e^{\omega' t} + \sum_{j=1}^{m} \hat{M}_j \int\limits_{s_1+\ldots+s_j\le t} c(S(s_1))\ldots c(S(s_j)) \\
& \qquad\qquad \cdot c(S(t-s_1-\ldots-s_j))ds_1\ldots ds_j \\
&\leq \hat{M}e^{\omega' t} + \sum_{j=1}^{m} \hat{M}^j M_j t^j e^{\omega' t} \leq C\cdot e^{\omega'' t},
\end{aligned}$$

$$\text{with } M_j \quad := \quad M\tilde{M}^{2(j+1)}c(B)^j,\ M \text{ according to 2.5 and } \tilde{M} \text{ as in 2.6}$$

where C is sufficiently large. Because we have chosen $\omega'' > \omega_e$ arbitrarily, 1.1 b) gives

$$\tilde{\omega}_e \leq \omega_e.$$

Showing the reverse inequality $\omega_e \leq \tilde{\omega}_e$, we use the expansion

$$S(t) = \sum_{j=0}^{m} \tilde{W}_j(t) + \tilde{R}_m(t),$$

which is obtained by interchanging the role of $(S(t))$ and $(T(t))$ and replacing B by $(-B)$. Hence

$$\tilde{R}_m(t) = \int\limits_{s_1+\ldots+s_{m+1}} (-1)^{m+1} T(s_1)BT(s_2)\ldots BT(s_{m+1})B \cdot S(t-s-s_1-\ldots-s_{m+1})ds_1\ldots ds_{m+1}.$$

We can replace the $T(s_k)$ in the representation of $\tilde{R}_m(t)$ by $\sum\limits_{j=0}^{m} W_j(s_k)+R_m(s_k)$. Thus $\tilde{R}_m(t)$ can be written as a set of multiple integrals of operator products, each containing a factor of the form

$$BS(s_1)B\ldots BS(s_m)B.$$

These factors are weakly compact resp. conditionally weakly compact according to the assumption. Since those operators form an operator ideal, $\tilde{R}_m(t)$ is weakly compact resp. conditionally weakly compact. Now proceed as in the proof for the reverse inequality, checking that 2.1 resp. 2.5 can be applied to $(T(t))$, since $(T(t))$ is again a C_0-semigroup. This procedure leads to $\omega_e = \tilde{\omega}_e$ and hence, 1.1 b) tells us that $r_e(T(t)) = r_e(S(t))$. □

Corollary 2.8. *Let X have the property that either $SS(X,X) = W(X,X)$, $CW(X,X) = SS(X,X)$ or $SS(X,X) = K(X,X)$. Let $B \in L(X)$. If for $m \in \mathbb{N}$ and all tupels $(t_1,t_2,\ldots,t_m)$ of positive numbers the operator $BS(t_1)BS(t_2)\ldots BS(t_m)B$ is strictly singular , then*

$$\forall\ t\geq 0:\ r_e(S(t)) = r_e(T(t)).$$

In the following remark we will point out, that 2.7 and 2.8 are substantial extensions of the result of L. Weis.

Remark 2.9.

a) If X has the Dunford-Pettis Property, i.e. each $T\in W(X,Y)$ maps weakly convergent sequences into norm convergent sequences, then X fulfills the condition in 2.7. This is the case if either $X = L_1(\nu)$ for a positive measure space (Γ,Θ,ν) or $X = C(K)$ for a compact Hausdorff space K (see [DU, p.176, 5.Notes]). These spaces even satisfy the assumption of Corollary 2.8. For these cases, 2.7 resp. 2.8 extends the result of L. Weis [We, p.11,3.1.Th.]. If every infinite dimensional subspace of X contains a subspace isomorphic to ℓ_1, then $CW(X,X) = SS(X,X)$.

b) An analysis of the proof of 2.7 also gives the following result:

Let us suppose X has the property that for each non-strictly singular operator $T \in L(X)$, there is a complemented subspace $Z \subset X$, so that $T|_Z$ is an isomorphism and Z is isomorphic to ℓ_p, $(1 < p < \infty)$. The L_p-spaces $(1 < p < \infty)$ have this property (see e.g. [We, p.10,Prop.2.4]). Then if for some $m \in \mathbb{N}$ and all choices of $t_1, t_2, \ldots, t_m > 0$ the operator $BS(t_1)B(t_2)\ldots BS(t_m)B$ is strictly singular, we obtain

$$\forall\, t \geq 0 : \; r_e(S(t)) = r_e(T(t)).$$

The proof uses the fact that on ℓ_p the strictly singular operators coincide with the compact operators (see [GMF])]. The most important part in the proof of 2.7 is the fact that $R_m(t)$ resp. $\tilde{R}_m(t)$ are strictly singular. Since on an infinite dimensional Banach space every compact operator is strictly singular, we can deduce with the help of [KS, Cor. 3.8] that this is fulfilled on ℓ_p, here. Thus 2.7 is also true for L_p-spaces $(1 < p < \infty)$.

c) We remark that recently J. Voigt [Vo3] has obtained a result similar to 2.7, but with a different approach.

d) The gap in the original proof of [We, p.11,3.1.Th.] is filled by 2.1 resp. [KS, Th. 3.7]. It has turned out that the underlying integral inequality for the measure of non-compactness for operator-valued functions is more sophisticated.

e) The inequality which we stated in remark 1.1 d) is not true if we change codim $Z < \infty$ to dim $Z = \infty$. For this purpose, we define for an operator $T : X \longrightarrow Y$

$$u(T) := \inf\{\|T|_Z\|;\; Z \text{ is a closed subspace with } \dim Z = \infty\}.$$

Let $(\Omega, \Sigma, \mu) = ([0,1], \mathcal{L}, \lambda)$ and $(r_n)_{n\in\mathbb{N}}$ be the sequence of Rademacher functions on $[0,1]$ with values in $\{0,1\}$. We define for $t \in [0,1]$:

$$\begin{aligned} U(t) \;\; : \;\; & \ell_1 \longrightarrow \ell_1 \\ & e_n \longmapsto 2e_n r_n(t) \end{aligned}$$

Then for $x = \sum\limits_{n\in\mathbb{N}} a_n e_n \in S(\ell_1)$:

$$\|\int\limits_{[0,1]} U(t)(x)d\lambda(t)\| \;=\; 2\|\int\limits_{[0,1]} \sum_{n\in\mathbb{N}} a_n e_n r_n(t) d\lambda(t)\|$$

$$= 2\|\sum_{n\in\mathbb{N}}\int_{[0,1]} a_n e_n r_n(t)d\lambda(t)\|$$

$$= 2\sum_{n\in\mathbb{N}}\|\int_{[0,1]} a_n e_n r_n(t)d\lambda(t)\| = 2\sum_{n\in\mathbb{N}}\frac{1}{2}|a_n| = 1$$

Thus

$$\inf_{Z\subset X,\dim Z=\infty}\|\big(\int_{[0,1]} U(t)d\lambda(t)\big)|_Z\| = 1. \tag{1}$$

Now let $t\in[0,1]$ be irrational, $t=\sum_{n\in\mathbb{N}}\epsilon_n 2^{-n}$. Then $\big(r_n(t)\big)_{n\in\mathbb{N}} = (\epsilon_n)_{n\in\mathbb{N}}$. We define $Z(t) := \overline{\text{span}}\{e_n;\epsilon_n=0\}$. Then $\dim Z(t)=\infty$ since t is irrational. $U(t)|_{Z(t)}=0$ by construction of $Z(t)$. Thus

$$\int_{[0,1]}\inf_{Z\in X,\dim Z=\infty}\|U(t)|_Z\|d\lambda(t) = 0.$$

Comparing this with (1) we see that the same inequality as in 1.1 d) does not hold for u.

f) Application for this perturbation theorem can be found in the theory of neutron transport equations (see e.g. [Vi, Vo1, We]).

We close by deducing a useful result concerning time evolution of the perturbation. The succeeding result is implicitly contained in 2.7 and extends a result given by J. Voigt [Vo1].

Theorem 2.10. *We suppose that for all choices of $t_1,t_2>0$ the operator $S(t_1)BS(t_2)$ is weakly compact, resp. conditionally weakly compact. Then for all $t\geq 0$ the operators:*

$$\begin{aligned} T(t)-S(t) \;&:\; X\longrightarrow X \qquad\qquad \textit{and}\\ \overline{T}-\overline{S} \;&:\; X\longrightarrow L_1([0,t],X)\\ &\quad\; x\longmapsto (T(\cdot)-S(\cdot))x \qquad \textit{weakly compact,}\\ &\quad\; \textit{resp. conditionally weakly compact)}\ . \end{aligned}$$

Proof. From

$$T(t) = S(t) + \int_0^t S(t)BT(t-s)ds$$

we get

$$\begin{aligned}\forall\, t \geq 0: \quad T(t) &= S(t) + \int\limits_0^t S(s)BT(t-s)ds \\ &= S(t) + \int\limits_0^t S(s)BS(t-s)ds + \\ &\quad \int\limits_0^t \Big(\int\limits_0^{t-s_1} S(s_1)BS(s_2)BT(t-s_1-s_2)ds_2\Big)ds_1 .\end{aligned}$$

We note that the above representation implies that for all $t \geq 0$ the operator $T(t) - S(t)$ is weakly compact, resp. conditionally weakly compact by [S2]. Again by [S2] (with a slight modification of the proofs) , we deduce for all $t \geq 0$ that $\overline{T} - \overline{S}$ is weakly compact resp. conditionally weakly compact. □

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