# Renorming in the Space of Bochner integrable Functions $L_1(\mu, X)$

by G.Schlüchtermann

**Abstract**

A sufficient condition is given when a subspace $L \subset L_1(\mu, X)$ of the space of Bochner integrable function defined on a finite and positive measure space $(S, \Sigma, \mu)$ with values in a Banach space $X$ is locally uniformly convex renormable in terms of the integrable evaluation $\{\int_A f d\mu; f \in L\}$. This shows the lifting property that $L_1(\mu, X)$ is renormable if and only if $X$ is, and indicates a large class of renormable subspaces even if $X$ does not admit an equivalent locally uniformly convex norm.

## 1.Introduction and Preliminaries

Usually problems concerning the Bochner integrable function $L_1(\mu, X)$, defined for a finite and positive measure space $(S, \Sigma, \mu)$ and a Banach space $X$, are divided into two classes:

- Can properties of $X$ be lifted to the space $L_1(\mu, X)$?
- How can geometric resp. topological properties of a subset $K \subset L_1(\mu, X)$ be characterized by the integral evaluation $\{\int_A f d\mu; f \in K\}$ for $A \in \Sigma$?

The famous Talagrand result "$L_1(\mu, X)$ is weakly sequentially complete iff $X$ is" can be put in the first category. Characterisations of relatively weakly compact subsets under certain conditions to the Banach space $X$ like the Dunford-Pettis criterion (see e.g. **[DU,]**) belong to the second one. In this paper both categories are combined for the open question about locally uniformly renorming of subspaces $L \subset L_1(\mu, X)$ . Recall the definition: A Banach spce $X$ is called *locally uniformly convex* iff
For all sequences $(x_n)_{n\in\mathbf{N}} \subset S(X)$ and $x \in S(X)$ in the unit sphere of X, s.t. $\lim_{n\in\mathbf{N}} \|x_n + x\| = 2$ implies $\lim_{n\in\mathbf{N}} \|x_n - x\| = 0$.
In **[LS]** and **[SM]** it was proved that special smooth properties of the Banach space $X$ is passed to the space of Bochner integrable functions $L_p(\mu, X)$, for

$1 < p < \infty$. They didn't consider any renorming of the space $L_p(\mu, X)$ since the natural norm provided the properties. In the case $p = 1$ things changes, since even in the scalar case $L_1(\mu)$ doesn't have a locally uniformly convex norm. The usual renorming technic tries to find a suitable not necessarily linear map $T : L \longrightarrow c_0(\Delta)$ for the renorming space $L$ and to define the equivalent norm by using the Day norm on $c_0(\Delta)$ (see e.g.**[GTWZ2]**,**[Zi]**). This is mainly done by constructing mappings $S_\delta : L \longrightarrow L$ which was first used by S.Troyanski (see**[Tr1]**). But he introduced the technic for $S_\delta L$ being separable. This difficulty was surpassed by V.Zizler (see **[Zi]**) using a long sequence of projections approximating the identity. For a general Banach space the only known existing projections for the space $L_1(\mu, X)$ are a long sequence of expectation operators $(\mathbf{E}_\alpha)_{\omega \leq \alpha \leq \lambda}$,where $\omega$ is the first infinite ordinal number and $\lambda$ a certain ordinal number. For subspaces in general $\mathbf{E}_\alpha$ doesn't map $L$ into $L$. This difficulty will be overcome by using an isometric representation of $L_1(\mu, X)$ by more easy to handle measure spaces and to investigate the structure of the expectation operator in detail for this particular measure space (see section 2). The main result will be: Let $L \subset L_1(\mu, X)$ be a subspace s.t.

$$\forall\ A \in \Sigma : \text{ Closure of}\Big\{ \int\limits_A f\ d\mu; f \in L \Big\} \text{ admits an equivalent locally}$$

uniformly convex norm, then:

$\exists\ |||\cdot||| : L \longrightarrow \mathbf{R}$ uniformly convex norm with

$$\forall\ f \in L_1(\mu, X) : \frac{1}{2}\|f\|_1 \leq |||f||| \leq \|f\|_1.$$

In addition the $\sigma' = (\sigma(L_1(\mu, X), L_\infty(\mu, X^*)))$-topology will be part of the consideration, where $L_\infty(\mu, X^*)$ is the space of $\mu$-essentially bounded and $X^*$ -valued functions. In **[BS]** it was shown that for a $\sigma'-$compactly generated subspace $L = \overline{\text{span}}^{\sigma'} K, K \subset L_1(\mu, X)$ $\sigma'$- compact there exist an bounded injection $T : L \longrightarrow c_0(\Gamma)$, hence such spaces are strictly convex renormable. The criterion here enables to extend this result.

The proof of the main result will be split into two parts. In section 2 integrals of a chain of martingale difference sequences will be represented as the difference of integral evaluations of two disjoint subsets of the measure space. There, the special measure space $(D_{\lambda_0}, Q_{\lambda_0}, \nu_{\lambda_0})$ will be considered, where $D_{\lambda_0} = \{0,1\}^{\lambda_0}$ for $\lambda_0$ the first ordinal number of cardinality of $\lambda_0$, $Q_{\lambda_0}$ is the product measure space and $\nu_{\lambda_0}$ is the product Haar propability measure.

The third section is devoted to the proof of the main result and its application to certain subspaces.

Throughout the paper $(S, \Sigma, \mu)$ is a finite and positive measure space, $X$ a real or complex Banach space,$S(X) := \{x \in X; \|x\| = 1\}$ denote the unit sphere,$\|\cdot\|_1$ the usual norm on $L_1(\mu, X)$,$\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. For a subset $L \in L_1(\mu, X)$ and $A \in \Sigma$ abbreviate $\{\int_A f d\mu; f \in L\}$ by $\int_A(L)$.

# 2.Integrals of Martingal Difference Sequences

In this section an auxiliary result is derived which will enable to establish the Main Theorem more easy. Throughout this section the measure space $(D_{\lambda_0}, Q_{\lambda_0}, \nu_{\lambda_0})$ is considered for which some notations are needed.
Let $\Gamma \subset [0,\lambda_0[$ be an infinite subset of the ordinal interval $[0,\lambda_0[$. Set $\Sigma(\Gamma) := pr^{-1}(Q_\Gamma)$ where $pr: \ D_{\lambda_0} \longrightarrow D_\Gamma$ is the measureable projection on the components of $\Gamma$. For $\beta < \lambda_0, i \in \{0,1\}$ define $A_i^\beta := \prod_{\gamma<\lambda_0} B_\gamma$ where $B_\gamma = \{0,1\}$ for $\gamma \neq \beta$ and $B_\gamma = \{i\}$ for $\gamma = \beta$. Then easily seen for $\Gamma \subset [0,\lambda_0[, \aleph_0 \leq \text{card } \Gamma, \alpha \in [0,\lambda_0[\backslash\Gamma$:

$$\Sigma(\Gamma \cup \{\alpha\}) = \Big\{(A_0^\alpha \cap A) \cup (A_1^\alpha \cap B); A, B \in \Sigma(\Gamma)\Big\}.$$

For an infinite subset $\Gamma \subset [0,\lambda_0[$ consider the system

$$\mathcal{G}(\Gamma) := \Big\{ \bigcap_{k-1}^{m} A_{p_k}^{\beta_k}; p_k \in \{0,1\}, \beta_k \in \Gamma, \beta_k \neq \beta_j, \text{ if } k \neq j \ ,$$
$$j, k \in \{1,\ldots,m\}, m \in \mathbf{N}\Big\} \quad \cup \quad \{\emptyset\}.$$

The definition reveals that $\mathcal{G}(\Gamma)$ generates $\Sigma(\Gamma)$. The algebra $\mathcal{A}(\mathcal{G}(\Gamma))$ generated by $\mathcal{G}(\Gamma)$ is dense in $\Sigma(\Gamma)$ by the usual metric $d\nu_{\lambda_0}$.

The following algebraic problem has to be solved to gain the result about renorming in $L_1(\mu, X)$. It will be proved in several steps and may independantly be of interest since it gives a representation of integrals of a finite chain of martingale differences.

**2.1.Theorem:** Let $\Gamma_1,\ldots,\Gamma_n \subseteq [0,\lambda_0[, \aleph_0 \leq \text{card } \Gamma_i, \alpha_i \in [0,\lambda_0[\backslash\Gamma_i \ \ (i = 1,\ldots,n)$ s.t. $(\Gamma_i \cup \{\alpha_i\}) \subset (\Gamma_{i+1} \cup \{\alpha_{i+1}\})$, $(i = 1,\ldots,n-1)$, $A = \bigcap_{k=1}^{m} A_{p_k}^{\beta_k} \in \mathcal{G}(\Gamma_1 \cup \{\alpha_1\}), p_k \in \{0,1\}, \beta_k \in (\Gamma_1 \cup \{\alpha_1\}), k = 1,\ldots,m, m \in \mathbf{N}$. Define

$$n_0 := \begin{cases} \min\{i \in \{1,\ldots,n\}; \alpha_i \in \{\beta_1,\ldots,\beta_m\}\} & \text{if } \{i \in \{1,\ldots,n\}; \\ & \quad \alpha_i \in \{\beta_1,\ldots,\beta_m\}\} \neq \emptyset \\ n+1 & \text{else} \end{cases}$$

Assume $(\alpha_1,\ldots,\alpha_{n_0-1}) = (\beta_1,\ldots,\beta_{n_0-1})$. Then

*i)* If $\quad n_0 < n+1$ then $\forall\ f \in L_1(D_{\lambda_0}, X)$

$$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1\cup\{\alpha_1\})}-\mathbf{E}_{\Sigma(\Gamma_1)})\dots(\mathbf{E}_{\Sigma(\Gamma_n\cup\{\alpha_n\})}-\mathbf{E}_{\Sigma(\Gamma_n)})(f)d\nu_{\lambda_0}=0$$

*ii)* If $n_0 = n+1$ then

$\exists\ \Delta_1,\Delta_2\subset\{0,1\}^{n_0}, \Delta_1\cap\Delta_2=\emptyset$ s.t. $\forall\ f\in L_1(D_{\lambda_0},X)$ :

$$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1\cup\{\alpha_1\})}-\mathbf{E}_{\Sigma(\Gamma_1)})\dots(\mathbf{E}_{\Sigma(\Gamma_n\cup\{\alpha_n\})}-\mathbf{E}_{\Sigma(\Gamma_n)})(f))d\nu_{\lambda_0}=$$

$$2^{-n}(\int_C-\int_D)(f)d\nu_{\lambda_0} \text{ where}$$

$$\overline{A}:=\bigcap_{k=n_0}^{m}A^{\beta_k}_{p_k}, C:=\Big(\bigcup_{(\overline{p}_1,\dots,\overline{p}_{n_0-1})\in\Delta_1}\ \bigcap_{k=1}^{n_0-1}A^{\beta_k}_{\overline{p}_k}\Big)\cap\overline{A} \text{ and}$$

$$D:=\Big(\bigcup_{(\overline{q}_1,\dots,\overline{q}_{n_0-1})\in\Delta_2}\ \bigcap_{k=1}^{n_0-1}A^{\beta_k}_{\overline{q}_k}\Big)\cap\overline{A}$$

*Proof:* First note that it can be assumed $(\alpha_1,\dots,\alpha_{n_0-1})=(\beta_1,\dots,\beta_{n_0-1})$, since for $\alpha_i=\alpha_j, i<j$ just set $\beta_j:=\alpha_i, p_j:=p_i$ and hence $A^{\beta_j}_{p_j}=A^{\beta_i}_{p_i}$. Relabel $\beta_k, p_k$ for $k=j+1,\dots,m$ as $\beta_{k+1},p_{k+1}$.
Define for $j\in\{1,\dots,n\}$ and $f\in L_1(D_{\lambda_0},X)$ :

$$\begin{aligned} f_j &:= ((\mathbf{E}_{\Sigma(\Gamma_j\cup\{\alpha_j\})}-\ \mathbf{E}_{\Sigma(\Gamma_j)})\dots(\mathbf{E}_{\Sigma(\Gamma_n\cup\{\alpha_n\})}-\ \mathbf{E}_{\Sigma(\Gamma_n)})(f)),\\ f_{n+1} &:= f. \end{aligned}$$

First the following property (I) will be proved by induction over $1\le i<n_0$:

Let $\overline{A}^i:=\bigcap_{k=i+1}^{m}A^{\beta_k}_{p_k}$ then

$\exists\ \Delta_1^i,\Delta_2^i\subset\{0,1\}^i, \Delta_1^i\cap\Delta_2^i=\emptyset$ s.t. $\forall\ f\in L_1(D_{\lambda_0},X)$ :

$$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1\cup\{\alpha_1\})}-\mathbf{E}_{\Sigma(\Gamma_1)})\dots(\mathbf{E}_{\Sigma(\Gamma_i\cup\{\alpha_i\})}-\mathbf{E}_{\Sigma(\Gamma_i)})(f_{i+1}))d\nu_{\lambda_0}=$$

$$2^{-i}(\int_{C^i}-\int_{D^i})(f_{i+1})d\nu_{\lambda_0}$$

$$C^i:=\Big(\bigcup_{(\overline{p}_1,\dots,\overline{p}_i)\in\Delta_1^i}\ \bigcap_{k=1}^{i}A^{\beta_k}_{\overline{p}_k}\Big)\cap\overline{A}^i \text{ and } D^i:=\Big(\bigcup_{(\overline{q}_1,\dots,\overline{q}_i)\in\Delta_2^i}\ \bigcap_{k=1}^{i}A^{\beta_k}_{\overline{q}_k}\Big)\cap\overline{A}^i.$$

For the proof an elementary result (1) is needed which proof is just the application of the definition of conditional expectation operators. Hence the proof is omitted. The assertion of (1) itself will be used as auxiliary result several times.

Let $\Gamma \subset [0,\lambda_0[$, $\alpha \in [0,\lambda[\setminus\Gamma$, $f \in L_1(D_{\lambda_0},X)$, $E \in \Sigma(\Gamma)$ and $p \in \{0,1\}$. Then

$$(1)\quad \int\limits_{A_p^\alpha \cap E} \Big((\mathbf{E}_{\Sigma(\Gamma\cup\{\alpha\})} - \mathbf{E}_{\Sigma(\Gamma)})(f)\Big) d\nu_{\lambda_0} = \frac{1}{2}\Big(\int\limits_{A_p^\alpha \cap E} - \int\limits_{A_q^\alpha \cap E}\Big)(f) d\nu_{\lambda_0}$$

$(q \in \{0,1\}, p \neq q)$.

Back to the proof of the Theorem.

For $i = 1$ this follows directly by (1), where $\alpha = \alpha_1, p = p_1$, and $E = \overline{A}^1$. Assume now that (I) is correct for $i-1$ then write

$$\int\limits_A ((\mathbf{E}_{\Sigma(\Gamma_1\cup\{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_{i-1}\cup\{\alpha_{i-1}\})} - \mathbf{E}_{\Sigma(\Gamma_{i-1})})(f_i)) d\nu_{\lambda_0} =$$

$$2^{-(n-1)}\Big(\sum_{(\overline{p}_1,\ldots,\overline{p}_{i-1})\in\Delta_1^{i-1}} \int\limits_{\bigcap_{k=1}^{i-1} A_{\overline{p}_k}^{\beta_k} \cap \overline{A}^{i-1}} - \sum_{(\overline{q}_1,\ldots,\overline{q}_{i-1})\in\Delta_2^{i-1}} \int\limits_{\bigcap_{k=1}^{i-1} A_{\overline{q}_k}^{\beta_k} \cap \overline{A}^{i-1}}\Big)(f_i) d\nu_{\lambda_0}$$

Since $i < n_0 - 1$, $\{\beta_1,\ldots,\beta_m\} \setminus \{\beta_i\} \subset (\Gamma_1 \cup \{\alpha_1\}) \setminus \{\alpha_i\} \subset \Gamma_i$. Apply (1) first to $\alpha = \beta_i, \Gamma = \Gamma_i$ and $E = \Big(\bigcap_{k=1}^{i-1} A_{\overline{p}_k}^{\beta_k} \cap \bigcap_{k=i+1}^{m} A_{p_k}^{\beta_k}\Big)$ for $(\overline{p}_1,\ldots,\overline{p}_{i-1}) \in \Delta_1^{i-1}$ [ resp. to $\overline{E} = \Big(\bigcap_{k=1}^{i-1} A_{\overline{q}_k}^{\beta_k} \cap \bigcap_{k=i+1}^{m} A_{p_k}^{\beta_k}\Big)$ for $(\overline{q}_1,\ldots,\overline{q}_{i-1}) \in \Delta_2^{i-1}$]. Then compute for $(\overline{p}_1,\ldots,\overline{p}_{i-1}) \in \Delta_1^{i-1}$:

$$\int\limits_{\left(\bigcap_{k=1}^{i-1} A_{\overline{p}_k}^{\beta_k} \cap \overline{A}^{i-1}\right)} (f_i) d\nu_{\lambda_0} = \frac{1}{2}\Big(\int\limits_{\left(\bigcap_{k=1}^{i-1} A_{\overline{p}_k}^{\beta_k} \cap A_{p_i}^{\beta_i} \cap \overline{A}^{i}\right)} - \int\limits_{\left(\bigcap_{k=1}^{i-1} A_{\overline{p}_k}^{\beta_k} \cap A_{q_i}^{\beta_i} \cap \overline{A}^{i}\right)}\Big)(f_{i+1}) d\nu_{\lambda_0},$$

where $p_i \neq q_i, q_i \in \{0,1\}$.

For $(\overline{q}_1,\ldots,\overline{q}_{i-1}) \in \Delta_1^{i-1}$:

$$\int\limits_{\left(\bigcap_{k=1}^{i-1} A_{\overline{q}_k}^{\beta_k} \cap \overline{A^{i-1}}\right)} (f_i) d\nu_{\lambda_0} = \frac{1}{2}\Big(\int\limits_{\left(\bigcap_{k=1}^{i-1} A_{\overline{q}_k}^{\beta_k} \cap A_{p_i}^{\beta_i} \cap \overline{A}^{i}\right)} - \int\limits_{\left(\bigcap_{k=1}^{i-1} A_{\overline{q}_k}^{\beta_k} \cap A_{q_i}^{\beta_i} \cap \overline{A}^{i}\right)}\Big)(f_{i+1}) d\nu_{\lambda_0},$$

where $p_i \neq q_i, q_i \in \{0,1\}$.
Define $\Delta_1^i := (\Delta_1^{i-1} \times \{p_i\} \cup \Delta_2^{i-1} \times \{q_i\})$ resp. $\Delta_2^i := (\Delta_2^{i-1} \times \{p_i\} \cup \Delta_1^{i-1} \times \{q_i\})$.
Then evidently $\Delta_1^i \cap \Delta_2^i = \emptyset$ and

$$\begin{aligned}
\int_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_{i-1} \cup \{\alpha_{i-1}\})} - \mathbf{E}_{\Sigma(\Gamma_{i-1})})(f_i)) d\nu_{\lambda_0} &= \\
2^{-(i-1)} \Big( \sum_{(\overline{p}_1, \ldots, \overline{p}_{i-1}) \in \Delta_1^{i-1}} \int_{\bigcap_{k=1}^{i-1} A_{\overline{p}_k}^{\beta_k} \cap \overline{A}^{i-1}} - \sum_{(\overline{q}_1, \ldots, \overline{q}_{i-1}) \in \Delta_2^{i-1}} \int_{\bigcap_{k=1}^{i-1} A_{\overline{q}_k}^{\beta_k} \cap \overline{A}^{i-1}} \Big)(f_i) d\nu_{\lambda_0} &= \\
2^{-i} \Big( \sum_{(\overline{p}_1, \ldots, \overline{p}_i) \in \Delta_1^i} \int_{\bigcap_{k=1}^{i} A_{\overline{p}_k}^{\beta_k} \cap \overline{A}^{i}} - \sum_{(\overline{q}_1, \ldots, \overline{q}_i) \in \Delta_2^i} \int_{\bigcap_{k=1}^{i} A_{\overline{q}_k}^{\beta_k} \cap \overline{A}^{i}} \Big)(f_{i+1}) d\nu_{\lambda_0} &= \\
2^{-i} \Big( \int_{C^i} - \int_{D^i} \Big)(f_{i+1}) d\nu_{\lambda_0}, &
\end{aligned}$$

where

$$C^i := \Big( \bigcup_{(\overline{p}_1, \ldots, \overline{p}_i) \in \Delta_1^i} \bigcap_{k=1}^{i} A_{\overline{p}_k}^{\beta_k} \Big) \cap \overline{A}^i$$

and

$$D^i := \Big( \bigcup_{(\overline{q}_1, \ldots, \overline{q}_i) \in \Delta_2^i} \bigcap_{k=1}^{i} A_{\overline{q}_k}^{\beta_k} \Big) \cap \overline{A}^i.$$

Suppose now $n_0 < n$, then $C^{n_0}, D^{n_0} \in \Sigma(\Gamma_{n_0+1})$ since $\{\beta_1, \ldots, \beta_m\} \subset (\Gamma_1 \cup \{\alpha_1\}) \setminus \{\alpha_{n_0+1}\} \subset \Gamma_{n_0+1}$. Hence

$$\begin{aligned}
\int_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_n \cup \{\alpha_n\})} - \mathbf{E}_{\Sigma(\Gamma_n)})(f)) d\nu_{\lambda_0} &= \\
2^{-n_0} \Big( \int_{C^{n_0}} - \int_{D^{n_0}} \Big)(f_{n_0+1}) d\nu_{\lambda_0} &= 0
\end{aligned}$$

If $n_0 = n$ then the assertion follows by the property (I) .

## 3.Renorming of Subspaces of Bochner Integrable Functions

The following Lemma is just the restatement of a result of V.Zizler. The proof follows basically that in **[Zi,]**. It will be applied in the Main Theorem to the long sequence $(\mathbf{E}_\alpha)_{\omega\le\alpha\le\lambda}$ of expectation operators approximating the identity.

**3.1. Lemma:** Let $(Y, \|\cdot\|)$ be a normed space and $Z \subset Y^*$ be a total subspace, s.t. $\|\cdot\| : Y \longrightarrow \mathbf{R}$ is $\sigma(Y,Z)-$lower semicontinuous. Let for $\lambda_0 > \omega$, a limit ordinal number,, $(P_\alpha)_{\omega\le\alpha\le\lambda_0}$ a family of linear projections, s.t.

*i)* $\sup\limits_{\omega\le\alpha\le\lambda_0} \|P_\alpha\| < \infty$ and for all ordinal numbers $\alpha, \omega \le \alpha \le \lambda_0$ is $P_\alpha$ $\sigma(Y,Z) - \sigma(Y,Z) -$ continuous.

*ii)* $\forall\ \alpha.\beta, \omega \le \alpha \le \beta < \lambda_0 :\ P_\alpha P_\beta = P_\beta P\alpha = P_\alpha.$

*iii)* $\forall\ \beta, \omega \le \beta < \lambda_0$ limit ordinal number $\forall\ y \in Y$ : $P_\beta(y) = \lim\limits_{\alpha<\beta} P_\alpha(y)$ in $(Y, \|\cdot\|)$.

*iv)* $\forall\ y \in Y\ \exists\ \alpha\ , \omega \le \alpha < \lambda_0 :\ P_\alpha(y) = y.$

Furthermore let $L \subset Z$ be a subspace with the property:

*I)* $\exists\ K_\omega > 0\ \exists\ \|\cdot\|_\omega : P_\omega(L) \longrightarrow \mathbf{R}$ a $\sigma(Y,Z) \cap P_\omega(L) -$ lower semicontinous locally uniformly convex norm on $P_\omega(L)$ s.t. $\forall\ y \in L : K_\omega\|P_\omega(y)\| \le \|P_\omega(y)\|_\omega \le \|P_\omega(y)\|.$

*II)* $\forall\ \alpha, \omega \le \alpha < \lambda\ \exists\ K_{\alpha+1} \ge 0\ \exists\ \|\cdot\|_{\alpha+1} : (P_{\alpha+1} - P_\alpha)(L) \longrightarrow \mathbf{R}$ a $\sigma(Y,Z) \cap (P_{\alpha+1} - P_\alpha)(L) -$ lower semicontinous locally uniformly convex norm on $(P_{\alpha+1} - P_\alpha)(L)$ s.t. $\forall\ y \in L$ : $K_{\alpha+1}\|(P_{\alpha+1} - P_\alpha)(y)\| \le \|(P_{\alpha+1} - P_\alpha)(y)\|_{\alpha+1} \le \|(P_{\alpha+1} - P_\alpha)(y)\|.$

Then there exists on $L$ a $\sigma(Y,Z) \cap L$-lower semicontinuous locally uniformly convex norm $|||\cdot|||$ s.t.

$$\forall\ y \in L : \frac{1}{2}\|y\| \le |||y||| \le \|y\|.$$

In particular the $\sigma(Y,Z) \cap L$-topology conincide with the norm topology on $S(L, |||\cdot|||)$.

The next fact should be known, though it is not found stated somewhere. The proof is omitted since it mimics the proof to the main result in **[GTWY2]**.

**3.2.Lemma:** Let $Y_1, Y_2 \subset X$ be two closed subspaces of a Banach space. Suppose $Y_1, Y_2$ admit an equivalent locally uniformly convex norm. Then $Y_1 + Y_2$ has an equivalent locally uniformly convex norm.

The proof of the following Main Theorem is divided in three steps. The first step will prove the result for a separable measure space. Here it would suffice to require $\forall\ A \in \Sigma : I(A)$ admits an equivalent locally uniformly convex norm. The applcation of Lemma 3.1. to the long sequence of expectation operators will need an isomorhic representation of the measure space by the special measure spaces $D_{\lambda_0}$. Then the result of chapter 2 will come into interplay and basically the assertion is reduced to this special case.

**3.3.Main Theorem:** Let $L \subset L_1(\mu, X)$ be a subspace s.t.

$\forall\ A \in \Sigma :$ Closure of $\int\limits_A (L)$ admits an equivalent locally uniformly convex norm, then:
$\exists\ |||\cdot||| : L \longrightarrow \mathbf{R}\ \sigma` \cap L$-lower semicontinuous locally uniformly convex norm with $\forall\ f \in L_1(\mu, X)$ :
$\frac{1}{2}||f|_1 \leq |||f||| \leq ||f||_1.$

*Proof:* The presumption combined with 3.2.Lemma implies the following property (**):
Let $L \subset L_1(\mu, X)$ be a subspace s.t.

$\forall\ C, D \in \Sigma, C \cap D\emptyset : (\int\limits_C - \int\limits_D)(L)$ admits an equivalent locally uniformly convex norm, then:
$\exists\ |||\cdot||| : L \longrightarrow \mathbf{R}\ \sigma` \cap L$-lower semicontinuous locally uniformly convex norm with $\forall\ f \in L_1(\mu, X)$ :
$\frac{1}{2}||f|_1 \leq |||f||| \leq ||f||_1.$

Thus in the sequel (**) is assumed. **Step 1:** Let $(S, \Sigma, \mu)$ be separable.
First prove the following weaker assertion (C):

Let $\mathcal{G} = \{A_1, A_2, \ldots\}$ be a countable system which is stable under intersection, s.t.:

- The algebra $\mathcal{A}(\mathcal{G}) \subset \Sigma$ is dense resp.to the metric $d\mu$.
- $\forall \delta > 0 : \ \mathrm{card}\{A \in \mathcal{G}; \mu(A) > \delta\} < \infty$.

Let $L \subset L_1(\mu, X)$ a subspace, s.t. $\forall\ A \in \mathcal{G} :\ \int\limits_A (L)$ admits an equivalent locally uniformly convex norm. Then
$\exists\ |||\cdot||| : L \longrightarrow \mathbf{R}$ an equivalent $\sigma` \cap L$-lower semicontinuous locally uniformly convex norm, s.t.
$\forall\ f \in L : \frac{1}{2}\|f\|_1 \leq |||f||| \leq \|f\|_1$.

*Proof to (C):* Let for $A \in \mathcal{G}$ $\|\cdot\|_A : \int\limits_A (L) \longrightarrow \mathbf{R}$ be an equivalent locally uniformly convex norm on $\int\limits_A (L)$, s.t.:

$$\forall\ x \in \int\limits_A (L) :\ \|x\|_A \leq \|x\|.$$

Set for $n \in \mathbf{N} :\ \sigma_n := \sigma(A_1, \ldots, A_n)$, then the algebra $\mathcal{A}(\bigcup\limits_{n\in\mathbf{N}} \sigma_n) = \bigcup\limits_{n\in\mathbf{N}} \sigma_n$ is dense in $\Sigma$ by the usual metric. Define $\Delta := \{0\} \cup \mathbf{N} \cup \mathcal{G}$ and construct the map

$$Q :\ L \longrightarrow c_0(\Delta)$$
$$Qf(\delta) := \begin{cases} \|f\|_1 & \text{if } \delta = 0 \\ \frac{1}{2}\|f - \mathbf{E}_{\sigma_n} f\|_1 & \text{if } \delta = n \in \mathbf{N} \\ \|\int\limits_A (f) d\nu_{\lambda_0}\|_A & \text{if } \delta = A \in \mathcal{G} \end{cases}$$

It is easy to realize that $\Big(Qf(\delta)\Big)_{\delta\in\Delta} \in c_0(\Delta)$. Define now

$$\begin{aligned} |||\cdot||| :\ & L && \longrightarrow \mathbf{R} \\ & f && \longmapsto (p \circ Q)(f) \end{aligned}$$

where $p$ is the Day norm on $c_0(\Delta)$. Furthermore $\forall\ f \in L :\ \frac{1}{2}|||f||| \leq \frac{1}{2}\|Qf\|_\infty \leq (p\circ Q)(f) \leq \|Qf\|_\infty \leq \|f\|_1$. Since $\forall\ \delta \in \Delta :\ Q_\delta$ is a $\sigma`\cap L$-lower semicontinuous seminorm $|||\cdot||| :\ L \longrightarrow \mathbf{R}$ is a $\sigma` \cap L$- lower semicontinuous norm (see e.g. **[Si,Lemma 20.2.]**). $|||\cdot|||$ is also locally uniformly convex which can be verified by the usual technic (see e.g.**[Di,]**). Hence the proof to step 1 is finished.
**Step 2:** Consider the special measure space $(D_{\lambda_0}, Q_{\lambda_0}, \nu_{\lambda_0})$. By transfinite induction prove the following assertion $(\Lambda)$ :
Let $\Gamma \subset [0, \lambda_0[$ be an infinite set and $L \subset L_1(\Sigma(\Gamma), X)$ a subspace with the

properties:

$I^*)$ $\quad \forall\ A \in \mathcal{G}(\Gamma)\ \int\limits_A (L)$ admits an equivalenty locally uniformly convex norm.

$II^*)$ $\quad$ If $\Gamma_1, \ldots, \Gamma_n \subset \Gamma,\ \aleph_0 \leq \text{card } \Gamma_i,\ \alpha_i \in \Gamma \setminus \Gamma_i\ (i = 1, \ldots, n)$ s.t. $(\Gamma_i \cup \{\alpha_i\}) \subset (\Gamma_{i+1} \cup \{\alpha_{i+1}\})\ (i = 1, \ldots, n-1)$ and $A \in \mathcal{G}(\Gamma_1 \cup \{\alpha_1\})$ then $\int\limits_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_n \cup \{\alpha_n\})} - \mathbf{E}_{\Sigma(\Gamma_n)})(L))$ has an equivalent locally uniformly convex norm.

Then on $L$ exists a $\sigma' \cap L$-lower semicontinuous locally uniformly convex norm $|||\cdot|||$ s.t.

$$\forall\ f \in L:\ \frac{1}{2}||f||_1 \leq |||f||| \leq ||f||_1.$$

*Proof to* $(\Lambda)$ : The proof is a transfinite induction over the cardinality of $\Gamma$.
If $\text{card}\Gamma = \aleph_0$ then the system $\mathcal{G}(\Gamma)$ is countable, closed under intersection and fulfills the condition needed for the property (C). Hence by (C) $(\Lambda)$ is satisfied. Assume now that $(\Lambda)$ is true for all infinite subset $\Gamma' \subset [0, \lambda_0[$ s.t. $\text{card}\Gamma' < \text{card}\Gamma$ and for all subspaces $L \subset L_1(\Sigma(\Gamma'), X)$ which satisfy $I^*)$ and $II^*)$ respectively. Let $\lambda_\Gamma$ be the first ordinal of the cardinality of $\Gamma$ and $L \subset L_1(\Sigma(\Gamma), X)$ a subspace satisfying $I^*)$ and $II^*)$. Consider the bijection $j : \Gamma \to [0, \lambda_\Gamma[$. For an ordinal number $\alpha, \omega \leq \alpha \leq \lambda_\Gamma$ define $\Gamma_\alpha := j^{-1}([0, \alpha[)$. The bijection $j$ implies

$$\forall\ \alpha, \beta,\ \omega \leq \alpha \leq \beta\lambda_\Gamma:\ \Sigma(\Gamma_\alpha) \subset \Sigma(\Gamma_\beta).$$

Furthermore for a limit ordinal $\omega \leq \beta \leq \lambda_\Gamma$:

$$\Sigma(\Gamma_\beta) = \sigma\Big(\bigcup_{\alpha<\beta} \Sigma(\Gamma_\alpha)\Big).$$

Set for $\omega \leq \alpha \leq \lambda_\Gamma$ : $\mathbf{E}_\alpha := \mathbf{E}_{\Sigma(\Gamma_\alpha)}$.
Apply Lemma (3.1.) to the long sequence of expectation operators $(\mathbf{E}_\alpha)_{\omega \leq \alpha < \lambda_\Gamma}$ and set $Y := L_1(\Sigma(\Gamma), X), Z := L_\infty(\Sigma(\Gamma), X^*)$. The properties of the conditional expectation operators implies with the classical martingal convergence theorem (see e.g. **[DU,p.125 Theorem 1]**) the properties $i) - iv)$ in Lemma (3.1.). To prove $(\Lambda)$ it suffices to use the induction hypothesis and to show that $\forall\ \omega \leq \alpha < \lambda_\Gamma$ : $\mathbf{E}_\omega(L)$ and $(\mathbf{E}_{\alpha+1} - \mathbf{E}_\alpha)(L)$ satisfy $I^*)$ and $II^*)$.
First consider $I^*)$.

- $\forall\ A \in \mathcal{G}(\Gamma_\omega)$ : $\int\limits_A \Big(\mathbf{E}_\omega(L)\Big) = \int\limits_A (L)$. Since $\mathcal{G}(\Gamma_\omega) \subset \mathcal{G}(\Gamma)$ and $L$ satisfies $I^*)$ the subspace $\mathbf{E}_\omega(L)$ fulfills $I^*)$ too.

- Let $\omega \leq \alpha < \lambda_\Gamma$ and $A \in \mathcal{G}(\Gamma_{\alpha+1}) \subset \mathcal{G}(\Gamma)$ and set in $II^*$) for $L$:

  $$\Gamma_1 := \Gamma_\alpha, \alpha_1 := j^{-1}(\{\alpha\}),$$

  then by $II^*$):
  $\int_A \Big((\mathbf{E}_{\alpha+1} - \mathbf{E}_\alpha)(L)\Big) = \int_A \Big((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha\})} - \mathbf{E}_{\Sigma(\Gamma_1)})(L)\Big)$ admits an equivalent uniformly convex norm.

For $II^*$) compute:

- Let $\Gamma_1, \ldots, \Gamma_n \subset \Gamma_\omega, \alpha_i \in \Gamma_\omega \setminus \Gamma_i \quad (i = 1, \ldots, n), (\Gamma_i \cup \{\alpha_i\}) \subset (\Gamma_{i+1} \cup \{\alpha_{i+1}\})$, then for $A \in \mathcal{G}(\Gamma_1 \cup \{\alpha_1\}) \subset \mathcal{G}(\Gamma_\omega)$ holds:

  $$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_n \cup \{\alpha_n\})} - \mathbf{E}_{\Sigma(\Gamma_n)})(\mathbf{E}_\omega(L)) =$$

  $$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_n \cup \{\alpha_n\})} - \mathbf{E}_{\Sigma(\Gamma_n)})(L)).$$

  and $II^*$) follows for $(\mathbf{E}_\omega)(L)$ since $L$ satisfies $II^*$).

- Let $\Gamma_1, \ldots, \Gamma_n \subset \Gamma_{\alpha+1}, \aleph_0 \leq \mathrm{card}\Gamma_i, \alpha_i \in \Gamma_{\alpha+1} \setminus \Gamma_i \quad (i = 1, \ldots, n), (\Gamma_i \cup \{\alpha_i\}) \subset (\Gamma_{i+1} \cup \{\alpha_{i+1}\})$, then for $A \in \mathcal{G}(\Gamma_1 \cup \{\alpha_1\}) \subset \mathcal{G}(\Gamma_{\alpha+1}) \subset \mathcal{G}(\Gamma)$ it follows:

  $$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_n \cup \{\alpha_n\})} - \mathbf{E}_{\Sigma(\Gamma_n)})(\mathbf{E}_{\alpha+1} - \mathbf{E}_\alpha)(L)) =$$

  $$\int_A ((\mathbf{E}_{\Sigma(\Gamma_1 \cup \{\alpha_1\})} - \mathbf{E}_{\Sigma(\Gamma_1)}) \ldots (\mathbf{E}_{\Sigma(\Gamma_n \cup \{\alpha_n\})} - \mathbf{E}_{\Sigma(\Gamma_n)})$$
  $$(\mathbf{E}_{\Sigma(\Gamma_{n+1} \cup \{\alpha_{n+1}\})} \quad - \quad \mathbf{E}_{\Sigma(\Gamma_{n+1})})(L))$$

  where $\Gamma_{n+1} := \Gamma_\alpha$ and $\alpha_{n+1} := \alpha$. It is $\alpha \in \Gamma \setminus \Gamma_\alpha, \Gamma_n \subset \Gamma_{\alpha+1} = \Gamma_\alpha \cup \{\alpha\} = \Gamma_{n+1} \cup \{\alpha_{n+1}\}$ and $\alpha_n \in \Gamma_{\alpha+1} = \Gamma_{n+1} \cup \{\alpha_{n+1}\}$. Hence $(\Gamma_1, \ldots, \Gamma_{n+1})$ and $(\alpha_1, \ldots, \alpha_{n+1})$ satisfy the presumptions in $II^*$) for $L$ and thus $(\mathbf{E}_{\alpha+1} - \mathbf{E}_\alpha)(L)$ fulfills $II^*$) too.

This finishes the proof of property $(\Lambda)$. But $(\Lambda)$ combined with (2.1.) Theorem asssures the assertion for the special measure space$(D_{\lambda_0}, Q_{\lambda_0}, \nu_{\lambda_0})$.
**Step 3:**For an arbitrary finite measure space use the following isometric representation of $L_1(\mu, X)$ which is based on the well known measure algebraic representation of D.Maharam (see e.g.[**La,p.127 Theorem 9**]):

$\exists$ disjoint and countable sets $I_1, I_2, \exists$ cardinal numbers $\mathcal{C}_i (i \in I_2)$,
$\exists$ isometry

$$T : L_1(\mu, X) \longrightarrow \Big( (\bigoplus_{i\in I_1} X) \oplus (\bigoplus_{i\in I_2} L_1(D_{\lambda_i}, X) \Big)_{\ell_1(I_1 \cup I_2)}$$

(where $\lambda_i$ is the first ordinal number of cardinality $\mathcal{C}_i$ $(i \in I_2)$), which is a

$$\sigma`-\sigma\Big(\Big((\bigoplus_{i\in I_1} X)\oplus(\bigoplus_{i\in I_2} L_1(D_{\lambda_i}, X)\Big)_{\ell_1}, \Big((\bigoplus_{i\in I_1} X^*)\oplus(\bigoplus_{i\in I_2} L_\infty(D_{\lambda_i}, X^*)\Big)_{\ell_\infty}\Big)-$$

homeomorphism.

The isometry $T$ is the tensor product $T = \overline{T} \otimes id_X$ where

$$\overline{T} : L_1(\mu) \longrightarrow \Big( (\bigoplus_{i\in I_1} \mathbf{K}) \oplus (\bigoplus_{i\in I_2} L_1(D_{\lambda_i}) \Big)_{\ell_1(I_1 \cup I_2)}$$

according to **[La,p.127 Theorem 9]**. Thus $T$ indicates the following properties:

- $\exists\ (A_i)_{i\in I_1} \subset \Sigma\ \forall\ i \in I_1\ \forall\ f \in L_1(\mu, X) : pr_i\Big((Tf)\Big) = \int\limits_{A_i} f d\mu.$
- $\forall\ i \in I_2\ \forall\ A, B \in Q_{\lambda_i}, A \cap B = \emptyset \exists\ \overline{A}, \overline{B} \in \Sigma, \overline{A} \cap \overline{B} = \emptyset,\ \forall\ f \in L_1(\mu, X) :$
  $\Big(\int\limits_A - \int\limits_B\Big)\Big(pr_i(Tf)\Big) d\nu_{\lambda_0} = \Big(\int\limits_{\overline{A}} - \int\limits_{\overline{B}}\Big)\ \ d\mu.$

This implies immediately with step 2 the Theorem.

The preceding thTheorem immidiately implies the following lifting prperty.

**3.4.Corollary:** Let $(S, \Sigma, \mu)$ be a finite measure space. Then $L_1(\mu, X)$ admits an equivalent uniformly convex norm if and only if $X$ does. In this case on $S(L_1(\mu, X))$ the norm and $\sigma`$-topology coincide.

Taken the result of l.Vasak **[Va]** into accout the next Corollary follows.

**3.5.Corollary:** Let $(S, \Sigma, \mu)$ be a finite measure space. Let $L \subset L_1(\mu, X)$ be a subspace s.t. $\forall\ A \in \Sigma :\ \{\int\limits_A (f) d\mu; f \in L\}$ is a subspace of a WCD-Banach space, then $L$ has an equivalent locally uniformly convex norm, s.t. on $S(L)$ the norm topology agrees with the $\sigma`$- topology.
In particular every $\sigma`$-compactly generated subspace admits an equivalent locally uniformly convex norm.

*3.6.Remark:* a) The result can be extended to an arbitrary not necessarily finite measure space by using the well known isometric representation.
b) If one exchange *locally uniformly convex* by *Kadec norm*, which means that on the unit sphere the weak and norm topology coincide, then the same result hold too,i.e. on the unit spehere the norm and $\sigma$'-topology are the same. c) It seems not known if the locally uniform convexity of a normed space $Y$ allows to define an equivalent locally uniformly convex norm on the completion. A positive answer would enable to lift the assumption that the closure of $\int_A(L)$ has an equivalent locally uniformly convex norm (see 3.2.Lemma).

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## Mailing Address:
**Georg Schlüchtermann**
**Mathematisches Institut der Universität München**
**Theresienstrasse 39**
**80333 München Germany**