Factorization of operator-valued Functions

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Abstract

For a finite and positive measure space (Ω, Σ, μ) we present results on the factorization of strongly measurable operator-valued functions. This leads on the one hand to characterizations of weakly compact and condtionally compact operators $T : X \longrightarrow L_\infty(\mu, Y)$, and on the other hand to structural properties, like the strong WCG-spaces introduced by G. Schüchtermann and R. Wheeler.

46E10, 46G10

1 Introduction and Preliminaries

Since in 1974 Davis et al. came up with their well-celebrated factorization theorem for weakly compact operators, their scheme has been imitated for a lot of other classes of operators. Here we are going to investigate this scheme in a parametric fashion, like the parametric ℓ_1-Rosenthal dichotomy of M. Talagrand [Ta].

Let us first fix the notation and outline the problem. For a finite and positive measure space (Ω, Σ, μ) and Banach spaces X, Y we consider bounded operator-valued functions $U : \Omega \longrightarrow L(X, Y)$, which are strongly measurable, i.e. $\Omega \ni \omega \longmapsto U(\omega)(x) \in Y$ is $\mu-$measurable for all $x \in X$. Suppose $A(X, Y)$ is an operator ideal, which admits a factorization according to the factorization scheme of Davis et al., and $U(\Omega) \subset A(X, Y)$, then we are going to investigate the following two problems: For $\omega \in \Omega$ let $Z(\omega)$ be a Banach space, such that $id_{Z(\omega)} \in A(Z(\omega))$ and let $U_1(\omega) \in L(X, Z(\omega)), U_2 \in L(Z(\omega), Y)$ be such that $U(\omega) = U_2(\omega) \circ U_1(\omega)$.

I) When do we have one and only one Banach space Z and a $T \in L(X, Z)$, such that $Z = Z(\omega)$ and $T = U_1(\omega)$ for a.a. $\omega \in \Omega$?

II) When do we find for all $\epsilon > 0$ a set $\Omega_\epsilon \subset \Omega_\epsilon, \mu(\Omega \setminus \Omega_\epsilon) < \epsilon$, one and only one Banach space Z and a $T \in L(Z, Y)$, such that $Z = Z(\omega)$ and $T = U_2(\omega)$ for all $\omega \in \Omega_\epsilon$?

The answer to the first question will lead to characterizations of certain classes of operators into $L_\infty(\mu, Y)$, like the weakly compact operators, while the second one gives structural properties of the Banach space Y.

Finally $B(X)$ denotes the unit ball of X, X^* will be the dual space and for a bounded and strongly measurable function $U : \Omega \longrightarrow L(X,Y)$ let us define $\overline{U} : X \longrightarrow L_\infty(\mu, Y)$ by $\overline{U} = U(\cdot)(x),\ x \in\ X$. For a Banach space X, $\mathcal{B}(X), \mathcal{W}(X), \mathcal{A}(X)$ and $\mathcal{CW}(X)$ denote the set of bounded, relatively weakly compact, Asplund and conditionally weakly compact sets, respectively.

2 Factorizing operators $T : X \longrightarrow L_\infty(\mu, Y)$

For answering problem I) we are going to select certain factorizable operator ideals, which include major ideals like the weakly compact or conditionally weakly compact operators. We will consider a class $\mathcal{F}(X^*)$ of w^*-closed subsets in the dual space X^* such that:

i) $\forall$ Banach spaces Y and $T \in L(Y,X) :\ T^*(\mathcal{F}(X^*)) \subset \mathcal{F}(Y^*)$.

ii) $\forall$ convex $B \in \mathcal{F}(X^*)$ $\forall$ convex and w^*-closed $C \subset B :\ C \in \mathcal{F}(X^*)$.

An operator ideal A is called an $\mathcal{F}$-ideal (written $A_\mathcal{F}$), if it satisfies the following two conditions:

a) For all Banach spaces X, Y, an operator T belongs to $A(X,Y)$ if and only if there exists a Banach space Z, such that $id_Z \in A(Z)$ and there are $T_1 \in L(X,Z), T_2 \in L(Z,Y)$, such that $T = T_2 \circ T_1$.

b) For all Banach spaces X, Y we have $T \in A(X,Y) \Leftrightarrow T^*(B(Y^*)) \in \mathcal{F}(X^*)$.

For this class of operator ideals we can formulate a strong parametric version of the factorization principle. $\mathcal{F}$ will always be a class of sets considered above.

Let us first consider the following examples of $\mathcal{F}$-ideals:

1) $W(X,Y)$, the ideal of weakly compact operators together with the class $\mathcal{F}$ of weakly compact sets.
In fact this follows from the factorization theorem of Davis et al. (see [Di, p.227, Lemma 3]) and the Gantmacher theorem.

2) $D_\infty(X,Y)$, the Asplund operators together with the class $\mathcal{F}$ of absolutely convex, w^*-compact sets with the RNP.
In fact by the factorization scheme of Davis et al., we construct an Asplund space Z. The class of subsets $\mathcal{F}(X^*)$ are the w^*-compact and absolutely convex subsets in the dual with the RNP (see [Bog, Lemma 5.3.4]).

3) $CW(X,Y)$, the conditionally weakly compact operators:
Again by the factorization scheme according to Davis et al. we can contruct a space Z, such that $\ell_1 \not\subset Z$. The class $\mathcal{F}(X^*)$ is given by the class of absolutely convex, w^*-compact sets with the complete continuity property (see [Bog, Th. 7.4.12]).

Theorem 2.1 *Let $U : \Omega \longrightarrow A_{\mathcal{F}}(X,Y)$ be strongly measurable and bounded. Then the following conditions are equivalent:*

a) $\exists$ Banach space Z, such that $id_Z \in A_{\mathcal{F}}(Z)$ $\exists$ $T \in L(X,Z)$ $\exists$ $U_2 : \Omega \longrightarrow L(Z,Y)$ bounded and strongly measurable, such that

$$U(\omega) = U_2(\omega) \circ T \text{ for a.a } \omega \in \Omega.$$

b) $\overline{U} \in A_{\mathcal{F}}(X, L_\infty(\mu, Y))$.

c) There exists an absolutely convex set $K \in \mathcal{F}(X^*)$, such that $U(\omega)^*(B(Y^*)) \subset K$ for a.a. $\omega \in \Omega$.

Proof: a) $\Rightarrow$ b): We have $\overline{U} = \overline{U_2} \circ T$ and $\overline{U_2} : Z \longrightarrow L_\infty(\mu, Y)$. Since $id_Z \in A_{\mathcal{F}}(Z)$ and $A_{\mathcal{F}}$ is an operator ideal, we see that $\overline{U_2} \in A_{\mathcal{F}}(Z, L_\infty(\mu, Y))$. Thus $\overline{U} \in A_{\mathcal{F}}(X, L_\infty(\mu, Y))$.

b) $\Rightarrow$ a): If $\overline{U} \in A_{\mathcal{F}}(X, L_\infty(\mu, Y))$, then according to the definition of $A_{\mathcal{F}}$ there is a Banach space Z with $id_Z \in A_{\mathcal{F}}(Z)$, and operators $T_1 \in L(X,Z)$, $T_2 \in L(Z, L_\infty(\mu, Y))$ such that $\overline{U} = T_2 \circ T_1$. [S1, Th. 2.1] tells us that there is a bounded and strongly measurable $U_2 : \Omega \longrightarrow L(Z,Y)$ such that $\overline{U_2} = T_2$. But this gives a).

b) $\Rightarrow$ c): $\overline{U}$ factors through a Banach space Z such that $id_Z \in A_{\mathcal{F}}(Z)$, i.e. there exist bounded operators $T_1 : X \longrightarrow Z$ and $T_2 : Z \longrightarrow L_\infty(\mu, Y)$, such that $\overline{U} = T_2 \circ T_1$. By [S1, Th. 2.1], there exists a strongly measurable map $U_2 : \Omega \longrightarrow L(Z,Y)$ such that $\overline{U_2} = T_2$. We have $U(\omega) = U_2(\omega) \circ T_1$ for a.a. $\omega \in \Omega$. Thus $U(\omega)^* = T_1^* \circ U_2(\omega)^*$ for a.a. $\omega \in \Omega$. Without loss of generality we assume that $\|T_2\| \leq 1$. Then $\|U_2(\omega)^*\| \leq 1$ for a.a. $\omega \in \Omega$. Let $K := T_1^*(B(Z^*)$. Then for a.a. $\omega \in \Omega$:

$$U(\omega)^*(B(Y^*)) \subset K,$$

and K is absolutely convex and belongs to $\mathcal{F}(X^*)$.

c) $\Rightarrow$ b): Let K be as assumed. For all $y^* \in B(Y^*)$ the map $\Omega \ni \omega \longmapsto U(\omega)^*(y^*)$ is w^*-measurable. Hence $\omega \longmapsto U(\omega)^*(f(\omega))$ is w^*-integrable for all $f \in L_1(\mu, Y^*)$, since U is bounded and strongly measurable. Thus we may define

$$\begin{aligned} \tilde{T} \quad : \quad & L_1(\mu, Y^*) \longrightarrow X^* \\ & f \longmapsto \int\limits_\Omega U(\omega)^*(f(\omega))d\mu(\omega) \quad \text{(as Gelfand integral)}. \end{aligned}$$

It is easy to check that $\overline{U}^*|_{L_1(\mu,Y^*)} = \tilde{T}$.

If $f \in B(L_1(\mu, Y^*))$ is a simple function, $f = \sum\limits_{i=1}^n y_i^* \chi_{A_i}$, then

$\tilde{T}(f) = \overline{U}^*(f) = \sum\limits_{i=1}^n \int\limits_{A_i} U(\omega)^*(y_i) \in K.$

Since the simple functions are norm dense in $L_1(\mu, Y^*)$, we see that $\overline{U}^*$ maps $B(L_1(\mu, Y^*))$ into K. Hence, by Goldstine's theorem and the w^*-w^*-continuity of $\overline{U}^*|_{B(L_\infty(\mu,Y)^*)}$, the image of $B(L_\infty(\mu, Y)^*)$ under T^* is a convex and w^*-closed subset of K, hence belongs to $\mathcal{F}(X^*)$. Thus, $\overline{U} \in A_{\mathcal{F}}(X, L_\infty(\mu, Y))$. □

Theorem 2.2 *: Let $T : X \longrightarrow L_\infty(\mu, Y)$ be a bounded operator. Then $T \in A_{\mathcal{F}}(X, L_\infty(\mu, Y))$ if and only if there exists a strongly measurable density $U : \Omega \longrightarrow L(X, Y)$ of T and there exists an absolutely convex set $K \in \mathcal{F}(X^*)$, such that $U(\omega)^*(B(Y^*)) \subset K$ for a.a. $\omega \in \Omega$.*

Proof: This follows immediately by applying [S1, p.380,Th.2.1] and the previous theorem. □

Remark 2.3 Theorem 2.2 can be used to deduce conditions for weak convergence in $L_\infty(\mu, X)$ resp. weak Cauchy sequences (see [S1, S2]). Directly, we have criteria for weakly compact, Asplund and conditionally weakly compact operators.

3 Strongly generated Banach spaces

First we define a certain class of bounded subsets, the *Grothendieck class* $\mathcal{G}$:
A class of subsets $\mathcal{G}(X) \subset \mathcal{B}(X)$ is called a *Grothendieck-class* of subsets if it satisfies the following conditions:

1) $\forall\ B, C \in \mathcal{G}(X)\ \forall\ D \subset B\ \forall\ m \in \mathbb{N} :\ B \cup C \in \mathcal{G}(X),\ mB \in \mathcal{G}(X) D \in \mathcal{G}(X)$, $\mathrm{co}B,\ \mathrm{aco}B \in \mathcal{G}(X)$,
2) $\forall\ B \in \mathcal{B}(X) :\ B \in \mathcal{G}(X) \Leftrightarrow\ \exists (B_n)_{n\in\mathbb{N}} \subset \mathcal{G}(X) :$ $B \subset \bigcap_{n\ni\mathbb{N}} \left(B_n + \frac{1}{n}B(X)\right)$.
3) $\forall\ A \in \mathcal{G}(X)\ \forall$ Banach spaces $Y\ \forall$ weakly continuous $T : A \longrightarrow Y : T(A) \in \mathcal{G}(Y)$.

The classes $\mathcal{W}, \mathcal{A}$ resp. $\mathcal{CW}$ are examples of Grothendieck classes of subsets (see [Di, Bog]). For a given Grothendieck class $\mathcal{G}$ we define the ideal $A_{\mathcal{G}}$ by:
For all Banach spaces $X, Y :\ T \in A_{\mathcal{G}}(X, Y)$ if and only if T factors through a space Z, such that $B(Z) \in \mathcal{G}$.
Let us remark that the Grothendieck classes find there applications in finding criteria for solutions of differential equations in Banach spaces, using certain measures of non compactness (see [KS] for further details).
We turn now to the second question.
Let $\mathcal{G}$ be a Grothendieck class. Let $U : \Omega \longrightarrow A_{\mathcal{G}}(X, Y)$ be strongly measurable and bounded.

U has the *Collectionwise Grothendieck Property (CGP)*if

$$\forall \epsilon > 0\ \exists\ \Omega_\epsilon \in \Sigma, \mu(\Omega \setminus \Omega_\epsilon) < \epsilon\ \exists\ K_\epsilon \in \mathcal{G}(X)\quad U(\Omega_\epsilon)(B(X)) \subset\ K_\epsilon.$$

This definition for operator-valued functions will be used now to inevestigate structural properties for Banach spaces.

> A Banach spaces Y has the *Collectionwise Grothendieck Property (CGP)* if for all separable X every strongly measurable and bounded $U : \Omega \longrightarrow A_{\mathcal{G}}(X,Y)$ has the *(CGP)*.

The following theorem will develop the connection to the Bochner integrable functions and answers question II) of the introduction.

Theorem 3.1 *: For a Banach space Y the following conditions are equivalent:*

a) Y has the CGP.

b) Every uniformly bounded and strongly measurable $U : \Omega \longrightarrow A_{\mathcal{G}}(\ell_1, Y)$ has the CGP.

c) $\forall$ separable X $\forall$ strongly measurable and bounded $U : \Omega \longrightarrow A_{\mathcal{G}}(X,Y)$ and $\forall\, \epsilon > 0\ \exists$ Banach space Z_ϵ, $B(Z_\epsilon) \in \mathcal{G}(Z_\epsilon)\ \exists\, \Omega_\epsilon \subset \Omega$, $\mu(\Omega \backslash \Omega_\epsilon) < \epsilon\ \exists\, U_2 \in L(Z_\epsilon, Y)\ \exists\, U_1 : \Omega_\epsilon \longrightarrow L(X, Z_\epsilon)$ bounded and strongly measurable, such that $U(\omega) = U_2 \circ U_1(\omega)$, $\omega \in \Omega_\epsilon$.

*d) For all bounded and uniformly integrable sequences $(f_n)_{n\in\mathbb{N}} \subset L_1(\mu, Y)$, such that $(f_n(\omega))_{n\in\mathbb{N}} \in \mathcal{G}(Y)$ for a.a. $\omega \in \Omega$ we have:
$\forall\, \epsilon > 0\ \exists\, \Omega_\epsilon \subset \Omega, \mu(\Omega \setminus \Omega_\epsilon) < \epsilon\ \exists\, K_\epsilon = \overline{acoK_\epsilon} \in \mathcal{G}(Y) : \bigcup_{n\in\mathbb{N}} f_n(\Omega_\epsilon) \subset K_\epsilon$.*

Proof: The implication a)$\Rightarrow$b) is clear.
a)$\Leftrightarrow$c): This follows according to condition 3) ($K_\epsilon := U_2(B(Z))$) and the factorization scheme according to Davis et al. (see [Di, p.227,Lemma 3]).
b)$\Rightarrow$d): Let $(f_n)_{n\in\mathbb{N}} \subset L_1(\mu, Y)$ be given. By extracting a set of measure zero, we may assume that $(f_n(\omega)) \in \mathcal{G}(Y)$ for all $\omega \in \Omega$. The uniform integrability and the L_1-boundedness implies:
$\forall\, k \in \mathbb{N}\ \exists M > 0\ \forall n \in \mathbb{N}\ \exists A_n \in \Sigma : \|f_n \chi_{A_n}\|_\infty \leq M,\ (f_n \chi_{A_n}(\omega))_{n\in\mathbb{N}} \in \mathcal{G}(Y)$
$\int_{A_n} \|f_n\| d\mu \geq \|f_n\|_1 - \frac{1}{k}$.
We define for $\omega \in \Omega :\ U(\omega) : \ell_1 \longrightarrow Y,\ U(e_n)(\omega) := f_n \chi_{A_n}(\omega)$. Then
$U : \Omega \longrightarrow A_{\mathcal{G}}(\ell_1, Y)$ is strongly measurable and bounded. According to b):
$\forall\, \delta > 0\ \exists\, \Omega_{\delta,k} \subset \Omega, \mu(\Omega_{\delta,k}) \geq \mu(\Omega) - \frac{\delta}{2^k},\ \exists\, K_{\delta,k} \subset \mathcal{G}(Y)\ U(\Omega_{\delta,k})(B(\ell_1)) \subset K_{\delta,k}$.
If we define $\Omega_\delta := \bigcap_{k\in\mathbb{N}} \Omega_{\delta,k}$, then $\mu(\Omega_\delta) \geq \mu(\Omega) - \delta$ and $\bigcup_{n\in\mathbb{N}} (f_n(\Omega_\delta) \subset \bigcap_{k\in\mathbb{N}} \big(K_{\delta,k} + \frac{1}{k} B(Y)\big)$. Thus the definition of Grothendieck class of sets gives the assertion.
d)$\Rightarrow$a): We assume without loss of generality that $\mu(\Omega) = 1$. Let X be separable, $(x_n)_{n\in\mathbb{N}} \subset B(X)$ be dense and $U : \Omega \longrightarrow A_{\mathcal{G}}(X,Y)$ be bounded and strongly measurable. We define for $\omega \in \Omega :\ g_n(\omega) := U(\omega)(x_n)$. Then $(g_n)_{n\in\mathbb{N}} \subset L_\infty(\mu, Y)$. We can deduce according to d) :
$\forall\, \epsilon > 0\ \exists\, \Omega_\epsilon \subset \Omega, \mu(\tilde{\Omega}_\epsilon) \geq 1 - \epsilon\ \exists\, K_\epsilon \in \mathcal{G}(Y)$ such that $\bigcup_{n\in\mathbb{N}} g_n(\Omega_\epsilon) \subset K_\epsilon$.
Since $U(\Omega) \subset L(X,Y)$, K_ϵ is absolutely convex and closed and (x_n) is dense in the unit ball, we have $U(\Omega_\epsilon)B(X) \subset K_\epsilon$. □

Remark 3.2 *If $\mathcal{G} = \mathcal{W}, \mathcal{A}$ resp. $\mathcal{CW}$, then CGP is replaced by CWP, CAP resp. CCWP.*

a) If X has the CWP, then the set $\{f_n; n \in \mathbb{N}\} \subset L_1(\mu, X)$ in 3.1 d) is a δS-set in $L_1(\mu, X)$, i.e. $\forall\ \delta > 0\ \exists\ K_\delta \subset X$ weakly compact $\forall n \in \mathbb{N} : \mu(\{\omega \in \Omega; f_n(\omega) \notin K_\delta\}) < \delta$ (see [BH] for more details).

b) If X enjoys the CCWP, then for a set $\{f_n; n \in \mathbb{N}\} \subset L_1(\mu, X)$ as considered in 3.1 d) is a δC- set in $L_1(\mu, X)$, as introduecd by J.Bourgain [Bo], i.e. $\forall\ \delta > 0\ \exists\ K_\delta \subset X$ conditionally weakly compact $\forall n \in \mathbb{N} : \mu(\{\omega \in \Omega; f_n(\omega) \notin K_\delta\}) < \delta$.

c) Certainly every reflexive space is CWP, every Asplund space enjoys the CAP and all spaces not containing ℓ_1 satisfy the CCWP.

The following examples shows that there are spaces X without *CWP*, *CCWP* and *CAP*.

Example 3.3 *[BH, 2.4 Example] J. Batt and W. Hiermeyer constructed a weakly sequentially complete separable Banach lattice X_0 and a bounded sequence $(f_n)_{n\in\mathbb{N}} \subset L_\infty([0,1], X_0)$ such that $\forall\ t \in [0,1] :\ f_n(t) \to\ 0$ weakly if $n \to \infty$. But $\{f_n; n \in \mathbb{N}\} \subset L_1([0,1], X_0)$ is not a δS-set. Hence according to the above Remark 3.2 a), X_0 does not have the CWP.*

As we have seen in 3.3 a weakly sequentially complete Banach lattice is not sufficient to provide that X has the *CWP*. But there is a class of Banach spaces introduced by R. Wheeler and the author [SW1] which assures this property, called *strongly WCG* Banach spaces (SWCG). We will generalize this concept here.
Let $\mathcal{G}$ be a Grothendieck class of sets. A Banach space X is called *strongly Grothendieck generated (SGG)* if

$$\exists\ K_0 \in \mathcal{G}(X) \forall\ K \in \mathcal{G}(X)\ \forall\ \epsilon > 0\ \exists\ m = m(K, \epsilon) \in \mathbb{N} :\ K \subset\ mK_0 + \epsilon B(X).$$

We note that for $\mathcal{G}(X) = \mathcal{W}(X)$ e.g. reflexive spaces and separable Schur-spaces enjoy this property, and that SWCG-spaces are weakly sequentially complete (see [SW1]).
Similarly to the dual Mackey topology we define on X^* the locally convex topology $\tau_{\mathcal{G}}$ generated by the polars $\{K^o; K \in \mathcal{G}(X)\}$ (see [Kö, p.257 (2)] for construction). Using this locally convex topology we have the following factorization theorem, which will turn out to be slightly stronger than condition 3.1.b).

Theorem 3.4 *(see [SW1, Th. 2.1]): Let $\mathcal{G}$ be a Grothendieck class. For a Banach space X the following conditions are equivalent:*

a) X is SGG,

b) $\exists$ *Banach space* Z*, such that* $B(Z) \in \mathcal{G}(Z)$ $\exists$ $T_0 \in L(Z,X)$ $\forall$ $T \in A_{\mathcal{G}}(\ell_1, X)$ $\forall$ $\epsilon > 0$ $\exists$ $T_\epsilon \in L(\ell_1, Z)$: $\|T - T_0 \circ T_\epsilon\| < \epsilon$,

c) $(B(X^*), \tau_{\mathcal{G}})$ *is metrizable.*

Proof: The equivalence of a) and c) can be proved as in [SW1, Th. 2.1].
a) $\Rightarrow$ *b)*: Let $K_0 \in \mathcal{G}(X)$ be absolutely convex set and strongly generating X. We consider $Z := X_C$, where $Z \subset X$ as a set, $B(Z) \in \mathcal{G}(Z)$ and $K_0 \subset C = B(Z)$ (see e.g. [Di, p.227,Lemma 3] for construction according to the scheme of Davis et al.). The inclusion $\imath : Z \longrightarrow X$ is continuous. Let $T_0 := \imath$. Let $\epsilon > 0$ and $T \in A_{\mathcal{G}}(\ell_1, X)$ be given. Let $K := T(B(\ell_1))$. According to the definition of SGG-spaces, there is an $m \in \mathbb{N}$, such that $K \subset mK_0 + \epsilon B(X)$. We define $T_\epsilon : \ell_1 \longrightarrow Z$, $e_n \longmapsto mx_n$, where $x_n \in K_0$ and $\|Te_n - mx_n\| < \epsilon$. Then $\|T_\epsilon\| \leq m$. Hence for all $n \in \mathbb{N}$: $\|Te_n - (T_0 \circ T_\epsilon)(e_n)\| = \|Te_n - mx_n\| < \epsilon$, which shows b).

b) $\Rightarrow$ *a)* : Let $K_0 := T_0(B(Z))$. Then K_0 is absolutely convex and $K_0 \in \mathcal{G}(X)$ according to condition 3). We suppose for a certain $\epsilon > 0$ that there is a set $K \in \mathcal{G}(X)$, such that for all $n \in \mathbb{N}$: $K \not\subset nK_0 + \epsilon B(X)$. Then there is a sequence $(x_n) \subset K$, with such that $(*)$ $\quad \forall\, n \in \mathbb{N} : x_n \notin nK_0 + \epsilon B(X)$. Let us define $T : \ell_1 \longrightarrow X$, $e_n \longmapsto x_n$. Then $T \in A_{\mathcal{G}}(\ell_1, X)$. By the assumption, there is a $T_\epsilon \in L(\ell_1, Z)$, such that $\|T - T_0 \circ T_\epsilon\| < \epsilon$. Hence for all $n \in \mathbb{N}$: $\operatorname{dist}(x_n, \|T_\epsilon\| K_0) < \epsilon$, which contradicts $(*)$. $\Box$

Theorem 3.5 *: If* X *is an SGG-space then* X *has the CGP.*

Proof: Let $(f_k)_{k\in\mathbb{N}} \subset L_1(\mu, X)$ be as in 3.1 d). Since (f_k) is bounded and uniformly integrable, by the usual technic we may assume $(f_k)_{k\in\mathbb{N}} \subset L_\infty(\mu, X)$. Let $K_0 \in \mathcal{G}(X)$ be a strongly generating set according to the definition.
We fix $\delta > 0$ and define for $m, k \in \mathbb{N}$: $A^\delta_{k,m} := \{\omega \in \Omega;\ f_k(\omega) \in mK_0 + \delta B(X)\}$.
Since f_k is measurable for all $k \in \mathbb{N}$, we have that $A^\delta_{k,m} \in \Sigma$.
Let $A^\delta_m := \bigcap\limits_{k\in\mathbb{N}} A^\delta_{k,m}$. Without loss of generality we assume that $(f_k(\omega))_{k\in\mathbb{N}} \in \mathcal{G}(X)$ for all $\omega \in \Omega$. Since $\forall\ \omega \in \Omega\ \exists\ m(\omega) \in \mathbb{N}$: $(f_k(\omega))_{k\in\mathbb{N}} \subset m(\omega)K_0 + \delta B(X)$, we can conclude $\mu(A^\delta_m) \longrightarrow \mu(\Omega)$ for $m \to \infty$. Hence $\forall\ \epsilon > 0\ \exists\ m_\epsilon \in \mathbb{N}\ \exists\ \Omega_{\epsilon,\delta} \subset \Omega$, such that for all $k \in \mathbb{N}$: $f_k(\Omega_{\epsilon,\delta}) \subset m_\epsilon K + \delta B(X)$ and $\mu(\Omega \setminus \Omega_{\epsilon,\delta}) < \frac{\epsilon}{2}$. Thus for a sequence $(\delta_n)_{n\in\mathbb{N}}, \delta_n \to 0$, there is a sequence $(\Omega_n)_{n\in\mathbb{N}} \subset \Sigma$,
$\Omega_{n+1} \subset \Omega_n$, $\mu(\Omega \setminus \Omega_{n+1}) < \frac{\epsilon}{2}$ and $m_n \in \mathbb{N}$ such that for all $n \in \mathbb{N}$:
$\bigcup_{k\in\mathbb{N}} f_k(\Omega_n) \subset m_n K + \delta_n B(X)$. We have $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$, where $\Omega_\epsilon := \bigcap\limits_{n\in\mathbb{N}} \Omega_n$.
The Grothendieck condition 2) reveals that $\bigcup_{k\in\mathbb{N}} f_k(\Omega_\epsilon)$ belongs to $\mathcal{G}(X)$. $\Box$

The following result is stated in equivalence to the one for $SWCG$-spaces [SW2, Th. 5.10].

Theorem 3.6 *Let $\mathcal{G}$ be a Grothendieck class containing the weakly compact sets. Let (X_n) be a sequence of SGG-Banach spaces, Then $c_o(X_n)$ and $\ell_p(X_n)$ for $1 < p < \infty$ are again SGG-Banach spaces if and only if all but finitely many of the X_n are Banach spaces, such that $B(X_n) \in \mathcal{G}(X_n)$.*

Proof: The sufficiency is straightforward. For the necessity we remark that the case ℓ_p runs parallely to the one presented in [SW2, Th. 5.10], hence the c_0-case will only be proved.
Let us suppose that $B(X_n) \notin \mathcal{G}(X_n)$ for all $n \in \mathbb{N}$ and assume that $K_0 \subset X := c_0(X_n)$ is strongly Grothendieck generating X. Evidently $K_n := pr_n(K_0) \subset X_n$ is strongly generating X_n and $K_n \in \mathcal{G}(X_n)$ by condition 3) for $n \in \mathbb{N}$, where pr_n is the projection on the nth coordinate.
For $n \in \mathbb{N}$, we find an $x_n \in B(X_n)$ such that $x_n \notin nK_n + \frac{1}{2}B(X_n)$. Otherwise we have $B(X_n) \subset nK_n + \frac{1}{2}B(X_n)$, thus $\frac{1}{2}B(X_n) \subset \frac{n}{2}K_n + \frac{1}{4}B(X_n)$, and hence $B(X_n) \subset (n + \frac{n}{2})K_n + \frac{1}{4}B(X_n)$. Following this procedure, we find, since $K_n \in \mathcal{G}(X_n)$, a sequence of sets $(C_m) \in \mathcal{G}(X_n)$, such that $B(X_n) \subset \bigcap_{m \in \mathbb{N}} C_m + 2^{-m}B(X_n)$. An application of condition 2) shows that $B(X_n) \in \mathcal{G}(X_n)$, which is a contradiction.
Now let $L := \{(0, \ldots, , x_n, 0, \ldots), n \in \mathbb{N}\}$. Since $c_0(X_n)^* = \ell_1(X_n^*)$, the set L is relatively weakly compact, hence a Grothendieck set. Since K_0 is strongly generating, there is an $m \in \mathbb{N}$, such that $L \subset mK_0 + \frac{1}{2}B(X)$, which implies that $x_n \in mK_n + \frac{1}{2}B(X_n)$ for all $n \in \mathbb{N}$, a contradiction. $\square$

Our final aim is to show that the implication in Theorem 3.5 cannot be reversed. For this purpose we will demonstrate the stability of CWP under ℓ_p-sums in the following proposition.

Proposition 3.7 *If $(X_i)_{i \in \mathbb{N}}$ is a sequence of Banach spaces with CWP, then for $1 \leq p < \infty$ the ℓ_p-sum $\ell_p(X_i)$ also has CWP.*

Proof: Let $X := \ell_p(X_i)$. We select a sequence $(f_k)_{k \in \mathbb{N}} \subset L_\infty(\mu, X)$ as in 3.1 d). Let $M := \sup_{k \in \mathbb{N}} ||f_k||_\infty$. Let $\pi_i : \ell_p(X_i) \longrightarrow X_i$ be the (continuous) projection on the i-th coordinate.
Case 1: We consider first $1 < p < \infty$: Since X_i has CWP, for a given $\epsilon > 0$ there exist weakly compact sets $K^i \subset X_i$ and $\Omega^i \subset \Omega$, $\mu(\Omega \setminus \Omega^i) < \frac{\epsilon}{2^{i+1}}$ such that:

$$\forall i \in \mathbb{N} \ \forall \omega \in \Omega^i : (\pi_i \circ f_k(\omega))_{k \in \mathbb{N}} \subset K^i.$$

We define

$$K := \ell_p(K^i) = \{(y_i) \in \ell_p(X^i); ||(y_i)||_p \leq M, \ y_i \in K^i\}.$$

It is easy to see that $K \in \mathcal{W}(X)$. Let $\Omega_\epsilon := \bigcap_{i \in \mathbb{N}} \Omega^i$. Then $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ and $\bigcup_{k \in \mathbb{N}} f_k(\Omega_\epsilon) \subset K$.

Case 2: If $p=1$, we first note that for $m \in \mathbb{N}$:

$$X^m := \{(y_i) \in \prod_{i\in\mathbb{N}} X_i; y_i = 0 \text{ if } i > m\} \text{ has } CWP \text{ by case 1.}$$

Let $\epsilon > 0$ be given. Then

$$(1)\ \ \forall\, \delta > 0\ \forall\, \omega \in \Omega\ \exists\, m(\omega) \in \mathbb{N} : \sup_{k\in\mathbb{N}} \sum_{i>m(\omega)} ||\pi_i \circ f_k(\omega)|| < \delta,$$

since $(f_k(\omega))_{k\in\mathbb{N}} \subset \ell_1(X_i) \in \mathcal{W}(X)$. Let us define

$$A^\delta_{k,m} := \{\omega \in \Omega; \sum_{i>m} ||\pi_i \circ f_k(\omega)|| < \delta\},\ A^\delta_m := \bigcap_{k\in\mathbb{N}} A^\delta_{k,m}.$$

Then by (1) $\mu(\bigcup_{m\in\mathbb{N}} A^\delta_m) = 1$. Hence

$$\forall\, n \in \mathbb{N}\ \exists\, m_n \in \mathbb{N} : \mu(A^{\frac{1}{n}}_{m_n}) \geq 1 - \frac{\epsilon}{2^{n+1}}.$$

Let $\tilde{\Omega}_\epsilon := \bigcap_{n\in\mathbb{N}} A^{\frac{1}{n}}_{m_n}$. Then

$$\mu(\tilde{\Omega}_\epsilon) \geq 1 - \frac{\epsilon}{2} \text{ and } \forall\, \omega \in \tilde{\Omega}_\epsilon\ \forall\, n \in \mathbb{N} : \sup_{k\in\mathbb{N}} \sum_{i>m_n} ||\pi_i \circ f_k(\omega)|| < \frac{1}{n}.$$

The assumption that X_i has CWP provides

$\forall\, i \in \mathbb{N}\ \exists \Omega_{\epsilon,i} \subset \tilde{\Omega}_\epsilon, \mu(\Omega_{\epsilon,i}) \geq \mu(\tilde{\Omega}_\epsilon) - \frac{\epsilon}{i}\ \exists\, K_{\epsilon,i} \subset M \cdot B(X_i)$ weakly compact, such that $\bigcup_{k\in\mathbb{N}} (\pi_i \circ f_k)(\Omega_{\epsilon,i}) \subset K_{\epsilon,i}$. Let $\Omega_\epsilon := \bigcap_{i\in\mathbb{N}} \Omega_{\epsilon,i}$. Then $\mu(\Omega_\epsilon) \geq \mu(\tilde{\Omega}_\epsilon) - \frac{\epsilon}{2} \geq \mu(\Omega) - \epsilon$. Choose $K^n_\epsilon \subset X^{m_n}$, such that $\forall i \leq m_n : \pi_i(K^n_\epsilon) = K_{\epsilon,i}$. Evidently $K^n_\epsilon \in \mathcal{W}(X)$. By the above it follows $\bigcup_{k\in\mathbb{N}} f_k(\Omega_\epsilon) \subset \bigcap_{n\in\mathbb{N}} \big(K^n_\epsilon + \frac{1}{n} B(\ell_1(X_i))\big)$. The Grothendieck condition [Di, p.227, Lemma 2]) shows that $\bigcup_{k\in\mathbb{N}} f_k(\Omega_\epsilon)$ is relatively weakly compact. □

Proposition 3.8 *If $(X_i)_{i\in\mathbb{N}}$ is a sequence of Banach spaces with CCWP, then $c_0(X_i)$ also enjoy CCWP.*

Proof: Again we select a sequence $(f_k)_{k\in\mathbb{N}} \subset L_\infty(\mu, X)$ as in 3.1.d). Let $M := \sup_{k\in\mathbb{N}} ||f_k||_\infty$. Let $\pi_i : c_0(X_j) \longrightarrow X_i$ be the (continuous) projection on the $\imath$th coordinate.
Since X_i has $CCWP$, for a given $\epsilon > 0$ and for all $i \in \mathbb{N}$, there exist conditionally weakly compact sets $K^i \subset X_i$ and $\Omega^i \subset \Omega$, $\mu(\Omega \setminus \Omega^i) < \frac{\epsilon}{2^i+1}$, such that: $\forall\, \omega \in \Omega^i : (\pi_i \circ f_k(\omega))_{k\in\mathbb{N}} \subset K^i$. We define $K := c_0(K^i) = \{(y_i) \in c_0(X^i); ||(y_i)||_\infty \leq M,\ y_i \in K^i\}$. It is easy to see that $K \in \mathcal{CW}(X)$. Let $\Omega_\epsilon := \bigcap_{i\in\mathbb{N}} \Omega^i$. Then $\mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ and $\bigcup_{k\in\mathbb{N}} f_k(\Omega_\epsilon) \subset K$.

□

Example 3.9 *a) The space $\ell_2(\ell_1)$ has the CWP, since ℓ_1 is a SWCG-space as a separable Schur space. But $\ell_2(\ell_1)$ fails to be a SWCG-space according to [SW2, p.283,Th.5.10]. Thus 3.5 cannot be reversed.*
The space $c_0(\ell_1)$ enjoys the CCWP by the above proposition, but it is not SCWG, hence also not SCWG by 3.6.

b) Evidently every closed subspace of a space with CWP (resp. CAP, CCWP) also has CWP (resp. CAP,CCWP). But quotients of CWP-spaces need not enjoy this property, as 3.3 reveals.

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