ON NONCOMMUTATIVE VECTOR-VALUED INTEGRATION

Georg Schlüchtermann & Florian Schwingenschlögl

Abstract

For a semifinite v. Neumann algebra M we define the noncommutative vector-valued function space as the projective Banach space tensor product $M_* \hat{\otimes} X$, where M_* is the predual and X a Banach space. In this setting we prove an Egorov type result, give a complete answer for the Dunford-Pettis property for the predual of a v. Neumann algebra and show that an example of Pisier can be used to demonstrate that the vector valued function space $M_* \hat{\otimes} X$ is not necessarily weakly sequentially complete, even if X is.

1 Introduction and Preliminary Results

Recently G.Pisier [10] made the first step towards a theory of noncommutative vector-valued integration and introduced the noncommutative vector-valued L_p-spaces in the category of operator spaces. He and M. Junge [8] derived several results concerning completely $p-$summing maps.

We will stick to the category of Banach spaces, consider basically the case $p = 1$ and define the noncommutative L_1- space as the projective tensor product $L_1(M, X) := M_* \hat{\otimes} X$, where M_* is the predual of a semifinite v. Neumann algebra and X a Banach space. Fundamental tools in the commutative theory of vector-valued integration are the well known theorems of Egorov and Lebesgue. The noncommutative versions of Egorov resp. Lebesgue theorem in the scalar case are well known and due to M. Tomita [13] for the Egorov-case and Th. Fack/H. Kosaki [7] for the Lebesgue theorem. A formulation of a similar result for the vector valued case will hit natural problems, since the abstract definition of noncommutative L_1-spaces is done by tensor products. They do not allow a natural way of formulating these classical theorems. Thus, we use a well known version of the Egorov theorem, namely that every L_1- convergent sequence admits a subsequence converging *almost uniformly* (see section 2 for a definition).

In addition we will answer the characterization of the Dunford-Pettis property for the predual of a general v. Neumann algebra, which has been partly answered by C.-C. Chu and B. Iochum [3]. Independently J. Bunce [2] answered this question. With our approach finally we are able to show that in contrast to the commutative case the space $L_1(\tau)\hat{\otimes}\ell_2$ is not weakly sequentially complete, though $L_1(\tau)$ and ℓ_2 are.

Let us fix some notations. $B(X)$ resp. $S(X)$ denotes the unit ball resp. unit sphere of a Banach space X with dual X^*. We recall the fact that every semifinite v. Neumann algebra admits of a semifinite, faithful and normal trace $\tau : M_+ \longrightarrow [0,\infty]$. The reader is referred to the standard monographes for the theory of traces [11]. A v. Neumann algebra will be denoted by M, or by $L_\infty(\tau)$, if we stress the existence of the trace. Thus the predual M_* of a v. Neumann algebra M is written by $L_1(\tau)$. If $\eta,\xi \in H$, H a Hilbert space, then $\langle \xi | \eta \rangle$ is the inner product. For $m \in M$ the adjoint is denoted by $m^\star$. $L_r(\mu, X)$, $1 \leq r \leq \infty$ stands for the $X-$valued Bochner-Lebesgue space. The projective resp. injective tensor product is denoted by "$\hat{\otimes}$" resp. "$\check{\otimes}$." For Banach spaces X, Y we write $\mathcal{L}(X,Y)$ for the space of continuous linear operators.

2 Almost uniform convergence

Let X be a Banach space and let M be a v. Neumann algebra equipped with a semifinite, faithful and normal trace τ, acting on a Hilbert space H. For a projection $p \in M$ we define

$$\mathbf{C}_p : M \longrightarrow pMp, \ m \longmapsto pmp$$

as the compression of M to the v. Neumann algebra pMp.

Definition 2.1 *Define for* $f \in L_1(\tau)\hat{\otimes}X$

$$\|f\|_\infty = \sup\{\|(\tau(m\cdot)\hat{\otimes}Id_X)(f)\|_X \mid m \in B(L_1(\tau)) \cap L_\infty(\tau)\}.$$

Remark 2.2 *Since* $B(L_1(\tau)) \cap L_\infty(\tau)$ *is dense in* $B(L_1(\tau))$, $\|f\|_\infty$ *equals the injective cross-norm* $\|f\|_{L_\infty(\tau)\check{\otimes}X}$ *for any* $f \in (L_1(\tau) \cap L_\infty(\tau)) \otimes X$ *. In general* $\|f\|_\infty$ *need not be finite. If it is, we identify* f *with the continuous extension of the map* $m \mapsto (\tau(m\cdot)\hat{\otimes}Id_X)(f)$*. Due to this identification we have* $\|f\|_\infty = \|f\|_{\mathcal{L}(L_1(\tau),X)}$*.*

Definition 2.3

1. *Let* (Ω, Σ, μ) *be a measure space. Call a sequence of* μ*-measurable* X*-valued maps* almost uniformly convergent*, if for all* $\epsilon > 0$ *there exists* Σ*-measurable set* A*, such that* $\mu(\Omega \setminus A) \leq \epsilon$ *and the sequence converges uniformly on* A*.*

2. *We call a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $L_1(\tau)\hat{\otimes}X$ almost uniformly convergent *w.r.t. a v. Neumann algebra* M *with a semifinite, faithful and normal trace* τ*, if for all* $\epsilon > 0$ *there exists a projection* $p \in M$*, such that* $\tau(1-p) \leq \epsilon$ *and* $((\mathbf{C}_p\hat{\otimes}Id_X)f_n)_{n\in\mathbb{N}}$ *converges in the topology generated by* $\|\cdot\|_\infty$*.*

A classical result in the theory of Bochner-Lebesgue spaces is the following

Proposition 2.4 *Let* μ *be a measure. Then for* $1 \leq r < \infty$ *every convergent sequence* $(f_n)_{n\in\mathbb{N}} \subset L_r(\mu, X)$ *admits of an almost uniformly convergent subsequence [9].*

We will prove a similar result concerning 2 of Definition 2.3 for the noncommutative case. We refer the reader to [7] for the theory of generalized singular numbers:

For $a \in L_1(\tau)$ and $t > 0$ define

$$\mu_t(a) := \inf\{\|ap\|_{L_\infty(\tau)};\ p \in M^p \text{ with } \tau(\mathbf{1} - p) < t\}.$$

Then we have

$$\|a\|_{L_1(\tau)} = \int_{\mathbb{R}_+} \mu_t(a)dt.$$

Lemma 2.5 *Let $m \in L_1(\tau) \cap L_\infty(\tau)$. Then for all $\epsilon > 0$ there exists a projection $p \in M$, such that*

$$\|m\|_{L_1(\tau)} \geq \frac{2\epsilon}{3}\|mp\|_{L_\infty(\tau)} \text{ with } \tau(1-p) < \epsilon. \tag{2.1}$$

Proof: If $m = 0$, then choose $p = 1$. Otherwise we assume that there is an $\epsilon > 0$, such that $\mu_\epsilon(m) \neq 0$. We have $\|m\|_{L_1(\tau)} = \tau(|m|) = \int_0^\infty \mu_t(m)\, dt$. Consider $\int_0^\epsilon \mu_t(m)\, dt > 0$. Since the integrand is nonincreasing, we can estimate $\int_0^\epsilon \mu_t(m)\, dt \geq \epsilon\mu_\epsilon(m)$. By definition we conclude $\mu_\epsilon(m) = \inf\{\|mp\|_{L_\infty(\tau)} |\ p \in M^p,\ \tau(1-p) \leq \epsilon\}$. Choose a projection p with $\tau(1-p) \leq \epsilon$, such that $2\|mp\|/3 \leq \inf\{\|mq\|_{L_\infty(\tau)} | q \in M^q \text{ with } \tau(1-q) \leq \epsilon\}$. p is the desired projection. □

For $m \in M$ and a complex number j with $j^4 = 1$, we denote by $m^{(j)}$ the positive elements from the decomposition $m = m^{(1)} - m^{(-1)} + im^{(i)} - im^{(-i)}$. $\Re(m)$ resp. $\Im(m)$ denotes the real resp. imaginary part of an $m \in M$.

Lemma 2.6 *Let $m \in L_1(\tau) \cap L_\infty(\tau)$. Then we have*

$$\|m\|_{L_1(\tau)} \geq \|\Re(m)\|_{L_1(\tau)} \vee \|\Im(m)\|_{L_1(\tau)}.$$

If m is selfadjoint, then $\|m\|_{L_1(\tau)} = \|m^{(1)}\|_{L_1(\tau)} + \|m^{(-1)}\|_{L_1(\tau)}$.

Proof: Using the polar decomposition we have $\|a\|_{L_1(\tau)} = \|a^*\|_{L_1(\tau)}$ for $a \in M_*$. We compute for the real part

$$\|\Re m\|_{L_1(\tau)} = \|\frac{m + m^\star}{2}\|_{L_1(\tau)} \leq \frac{1}{2}(\|m\|_{L_1(\tau)} + \|m^\star\|_{L_1(\tau)}) = \|m\|_{L_1(\tau)},$$

and similar for the imaginary part. The equation for selfadjoint m is a consequence of the Jordan decomposition. □

Lemma 2.7 *Let $h \in (L_1(\tau) \cap L_\infty(\tau)) \otimes X$. Then for all $\epsilon > 0$ there exists a projection $p \in M$, such that*

$$\|h\|_{L_1(\tau)\hat{\otimes}X} \geq \frac{\epsilon}{8}\|(\mathbf{C}_p \otimes Id_X)h\|_{L_\infty(\tau)\check{\otimes}X} \text{ with } \tau(1-p) < \epsilon \tag{2.2}$$

Proof: For $h = 0$ use $p = 1$. The definition of the projective tensor norm on $L_1(\tau) \otimes X$ implies, that for each $\delta > 0$ there exists a representation of $h \neq 0$ as a finite sum of simple tensors $m_i \otimes x_i$ with $m_i \in L_1(\tau) \cap L_\infty(\tau)$ and $x_i \in X$ for $i = 1, 2, \ldots, n$, such that:

$$\|h\|_{L_1(\tau)\hat{\otimes}X} \quad \geq \quad (1-\delta)\sum_{i=1}^{n}\|m_i\|_{L_1(\tau)}\|x_i\|_X \overset{2.6}{\geq} (1-\delta)\frac{1}{2}\sum_{i=1}^{n}\sum_{j^4=1}\|m_i^j\|_{L_1(\tau)}\|x_i\|_X.$$

Using the additivity and positivity of the $\|\cdot\|_{L_1(\tau)}$-norm for positive elements in M, we conclude

$$\|h\|_{L_1(\tau)\hat{\otimes}X} \geq (1-\delta)\frac{1}{2}\left\|\sum_{i=1}^{n}\sum_{j^4=1}\|x_i\|_X m_i^j\right\|_{L_1(\tau)}.$$

According to Lemma 2.5 there exists a projection $p \in M$ with $\tau(1-p) \leq \epsilon$, such that

$$\left\|\sum_{i=1}^{n}\sum_{j^4=1}\|x_i\|_X m_i^j\right\|_{L_1(\tau)} \geq \frac{2\epsilon}{3}\left\|\left(\sum_{i=1}^{n}\sum_{j^4=1}\|x_i\|_X m_i^{(j)}\right)p\right\|_{L_\infty(\tau)}.$$

With the help of the submultiplicativity of $\|\cdot\|_{L_\infty(\tau)}$ we can estimate, since $\|p\|_{L_\infty(\tau)} = 1$:

$$\begin{aligned}\|h\|_{L_1(\tau)\hat{\otimes}X} &\geq (1-\delta)\frac{\epsilon}{3}\left\|p\left(\sum_{i=1}^{n}\sum_{j^4=1}\|x_i\|_X m_i^{(j)}\right)p\right\|_{L_\infty(\tau)} \\ &= (1-\delta)\frac{\epsilon}{3}\left\|\sum_{i=1}^{n}\sum_{j^4=1}\|x_i\|_X p m_i^{(j)} p\right\|_{L_\infty(\tau)}. \ (*)\end{aligned}$$

Now let $x^* \in B(X^*)$. For every $\xi \in H$ using the positivity of $pm_i^{(j)}p$ for $i \in \{1, 2, \ldots, n\}$ and j with $j^4 = 1$

$$\left|\left\langle \xi \,\middle|\, \Re\left(\sum_{i=1}^{n}\sum_{j^4=1}x^*(jx_i)pm_i^{(j)}p\right)(\xi)\right\rangle\right| = \left|\left\langle \xi \,\middle|\, \left(\sum_{i=1}^{n}\sum_{j^4=1}\Re x^*)(jx_i)pm_i^{(j)}p\right)(\xi)\right\rangle\right|$$

$$
\begin{aligned}
&= \left| \sum_{i=1}^{n} \sum_{j^4=1} (\Re x^*)(jx_i) \langle \xi \mid pm_i^{(j)} p(\xi) \rangle \right| \quad \leq \quad \sum_{i=1}^{n} \sum_{j^4=1} |(\Re x^*)(jx_i)| \langle \xi \mid pm_i^{(j)} p(\xi) \rangle \\
&\leq \sum_{i=1}^{n} \sum_{j^4=1} \|jx_i\| \langle \xi \mid pm_i^{(j)} p(\xi) \rangle \quad = \quad \left\langle \xi \;\middle|\; \left(\sum_{i=1}^{n} \sum_{j^4=1} \|x_i\| pm_i^{(j)} p \right) (\xi) \right\rangle .
\end{aligned}
$$

Taking the supremum over all $\xi \in B(H)$, we get

$$
\left\| \Re \left(\sum_{i=1}^{n} \sum_{j^4=1} x^*(jx_i) pm_i^{(j)} p \right) \right\|_{L_\infty(\tau)} \leq \left\| \sum_{i=1}^{n} \sum_{j^4=1} \|x_i\|_X pm_i^{(j)} p \right\|_{L_\infty(\tau)} . \tag{2.3}
$$

We infer a similar estimate for the imaginary part, which implies:

$$
\left\| \sum_{i=1}^{n} \sum_{j^4=1} x^*(jx_i) pm_i^{(j)} p \right\|_{L_\infty(\tau)} \leq 2 \left\| \sum_{i=1}^{n} \sum_{j^4=1} \|x_i\|_X pm_i^{(j)} p \right\|_{L_\infty(\tau)} . \tag{2.4}
$$

Since $x^* \in B(X^*)$ is arbitrary, finally we have, using the inequality (*)

$$
\begin{aligned}
\|h\|_{L_1(\tau)\hat{\otimes} X} &\geq (1-\delta)\frac{\epsilon}{6} \sup_{x^* \in B(X^*)} \left\| \sum_{i=1}^{n} \sum_{j^4=1} x^*(jx_i) pm_i^{(j)} p \right\|_{L_\infty(\tau)} \\
&= (1-\delta)\frac{\epsilon}{6} \sup_{x^* \in B(X^*)} \left\| \sum_{i=1}^{n} x^*(x_i) p \left(\sum_{j^4=1} jm_i^{(j)} \right) p \right\|_{L_\infty(\tau)} \\
&= (1-\delta)\frac{\epsilon}{6} \sup_{x^* \in B(X^*)} \left\| \sum_{i=1}^{n} x^*(x_i) pm_i p \right\|_{L_\infty(\tau)} \\
&= (1-\delta)\frac{\epsilon}{6} \left\| \sum_{i=1}^{n} x_i \otimes pm_i p \right\|_{L_\infty(\tau)\check{\otimes} X} \\
&= (1-\delta)\frac{\epsilon}{6} \|(\mathbf{C}_p \otimes Id_X) h\|_{L_\infty(\tau)\check{\otimes} X} .
\end{aligned}
$$

Choosing $\delta := \frac{1}{4}$ proves the inequality(2.2) $\square$

Lemma 2.8 *Suppose $(f_n)_{n\in\mathbb{N}} \to f$ in $L_1(\tau)\hat{\otimes} X$ and $(f_n)_{n\in\mathbb{N}} \to f$ is a $\|\cdot\|_\infty$-Cauchy sequence. Then $\|f\|_\infty < \infty$, and $(f_n)_{n\in\mathbb{N}}$ converges to f w.r.t.. $\|\cdot\|_\infty$.*

Proof: Since

$$
(f_n)_{n\in\mathbb{N}} \to f \text{ in } L_1(\tau)\hat{\otimes} X,
$$

we have

$$((\tau(m\cdot)\hat{\otimes}Id_X)(f_n))_{n\in\mathbb{N}} \to (\tau(m\cdot)\hat{\otimes}Id_X)(f) \text{ in } X$$

for every $m \in L_\infty(\tau)$. Since $(f_n)_{n\in\mathbb{N}}$ is also $\|\cdot\|_\infty$-Cauchy sequence, it converges uniformly on $B(L_1(\tau)) \cap L_\infty(\tau)$. □

Lemma 2.9 *Let $(s_n)_{k\in\mathbb{N}} \subset (L_1(\tau) \cap L_\infty(\tau)) \otimes X$, such that $\|s_m - s_n\|_{L_1(\tau)\hat{\otimes}X} \leq 2^{-2k}$ for all $k \in \mathbb{N}$ and $m, n \geq k$. Then for all $\epsilon > 0$ there exist a projection $p \in M$ with $\tau(1-p) < \epsilon$, and a $k \in \mathbb{N}$, with*

$$\|(\mathbf{C}_p\hat{\otimes}Id_X)(s_m - s_n)\|_{L_\infty(\tau)\check{\otimes}X} \leq 2^{4-k} \text{ for } m, n \geq k$$

Proof: Let $h_k := s_{k+1} - s_k$ for $k \in \mathbb{N}$. Then

$$\|h_k\|_{L_1(\tau)\hat{\otimes}X} \leq 2^{-2k} \tag{2.5}$$

for all $k \in \mathbb{N}$. Choose $N \in \mathbb{N}$, so that $2^{-N} \leq \epsilon$. With Lemma 2.7 for every $k > N$ we find a projection $p_k \in M$, such that

$$\tau(1-p_k) \quad < \quad 2^{-k} \tag{2.6}$$

$$2^{-k-3}\|(\mathbf{C}_{p_k} \otimes Id_X)h_k\|_{L_\infty(\tau)\check{\otimes}X} \quad \leq \quad \|h_k\|_{L_1(\tau)\hat{\otimes}X}. \tag{2.7}$$

Let $p := \bigwedge_{k>N} p_k$. From subadditivity and inequality (2.6) we infer that

$$\tau(1-p) = \tau(1 - \bigwedge_{k>N} p_k) = \tau(\bigvee_{k>N}(1-p_k)) \leq \sum_{k>N} \tau(1-p_k) \leq \sum_{k>N} 2^{-k} = 2^{-N} < \epsilon. \tag{2.8}$$

If $k > N$, then $p \leq p_k$ and

$$\begin{aligned} \|(\mathbf{C}_p \otimes Id_X)(h_k)\|_{L_\infty(\tau)\check{\otimes}X} &= \|(\mathbf{C}_p \otimes Id_X)(\mathbf{C}_{p_k} \otimes Id_X)(h_k)\|_{L_\infty(\tau)\check{\otimes}X} \\ &\leq \|(\mathbf{C}_{p_k} \otimes Id_X)(h_k)\|_{L_\infty(\tau)\check{\otimes}X}. \end{aligned}$$

Combining inequality (2.5) and (2.7) we get

$$\|(\mathbf{C}_p \otimes Id_X)(h_k)\|_{L_\infty(\tau)\check{\otimes}X} \leq 2^{3-k} \ (\forall k > N).$$

Thus $m, n \geq k \geq N$ implies, w.l.o.g. $m \geq n$, that

$$\begin{aligned} \|(\mathbf{C}_p \hat{\otimes} Id_X)(s_m - s_n)\|_{L_\infty(\tau)\check{\otimes} X} &= \left\| \sum_{k=n}^{m-1} (\mathbf{C}_p \hat{\otimes} Id_X) h_k \right\|_{L_\infty(\tau)\check{\otimes} X} \\ \leq \sum_{k=n}^{m-1} \|(\mathbf{C}_p \hat{\otimes} Id_X) h_k\|_{L_\infty(\tau)\check{\otimes} X} &\leq \sum_{k=n}^{m-1} 2^{3-k} < 2^{4-n}. \end{aligned}$$

□

Lemma 2.10 *Let $p, q \in M$ with $p \leq q$. For each $f \in L_1(\tau)\hat{\otimes} X$ it follows*

$$\|(\mathbf{C}_p \hat{\otimes} Id_X) f\|_\infty \leq \|(\mathbf{C}_q \hat{\otimes} Id_X) f\|_\infty.$$

Proof: Since $(\mathbf{C}_p \hat{\otimes} Id_X) = (\mathbf{C}_p \hat{\otimes} Id_X)(\mathbf{C}_q \hat{\otimes} Id_X)$ it suffices to consider $q = 1$. On ground of the identity $\tau(m\cdot)\mathbf{C}_p = \tau(\mathbf{C}_p(m)\cdot)$ on $L_1(\tau)$ for all $m \in L_1(\tau) \cap L_\infty(\tau)$ we conclude

$$(\tau(m\cdot)\hat{\otimes} Id_X)(\mathbf{C}_p \hat{\otimes} Id_X) = \tau(\mathbf{C}_p(m)\cdot)\hat{\otimes} Id_X).$$

$\mathbf{C}_p$ maps the set $B(L_1(\tau)) \cap L_\infty(\tau)$ into itself. Thus for all $f \in L_1(\tau)\hat{\otimes} X$ we arrive at the estimate

$$\begin{aligned} \|(\mathbf{C}_p \hat{\otimes} Id_X)\|_\infty &= \sup\{\|\tau(m\cdot)\hat{\otimes} Id_X)(f)\|_X \mid m \in \mathbf{C}_p[B(L_1(\tau)) \cap L_\infty(\tau)]\} \\ &\leq \sup\{\|\tau(m\cdot)\hat{\otimes} Id_X)(f)\|_X \mid m \in B(L_1(\tau)) \cap L_\infty(\tau)\} = \|f\|_\infty. \end{aligned}$$

□

Theorem 2.11 *Every Cauchy sequence $(g_n)_{n\in\mathbb{N}} \subset L_1(\tau)\hat{\otimes} X$ contains a subsequence $(f_n)_{n\in\mathbb{N}}$, converging in the sense of Definition 2.3.2 almost uniformly.*

Proof: For each $k \in \mathbb{N}$ select an $N_k \in \mathbb{N}$, such that for all $m, n \geq N_k$:

$$\|g_m - g_n\|_{L_1(\tau)\hat{\otimes} X} \leq \frac{2^{-2k}}{3}.$$

Let $f_k := g_{N_k}$ for all $k \in \mathbb{N}$, so that

$$\|f_m - f_n\|_{L_1(\tau)\hat{\otimes}X} \leq \frac{2^{-2k}}{3} \tag{2.9}$$

for all $m, n \geq k$.

Let $\epsilon > 0$. We will reduce the proof to Lemma 2.9 by an approximation argument. For this let $(g_n)_{n\in\mathbb{N}} \subset (L_1(\tau) \cap L_\infty(\tau)) \otimes X$. Lemma 2.9 implies immediately the almost uniform convergence of the upper sequence $(f_k)_{k\in\mathbb{N}}$.
For the general case recall that $(L_1(\tau) \cap L_\infty(\tau)) \otimes X$ is dense in $L_1(\tau)\hat{\otimes}X$. Hence, there exists for all $k \in \mathbb{N}$ a sequence $(s_n^k)_{n\in\mathbb{N}} \subset L_1(\tau)\hat{\otimes}X$ converging to f_k, whose terms are in $(L_1(\tau) \cap L_\infty(\tau)) \otimes X$. ¿From each sequence $(s_n^k)_{n\in\mathbb{N}}$, $n \in \mathbb{N}$ we can select an almost uniformly convergent subsequence. This enables us to select for each $k \in \mathbb{N}$ an element t_k from the sequences $(s_n^k)_{n\in\mathbb{N}}$, such that

$$\begin{aligned} \|t_k - f_k\|_{L_1(\tau)\hat{\otimes}X} &\leq \frac{2^{-2k}}{3} \\ \|(\mathbf{C}_{p_k}\hat{\otimes}Id_X)(t_k - f_k)\|_\infty &\leq \frac{1}{k}, \end{aligned}$$

where $p_k \in M$ is a projection with $\tau(1 - p_k) < \epsilon 2^{-1-k}$. Using equalities (2.9) and (2.10) we obtain for $k \in \mathbb{N}$ and $m, n \geq k$

$$\begin{aligned} \|t_m - t_n\|_{L_1(\tau)\hat{\otimes}X} &\leq \|t_m - f_m\|_{L_1(\tau)\hat{\otimes}X} \\ &+ \|f_m - f_n\|_{L_1(\tau)\hat{\otimes}X} + \|f_m - t_n\|_{L_1(\tau)\hat{\otimes}X} \leq 2^{-2k}. \end{aligned}$$

Thus, $(t_k)_{k\in\mathbb{N}}$ fulfills the assumptions of Lemma 2.9, which implies the existence of a projection $q \in M$ with $\tau(1 - q) < \epsilon/2$. For sufficiently large $k \in \mathbb{N}$ and $m, n \geq k$ we can deduce

$$\|(\mathbf{C}_q\hat{\otimes}Id_X)(t_m - t_n)\|_\infty \leq 2^{4-k} \tag{2.10}$$

Let $p = q \wedge \bigwedge_{k\in\mathbb{N}} p_k$. An argument analogous to (2.8) shows, that $\tau(1-p) < \epsilon$. Lemma 2.10 and inequalities (2.10) and (2.10) imply

$$\begin{aligned} \|(\mathbf{C}_p \hat{\otimes} Id_X)(f_n - f_m)\|_\infty &\leq \|(\mathbf{C}_p \hat{\otimes} Id_X)(f_n - t_n)\|_\infty + \|(\mathbf{C}_p \hat{\otimes} Id_X)(t_n - t_m)\|_\infty + \\ &\quad \|(\mathbf{C}_p \hat{\otimes} Id_X)(t_m - f_m)\|_\infty \\ &\leq \|(\mathbf{C}_{p_n} \hat{\otimes} Id_X)(f_n - t_n)\|_\infty + \|(\mathbf{C}_{p_q} \hat{\otimes} Id_X)(t_n - t_m)\|_\infty + \\ &\quad \|(\mathbf{C}_{p_m} \hat{\otimes} Id_X)(t_m - f_m)\|_\infty \leq \frac{1}{n} + 2^{4-k} + \frac{1}{m} \leq \frac{2}{k} + 2^{4-k}. \end{aligned}$$

This proves, considering Lemma 2.8, the assertion. □

By $m \otimes x \mapsto \tau(m\cdot)$ we have an isometric embedding of $L_\infty(\tau)\check{\otimes}X \to \mathcal{L}(L_1(\tau), X)$. The image of this map consists of compact operators. We can use Theorem 2.11 to construct arbitrary operators $T \in L(L_1(\tau), X)$, which have a representation in $L_1(\tau)\hat{\otimes}X$. These operators are only τ-a.e. defined and are bounded on arbitrary "large" subspaces. We will explain this in more detail:

Proposition 2.12 *Let $f \in L_1(\tau)\hat{\otimes}X$. Then for each $\epsilon > 0$ there exists a projection $p \in M$, $\tau(1-p) < \epsilon$, such that $\|(\mathbf{C}_p \hat{\otimes} Id_X)(f)\|_\infty < \infty$.*

Proof: First select a sequence $(h_n)_{n\in\mathbb{N}} \subset L_1(\tau)\hat{\otimes}X$, converging to f. Due to Theorem 2.11 we find an almost uniform convergent subsequence of $(h_n)_{n\in\mathbb{N}}$, say $(h_n)_{n\in\mathbb{N}}$ itself. Then pick a projection $p \in M$, $\tau(1-p) < \epsilon$, such that $((\mathbf{C}_p \hat{\otimes} Id_X)(h_n))_{n\in\mathbb{N}}$ is a $\|\cdot\|_\infty$-Cauchy sequence. This sequence converges in $L_1(\tau)\hat{\otimes}X$ to $(\mathbf{C}_p \hat{\otimes} Id_X)(f)$. Lemma 2.8 implies the assertion. □

3 The Dunford-Pettis Property and weak sequential completeness

Recall that for Banach spaces X and Y a linear bounded operator $T : X \longrightarrow Y$ is called a *Dunford Pettis operator* if T maps weakly compact sets into compact sets. In addition a Banach space enjoys the *Dunford Pettis property* provided for all Banach spaces Y every weakly compact operator $T : X \longrightarrow Y$ is a Dunford Pettis operator.

Standard example for Banach spaces with the Dunford Pettis property are the spaces $L_1(\mu)$, $L_\infty(\mu)$ for a positive measure μ and $C(K)-$spaces. We refer the reader to the monographes [5, 6] for a detailed description of this topic.

We have the following well-known fact

$$(*)\left\{\begin{array}{l}\text{A Banach space } X \text{ has the Dunford-Pettis property if and only}\\ \text{if } \lim\langle x_n, x_n^*\rangle = 0 \text{ for all weak zero sequences } (x_n) \subset X \text{ and } (x_n^*) \subset X^*.\end{array}\right.$$

An immediate consequence of the embedding of a Banach space into its bidual is the following

Corollary 3.1 *If the dual space X^* has Dunford-Pettis property, so does X.*

A straightforward verification leads to the fact that for a Hilbert space H the space $K(H)$ of compact operators does not have the Dunford-Pettis property.

For $a, b \in M$ we define the anti-commutator $[a, b]_+ := ab + ba$.

Proposition 3.2 *For a v. Neumann algebra M the following conditions are equivalent to the fact that M_* has the Dunford-Pettis property.*

1. *For each $\sigma(M, M^*)$-zero sequences $(x_n)_{n\in\mathbb{N}}$ the sequence $(x_n^\star x_n)_{n\in\mathbb{N}}$ is a $\sigma(M, M_*)$-zero sequence.*
2. *For each $\sigma(M, M^*)$-zero sequences $(x_n)_{n\in\mathbb{N}}$ the sequence $(x_n x_n^\star)_{n\in\mathbb{N}}$ is a $\sigma(M, M_*)$-zero sequence.*
3. *For each $\sigma(M, M^*)$-zero sequences $(x_n)_{n\in\mathbb{N}}$ the sequence $([x_n^\star, x_n]_+)_{n\in\mathbb{N}}$ is a $\sigma(M, M_*)$-zero sequence.*

Proof: Suppose M_* has the Dunford-Pettis property and let $(x_n)_{n\in\mathbb{N}}$ be a $\sigma(M, M^*)$-zero sequence. For fixed $y \in M_*$ we have that $(y(x_n^\star\cdot))_{n\in\mathbb{N}}$ is a $\sigma(M_*, M)$-zero sequence because of the

$\sigma(M, M_*)$-continuity of the adjoint and multiplication. Since M_* has the Dunford Pettis property, $(y(x_n^\star x_n))_{n\in\mathbb{N}}$ converges to zero. Since that is true for all $y \in M_*$, the sequence $(x_n^\star x_n)_{n\in\mathbb{N}}$ converges to zero in the $\sigma(M, M_*)$-topology. If $(x_n)_{n\in\mathbb{N}}$ is a $\sigma(M, M^*)$-zero sequence, so is the adjoint one $(x_n^\star)_{n\in\mathbb{N}}$. Thus, we have the third implication.

Suppose M fulfills the third condition. Then the implication follows from the following lemma.

Lemma 3.3 *Let $(y_n)_{n\in\mathbb{N}} \subset M_*$ be a $\sigma(M_*, M)$-zero sequence and let $(x_n)_{n\in\mathbb{N}} \subset M$ be norm bounded, such that $([x_n^\star, x_n]_+)_{n\in\mathbb{N}}$ is a $\sigma(M, M_*)$-zero sequence. Then*

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} |y_n(x_m)| = 0$$

Proof: See [11] S. 151 f. for a proof. □

Corollary 3.4 *If M_* has the Dunford-Pettis property, then every predual of every v.Neumann subalgebra $N \subset M$ enjoys it, too.*

Proof: Let M_* have the Dunford-Pettis property and let $(x_n)_{n\in\mathbb{N}} \subset N$ be a $\sigma(M, M^*)$-zero sequence. Then $(x_n^\star x_n)_{n\in\mathbb{N}}$ is $\sigma(M, M_*)$-zero convergent, because of Proposition 3.2., thus also $\sigma(N, N_*)$-zero convergent. Proposition 3.2 shows that N has the Dunford Pettis property. □

We want to characterize those v. Neumann algebras, whose predual enjoys the Dunford Pettis property.

Lemma 3.5 *Let M_* have the Dunford Pettis property. Then M is finite.*

Proof: Let H be a infinite dimensional Hilbert space. Then $\mathcal{K}(H)$ does not have the Dunford Pettis property. Thus, according to Corollary 3.1 the bidual $\mathcal{L}(H)$ has it neither. Every infinite von Neumann Algebra has a v. Neumann subalgebra of the type $\mathcal{L}(H)$, such that by Corollary 3.4 every infinite v. Neumann algebra does not have the Dunford Pettis property. □

Lemma 3.6 *Let $a, b \in M$. For any trace τ on M: $\tau(ab) = \tau([a, b]_+)/2$.*

Proof: $\tau([a,b]_+) = \tau(ab+ba) = \tau(ab)+\tau(ba) = 2\tau(ab)$. □

Lemma 3.7 *The predual of the hyperfinite II_1-factor CAR does not have the Dunford Pettis property.*

Proof: First we refer the reader to [1] for a detailed description of a hyperfinite II_1−factor. This implies that there is an isometry $a : \ell_2 \longrightarrow$ CAR such that

i) $\forall\, \xi, \eta \in \ell_2 :\ [a(\xi), a(\eta)]_+ = 0.$

ii) $\forall\, \xi, \eta \in \ell_2 :\ [a(\xi), a(\eta)^*]_+ = \langle \xi | \eta \rangle id.$

We will construct weak zero sequences $(x_n) \subset \mathrm{CAR}_*$, $(x_n^*) \subset$ CAR, such that $(*)$ fails. CAR will be equipped with its normalized trace τ. For $n \in \mathbb{N}$ let ξ_n be the n-th vector of an orthonormal basis of ℓ_2. Define $x_n^* := \tau(a(\xi_n)^\star \cdot)$, which defines an element of the predual of CAR , and $x_n = a(\xi_n)$ for all $n \in \mathbb{N}$.

$(x_n^*)_{n\in\mathbb{N}}$ is a weak zero sequence: As $a(\ell_2)$ generates CAR, by linearity and the trace inequality it suffices to prove that $\langle x_n^*, y \rangle \to 0$ for $y \in a(\ell_2)$. By i) and ii) above we have for $n \in \mathbb{N}$:

$$x_n^*(\mathbf{1}) = \tau(a(\xi_n)^\star \mathbf{1}) = \tau(a(\xi_n)[a(\xi_n), a(\xi_n)^\star]_+) = 0.$$

Furthermore for $n \in \mathbb{N}$ and $\xi \in \ell_2$:

$$\begin{aligned} x_n^*(a(\xi)) &= \tau(a(\xi_n)^\star a(\xi)) = \frac{1}{2}\tau([a(\xi_n)^\star, a(\xi)]_+) \\ &= \frac{1}{2}\tau(\langle \xi \mid \xi_n \rangle \mathbf{1}) = \frac{1}{2}\langle \xi \mid \xi_n \rangle. \end{aligned}$$

The last term is the n−th part of a zero sequence for $\xi \in \ell_2$ fixed.

$(x_n)_{n\in\mathbb{N}}$ is a weak zero sequence: $a(\ell_2)$ is isomorphic to ℓ_2. Thus, $a(\xi_n)$ is weakly zero, since (ξ_n) is weakly zero in ℓ_2. ¿From the canonical anticommutation relations

$$x_n^*(x_n) = \tau(a(\xi_n)^\star a(\xi_n)) = \frac{1}{2}\tau([a(\xi_n)^\star, a(\xi_n)]_+) = \frac{1}{2}\tau(\mathbf{1}) = \frac{1}{2}\ (n \in \mathbb{N}).$$

Hence, $(x_n^*(x_n))_{n\in\mathbb{N}}$ is not a zero sequence, and the proof is done. □

Theorem 3.8 *The predual of a v. Neumann algebra M has the Dunford Pettis property if and only if M is finite of type I.*

Proof: The "if-part" is contained in [3]. Suppose M_* has the Dunford Pettis property. Then there are only factors of type I and II_1 according to the partition into types. In every type II_1 algebra we can embed CAR according to [4]. By Lemma 3.7 CAR does not have the Dunford Pettis property. Thus, according to the previous lemma the II_1 part disappears and hence, M is of type I and finite. □

The space CAR can be used for the investigation of weak sequential completeness of $L_1(\tau)\hat{\otimes}X$. Recall that a Banach space is said to be *weakly sequentially complete,* if every weak Cauchy sequence converges weakly. Reflexive spaces, $L_1(\mu)$ for a measure μ and the predual M_* are weakly sequentially complete. The property is inherited by subspaces. All spaces containing an isomorphic copy of c_0 fail this property.

A deep result of M. Talagrand (cf. [12]) states:

> Let X be a Banach space, let μ be a measure. Then the space of Bochner integrable $X-$ valued functions $L_1(\mu, X)$ is weakly sequentially complete if and and only if X is.

Example 3.9 *The reflexive space ℓ^2 is weakly sequentially complete. Let the hyperfinite II_1-factor CAR be equipped with a normalized trace τ. Then $L_1(\tau)\hat{\otimes}\ell^2$ is not weakly sequentially complete.*

Proof: We use an observation by G. Pisier in the category of *operator spaces,* which can be transferred to the Banach space category. He proved (cf. [10, p. 56]):

> Let ξ_n, $n \in \mathbb{N}$ the canonical orthonormal basis in ℓ_2 and let (e_n) be the canonical unit vector basis in c_0. Then, with the above notations concerning the CAR-algebra:

$$c_0 \to L_1(\tau)\hat{\otimes}\ell^2, e_i \mapsto a(\xi_i) \otimes \xi_i$$

is an embedding of the space c_0 into $L_1(\tau)\hat{\otimes}\ell_2$.

Thus, $L_1(\tau)\hat{\otimes}\ell_2$ is not weakly sequentially complete.

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GEORG SCHLÜCHTERMANN & FLORIAN SCHWINGENSCHLÖGL

MATHEMATISCHES INSTITUT DER UNIVERSITÄT MÜNCHEN

THERESIENSTR. 39, D-80333 MÜNCHEN, GERMANY

e-mail: schluech@rz.mathematik.uni-muenchen.de