A Note on Multiplication of Strong Operator Measurable Functions *

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Abstract

Let (Ω, Σ, μ) be a finite and positive measure space and $U_1, \ldots, U_n$ be strongly measurable functions with values in the space of bounded linear operators on a Banach space, then the product $U_1 \cdots U_n$ is again strongly measurable.

In the paper "The product of strong operator measurable functions is strong operator measurable" recently G.W.Johnson (see [**Jo**]) proved for a Hausdorff space K equipped with a finite Radon measure μ and a separable Hilbert space H that the finite product $U_1 \cdots U_n$ of strongly measurable operator valued mappings $U_i : K \longrightarrow L(H)$ is again strongly measurable, where $L(H)$ is the space of all linear bounded operators on H. As mentioned there, the general result seems not to be known in the literature.
Here for a general Banach space X the above property will be established. First some notations and standard definitions. Let (Ω, Σ, μ) be a finite and positive measure space. X, Y, X_1 are Banach spaces,

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where $S(X)$ is the unit sphere of X and $L(X,Y)$ will denote the space of bounded linear operators $T : X \longrightarrow Y$.
Definition: (see e.g. **[Jo]**)

1) A function $g : \Omega \longrightarrow X$ is $\mu-$*measurable*, if there exists a sequence (g_n) of simple functions $g_n : \Omega \longrightarrow X$, such that $\lim_{n\to\infty} \|g(\omega) - g_n(\omega)\| = 0$ f.a.a. $\omega \in \Omega$.

2) An operator-valued function $U : \Omega \longrightarrow L(X,Y)$ is *strongly measurable*, if $U(\cdot)(x) : \Omega \longrightarrow Y$ is $\mu-$measurable for all $x \in X$.

Remark 1: Let X be separable, $(x_k) \subset S(X)$ be dense, let Y be a Banach space and $U : \Omega \longrightarrow L(X,Y)$ be strongly measurable. For $\omega \in \Omega$ we have $\|U(\omega)\| = \sup_{k\in\mathbf{N}} \|U(\omega)(x_k)\|$. Thus the function $\omega \longmapsto \|U(\omega)\|$ is measurable.

Lemma 1 *Let $f : \Omega \longrightarrow X_1$ be $\mu-$measurable, let Y be a Banach space and $U : \Omega \longrightarrow L(X_1,Y)$ be strongly measurable operator valued functions, such that $\omega \longmapsto \|U(\omega)\|$ is measurable. Then $U \circ f : \Omega \longrightarrow Y$ is $\mu-$measurable.*

Proof: Let $\epsilon > 0$ be given. Since $\omega \longmapsto \|f(\omega)\|$ is measurable, there is an $n \in \mathbf{N}$, such that for $A_n := \{\omega \in \Omega; \|f(\omega)\| \leq n\} :\ \mu(\Omega \setminus A_n) < \frac{\epsilon}{3}$. Similar there is an $m \in \mathbf{N}$, such that for $D_m := \{\omega \in A_n; \|U(\omega)\| \leq m\} :\ \mu(A_n \setminus D_m) < \frac{\epsilon}{3}$.
Since $f(\cdot)$ is measurable, by the Egorov theorem (see **[Din,p.94]**) there is a set $\Omega_\epsilon \subset D_m$, $\mu(D_m \setminus \Omega_\epsilon) < \frac{\epsilon}{3}$ and a simple function $g_\epsilon : \Omega \longrightarrow X_1$, $g_\epsilon = \sum_{i=1}^{l} x_i \chi_{B_i}$, $B_i \in \Omega_\epsilon \cap \Sigma$, such that

$$\|f(\cdot)\chi_{\Omega_\epsilon} - g_\epsilon(\cdot)\|_\infty < \frac{\epsilon}{m}.$$

$G(\cdot) := U(\cdot)g_\epsilon(\cdot)$ is $\mu-$measurable, since $G(\cdot) = \sum_{i=1}^{l} U(\cdot)(x_i)\chi_{B_i}$ and U is strongly measurable. Since $\Omega_\epsilon \subset D_m$, it follows that $\|U(\omega)\| \leq m$ for $\omega \in \Omega_\epsilon$. Thus, $\|U(\cdot) \cdot f(\cdot)\chi_{\Omega_\epsilon} - G(\cdot)\|_\infty < \epsilon$.
Hence one obtains:

$\forall\, \epsilon > 0\ \exists\, \Omega_\epsilon \subset \Omega, \mu(\Omega \backslash \Omega_\epsilon) < \epsilon\ \exists\, G : \Omega \longrightarrow Y, \mu-$measurable, such that

$$\|U(\cdot)\cdot f(\cdot)\chi_{\Omega_\epsilon} - G(\cdot)\|_\infty < \epsilon.$$

But this implies the measurability (see e.g. **[Din,p.94,Th.1]**).

Theorem 1 : *Let X be a Banach space, then for $U_i : \Omega \longrightarrow L(X)$ strongly measurable $(i = 1,\ldots,n)$ the product $U_1 \cdots U_n$ is again strongly measurable*

Proof: It is obvious that by an induction argument one may restrict oneself to the case $n = 2$.
Hence let $x \in S(X)$ be fixed. Since $U_2(\cdot)(x)$ is $\mu-$measurable, there is a separable space $X_0 \subset X$, such that $U_2(\cdot)(x) : \Omega \longrightarrow X_0$. Using the remark for $\tilde{U}_1(\cdot) := U_1(\cdot)|_{X_0} : \Omega \longrightarrow L(X_0, X)$, the map $\omega \longmapsto \|\tilde{U}_1(\omega)\|$ is measurable. Thus the lemma is applicable with $X_1 := X_0, Y := X, f(\cdot) := U_2(\cdot)(x)$ and $U := \tilde{U}_1$, which shows that $\tilde{U}_1(\cdot)\cdot U_2(\cdot)(x) = U_1(\cdot)\cdot U_2(\cdot)(x)$ is $\mu-$measurable.

Remark 2: In the paper "The composition of operator-valued measurable functions is measurable" A.Badrikian, G.W.Johnson and I.Yoo proved for separable Fréchet spaces E, F and G that for a measurable space (Ω, Σ) and measurable maps $T_1 : \Omega \longrightarrow L(E, F)$ and $T_2 : \Omega \longrightarrow L(F, G)$, the product $T_2 \circ T_1 : \Omega \longrightarrow L(E, G)$ is measurable again, where on $L(E, F)$ resp.on $L(F, G), L(E, G)$ the Borel-σ-algebra of the strong operator topology is considered. As a corollary **[BJY,Cor.3]** they could deduce the main result of this paper for separable Banach spaces. Thus, their results overlap with the main result of this paper, neither one containing the other, and the proofs are different, too.

References:

[BJY] Badrikian,A.,Johnson,G.W.,Yoo,I.: The composition of operator-valued measurable functions is measurable, to appear in PAMS

[Din] Dinculeanu,N.: Vector Measures, VEB Deutscher Verlag der Wissenschaften, Berlin (1966)

[Jo] Johnson,G.W.: The product of strong operator measurable functions is strong operator measurable, PAMS **117,4** (1993), p.1097-1104