The Mazur Property and Completeness in the Space of Bochner- integrable Functions $L_1(\mu, X)$

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Abstract

The Banach space $L_1(\mu, X)$ of Bochner-integrable functions over a finite and positive measure space with values in a Banach space X is a Mazur space if X is also, and either the measure space is separable or ℓ_1 does not embed in X. Furthermore results concerning the completeness in the dual pairing $\langle L_1(\mu, X), L_\infty(\mu, X^*)\rangle$ are derived.

1.Introduction and Preliminaries

A locally convex space $(E, \mathcal{T})$ has the *Mazur Property* iff every linear $\mathcal{T}$-sequential continuous functional is $\mathcal{T}$-continuous (see **[12]**).
In the Banach space setting, a Banach space X is a *Mazur Space* iff the dual space X^* endowed with the w^*-topology has the Mazur property. The Mazur property was introduced by S.Mazur, and for Banach spaces it is investigated in detail in **[4]**, where relations to other properties and application to measure theory are lined up. T.Kappeler gained (see **[8]**) certain results for the injective tensor product and showed that $L_1(\mu, X)$, the space of Bochner-integable functions over a finite and positive measure space (S, Σ, μ), is a Mazur space provided X is also, and ℓ_1 does not embed in X.
In the second paragraph, the same result is received using a different technique. Extending this result it is shown that, for every separable measure space, $L_1(\mu, X)$ enjoys the Mazur property iff X does. The deductions are based on the consideration of the dual pairing $\langle L_1(\mu, X), L_\infty(\mu, X^*)\rangle$, where $L_\infty(\mu, X^*)$ is the space of X^*-valued μ-essentially bounded functions. If σ' (resp.σ^*) denote the weak topology of this pairing on $L_1(\mu, X)$ (resp.$L_\infty(\mu, X^*)$) then the locally convex space $(L_1(\mu, X), \sigma')$ (resp.$(L_\infty(\mu, X^*), \sigma^*)$), has the Mazur property (resp. iff (X^*, w^*) has the Mazur property). The fact that $(L_1(\mu, X), \sigma')$ has the Mazur property will establish a simple alternative proof to a result of

N.Ghoussoub and P.Saab (see **[6,Theorem 1]**).
The third paragraph will show that the locally convex spaces $(L_1(\mu,X),\tau')$ resp.$(L_\infty(\mu,X^*),\tau^*)$ are complete, where τ' resp. τ^* denote the Mackey-topology of the dual pairing $\langle L_1(\mu,X),L_\infty(\mu,X^*)\rangle$ on $L_1(\mu,X)$ resp. $L_\infty(\mu,X^*)$. This extends a result of T.Kappeler who proved that $(L_1(\mu,X),\tau')$ is complete provided that $L_1(\mu,X)$ is a Mazur space.
Throughout the paper $B(Y)$ denote the unit ball of a Banach space Y, while $co(A)$ resp. $aco(A)$ is the convex resp. absolutely convex hull of a subset $A\subset Y$ of a Banach space Y. For a finite measure space (S,Σ,μ) , π is a finite Σ-partition and

$$\Pi := \{(\pi_n)_{n\in\mathbf{N}};(\pi_n)_{n\in\mathbf{N}} \text{ is an increasing sequence of finite partitions}\}.$$

For a finite partition π set $\mathbf{E}_\pi$ is the conditional expectation operator to the σ-algebra generated by π. Define

$$V_\infty(\mu,X^*) := \{m:\Sigma\to X^*, \exists\ K>0\ \forall A\in\Sigma\ \|m(A)\|\le K\mu(A)\}$$

the Banach space of μ-Lipschitz continuous X^*-valued vector measures equipped with the norm $\|m\|_{\text{Lip}} := \inf\{K>0,\forall\ A\in\Sigma\ \|m(A)\|\le K\mu(A)\}$. Note that $V_\infty(\mu,X^*)$ is isometrically isomorphic to the dual $L_1(\mu,X)^*$ (see **[2]**). All these results exploit the following characterization of certain X resp. X^* valued vector measures which was first given in this form by J.Batt and W.Hiermeyer (see **[1,2.4.Proposition]**).
If $(\pi_n)_{n\in\mathbf{N}}\in\Pi$ then set

$$\begin{aligned}\Lambda_1((\pi_n)_{n\in\mathbf{N}}) := \{T\ &:\ L_1(\mu,X)\to c_0;\exists\ (g_n)_{n\in\mathbf{N}}\subset B(L_\infty(\mu,X^*))\text{ s.t.}\\ g_n\ &=\ (\mathbf{E}_{\pi_{n+1}}-\mathbf{E}_{\pi_n})(g_n)\text{ and } Tf=(\langle f,g_n\rangle)_{n\in\mathbf{N}}\}\end{aligned}$$

$$\begin{aligned}\Lambda_1^*((\pi_n)_{n\in\mathbf{N}}) := \{T\ &:\ L_\infty(\mu,X^*)\to c_0;\exists\ (f_n)_{n\in\mathbf{N}}\subset B(L_\infty(\mu,X))\text{ s.t.}\\ f_n\ &=\ (\mathbf{E}_{\pi_{n+1}}-\mathbf{E}_{\pi_n})(f_n)\text{ and } Tg=(\langle f_n,g\rangle)_{n\in\mathbf{N}}\}\end{aligned}$$

These classes of operators help to characterize vector measures which have a Bochner density as shown in the next Lemmata; the proof is basically the same as in **[1,2.4.Proposition]**) .

1.1.Lemma: Let $m:\Sigma\to X$ be a finitely additive vector measure of bounded variation s.t. $\mu(A)=0$ implies $m(A)=0$. Then the following are equivalent:
a)m has a $L_1(\mu,X)$-density
b)$\forall(\pi_n)_{n\in\mathbf{N}}\in\Pi\ \forall T\in\Lambda_1((\pi_n)_{n\in\mathbf{N}})$ $T(K((\pi_n)_{n\in\mathbf{N}})$ is relatively weakly compact, where $K((\pi_n)_{n\in\mathbf{N}}) := \Big\{\sum_{A\in\pi_n}\frac{m(A)}{\mu(A)}\chi_A; n\in\mathbf{N}\Big\}$.

1.2.Lemma: For $m \in V_\infty(\mu, X^*)$ are equivalent:
a) m has a Bochner-density $g \in L_\infty(\mu, X^*)$
b)$\forall (\pi_n)_{n\in\mathbf{N}} \in \Pi \ \forall T \in \Lambda_1^*((\pi_n)_{n\in\mathbf{N}}) \ \ T(K((\pi_n)_{n\in\mathbf{N}}))$ is relatively weakly compact, where $K((\pi_n)_{n\in\mathbf{N}}) := \Big\{ \sum_{A\in\pi_n} \frac{m(A)}{\mu(A)}\chi_A; \ n \in \mathbf{N}\Big\}$.

2.The Mazur Property for $L_1(\mu, X)$.

2.1.Theorem: a) $(L_\infty(\mu, X^*), \sigma^*)$ has the Mazur property if and only if (X^*, w^*) has the Mazur property.
b) If $L_\infty(\mu, X^*)$ is w^*-sequential dense in $L_1(\mu, X)^*$ then $L_1(\mu, X)$ is a Mazur space if and only if X is.
Proof: The necessity in a) resp. b) is evident.
a): Let $F \in L_\infty(\mu, X^*)^*$ $\ \sigma^*$-sequential continuous. Then the map

$$(3) \quad X^* \ni x^* \longmapsto \langle F, x^*\chi_A\rangle$$

is w^*-sequential continuous for all $A \in \Sigma$.
X is a Mazur space, hence define the following additive vector measure

$$(1) \qquad \begin{array}{rll} m: & \Sigma & \longrightarrow X \\ & A & \longmapsto (x^* \mapsto \langle F, x^*\chi_A\rangle) \end{array}$$

which has bounded variation since $F \in L_\infty(\mu, X^*)^*$. Evidently $\mu(A) = 0$ implies $m(A) = 0 \ \ \forall\ A \in \Sigma$. So apply (1.1) to demonstrate that m has a Bochner derivative.
Take $(\pi_n)_{n\in\mathbf{N}} \in \Pi$ and $T \in \Lambda_1((\pi_n)_{n\in\mathbf{N}})$. It is easily seen that the martingale difference sequence $(g_n)_{n\in\mathbf{N}}$ associated to T is σ^*-convergent to 0. Define for $k \in \mathbf{N}$:

$$h_k := \sum_{A\in\pi_k} \frac{m(A)}{\mu(A)}\chi_A$$

then

$$(2) \qquad T(h_k) \longrightarrow (\langle h_{n+1}, g_n\rangle)_{n\in\mathbf{N}} \ \text{ weakly in } c_0 \text{ if } k \to \infty$$

To see this, note that

$$(3) \qquad \forall n \in \mathbf{N}: \ \langle F, g_n\rangle \ = \ \langle \mathbf{E}^{**}_{\pi_{n+1}} F, g_n\rangle \ = \ \langle h_{n+1}, g_n\rangle.$$

Since (g_n) is σ^*-converging to 0, the σ^*-sequential continuity of F implies

$$(4) \qquad \langle F, g_n\rangle \longrightarrow 0 \text{ if } n \to \infty.$$

Since $g_n = (\mathbf{E}_{\pi_{n+1}} - \mathbf{E}_{\pi_n})(g_n)$ it follows that

$$T(h_k) \longrightarrow (\langle h_{n+1}, g_n\rangle)_{n\in\mathbf{N}} \text{ pointwise if } k \to \infty.$$

Thus considering (3),(4) and the boundedness of $(h_k)_{k\in\mathbf{N}} \subset L_1(\mu, X)$ the sequence $(T(h_k))_{k\in\mathbf{N}}$ converges weakly and (2) is proved. Hence there is a $f \in L_1(\mu, X)$ s.t.

$$\begin{aligned} \langle f, g\rangle &= \langle F, g\rangle \text{ for all simple functions} \\ g &= \sum_{i=1}^{n} x_i^* \chi_{A_i} \ (x_i^* \in X^*, A_i \in \Sigma, i = 1 \ldots n). \end{aligned} \tag{5}$$

For an arbitrary $g \in L_\infty(\mu, X^*)$ take a sub-σ-algebra generated by a $(\pi_n)_{n\in\mathbf{N}} \in \Pi$ s.t. $\mathbf{E}^*_{\pi_n}(g) \to g$ in $(L_\infty(\mu, X^*), \sigma^*)$ if $n \to \infty$. Using (5) and the σ^*-sequential continuity of $F \in L_\infty(\mu, X^*)^*$ it is

$$\langle f, g\rangle \;=\; \lim_{n\to\infty} \langle f, \mathbf{E}^*_{\pi_n}(g)\rangle \;=\; \lim_{n\to\infty} \langle F, \mathbf{E}^*_{\pi_n}(g)\rangle \;=\; \langle F, g\rangle$$

and F is induced by f.
b):If $F \in L_1(\mu, X)^{**}$ is w^*-sequential continuous then the restriction F' to $L_\infty(\mu, X^*)$ is σ^*-sequential continuous. Hence the w^*-sequential density of $L_\infty(\mu, X^*) \subset L_1(\mu, X)^*$ combined with part "a)" indicates the assertion.

2.2.Corollary: If either the measure space (S, Σ, μ) is separable or ℓ_1 does not embed in X then

$L_1(\mu, X)$ is a Mazur space if and only if X is.

Proof: In the first case it is evident that $L_\infty(\mu, X^*) \subset L_1(\mu, X)^*$ is w^*- sequential dense. Suppose ℓ_1 is not contained in X, then **[9, Theorem 5]** combined with **[10,Proposition 4,Remark 1 and Corollary 1]** reveals that the simple functions in $L_\infty(\mu, X^*)$ are w^*-sequential dense in $L_1(\mu, X)^*$. Hence (2.1.) part b) can be applied.

The situation for $L_1(\mu, X)$ is quite similiar. Since X in the weak topology has always the Mazur property, the next theorem is not surprising and should be understood as a dual version to (2.1.a).

2.3. Theorem: $(L_1(\mu, X), \sigma')$ has the Mazur property.
Proof: Let $G \in L_1(\mu, X)^*$ be σ'-sequential continuous. The representation of the dual $L_1(\mu, X)^*$ shows (see the Introduction)

$$\exists m \in V_\infty(\mu, X^*): \ \langle f, G\rangle \;=\; \int f\,dm \text{ as the Bartle-integral.}$$

Hence the proof is done if m has a Bochner-density. Thus applying (1.2.) let $(\pi_n)_{n\in\mathbf{N}} \in \Pi$ and $T \in \Lambda_1^*((\pi_n)_{n\in\mathbf{N}})$. Define for $k \in \mathbf{N}$:

$$h_k := \sum_{A\in\pi_k} \frac{m(A)}{\mu(A)}\chi_A.$$

If $(f_n)_{n\in\mathbf{N}}$ is the martingale difference sequence related to T, then it is easy to realize that (f_n) converges to 0 in $(L_1(\mu, X), \sigma')$. To demonstrate that $(T(h_k))_{n\in\mathbf{N}}$ is relatively weakly compact one proves in the same way as in (2.1.a) that

$$T(h_k) \longrightarrow (\langle f_n, h_{n+1}\rangle)_{n\in\mathbf{N}} \text{ pointwise in } c_0$$

hence weakly and the proof is complete.

The next two Corollaries will deal with weakly resp. σ'-compact subsets of $L_1(\mu, X)$ and are applications of (2.3.).

2.4.Corollary: Every relatively σ'-compact subset of $L_1(\mu, X)$ is relatively weakly compact if and only if X^* has the Radon-Nikodym property with respect to (S, Σ, μ).

2.5.Corollary(N.Ghoussoub,P.Saab): All bounded and uniformly integrable subsets $K \subset L_1(\mu, X)$ s.t. $\left\{\int_A f d\mu; f \in K\right\}$ is relatively weakly compact in X for all $A \in \Sigma$ are relatively weakly compact in $L_1(\mu, X)$, if and only if both X and X^* have the Radon-Nikodym property with respect to (S, Σ, μ).

3.Completeness in the Dual Pairing $\langle L_1(\mu, X), L_\infty(\mu, X^*)\rangle$.

The space $(L_1(\mu, X), \sigma')$ is, due to an unpublished result of J.Batt and W.Hiermeyer, sequential complete if and only if X is weakly sequential complete and X has the Radon-Nikodym property with respect to (S, Σ, μ) (see also **[11,3.3.Lemma]**). Using **[3,IV.1 Theorem 1]** it is evident that $(L_\infty(\mu, X^*), \sigma^*)$ is sequential complete if and only if X^* has the Radon-Nikodym property with respect to (S, Σ, μ). The unknown fact if the corresponding Mackey topologies are complete was partially solved by T.Kappeler (see**[8,4.2. Proposition]**). The next theorem will show that they both are complete locally convex spaces.

3.1 Theorem: The spaces $(L_1(\mu,X),\tau')$ and $(L_\infty(\mu,X^*),\tau^*)$ are complete.
Proof: Let $(f_\alpha)_{\alpha\in J}$ be a τ'-Cauchy net in $L_1(\mu,X)$.
1.Step: Since the operator

$$\begin{aligned} I_A: \quad L_1(\mu,X) &\longrightarrow X \\ f \quad &\longmapsto \int_A f\ d\mu \end{aligned}$$

is $\tau'-\|\cdot\|$-continuous

$$\begin{aligned} m: \quad \Sigma &\longrightarrow X \\ A &\longmapsto \|\cdot\| - \lim_{\alpha\in J} I_A(f_\alpha) \end{aligned}$$

defines an additive vector measure s.t. $\mu(A)=0$ implies $m(A)=0$. To apply (1.1.) m has to have bounded variation. Consider $L_1(\mu,X)$ as a subspace of $L_\infty(\mu,X^*)^*$, then the net (f_α) is a Cauchy-net in the dual Mackey topology $\tau(L_\infty(\mu,X^*)^*,L_\infty(\mu,X^*))$. Since the dual Mackey topology is complete (see **[12,9-2-14 Theorem]**), this net has a limit $F\in L_\infty(\mu,X^*)^*$. In particular

$$\langle f_\alpha, g\rangle \to \langle F,g\rangle \quad \forall g\in L_\infty(\mu,X^*) \text{ if } \alpha\in J. \tag{6}$$

Thus F induces a finitely additive vector measure

$$\begin{aligned} m': \quad \Sigma &\longrightarrow X^{**} \\ A &\longmapsto (x^*\mapsto \langle F, x^*\chi_A\rangle). \end{aligned}$$

Evidently m and m' coincide by the definition of m and (6). Hence

$$\langle F,g\rangle \ = \ \int_S g\ dm \text{ for all simple functions } g=\sum_{i=1}^n x_i^*\chi_{A_i} \text{ in } L_\infty(\mu,X^*).$$

Since (f_α) converges in the dual Mackey topology to $F\in L_\infty(\mu,X^*)^*$, it follows

$$\langle F,g\rangle \ = \ \textstyle\int_S g\ dm \ \forall g\in L_\infty(\mu,X^*). \tag{7}$$

2.Step: m has a Bochner derivative:
Use (1.1.) and select a $(\pi_n)_{n\in\mathbf{N}}\in\Pi$ and $T\in\Lambda_1((\pi_n)_{n\in\mathbf{N}})$. Define for $n\in\mathbf{N}$:

$$h_n := \sum_{A\in\pi_n}\frac{m(A)}{\mu(A)}\chi_A.$$

Let $(g_n)_{n\in\mathbf{N}}\subset B(L_\infty(\mu,X^*))$ be the martingale difference sequence related to T. Then (g_n) converges to 0 in $(L_\infty(\mu,X^*),\sigma^*)$. With the help of (1.2.) it can be shown along the line of the proof of **[1,2.5.Theorem]** that $\overline{aco}^{\sigma^*}(g_n;n\in\mathbf{N})$ is σ^*-compact. Hence the set

$$D:=\Big\{\sum_{n=1}^\infty b_ng_n;\ (b_n)_{n\in\mathbf{N}}\subset B(\ell_1)\Big\}$$

is σ^*-compact.
Suppose now that $(T(h_n))_{n\in\mathbf{N}}$ is not relatively weakly compact in c_0. According to **[7,Theorem 1]**:

$$(8)\qquad \begin{array}{l} \exists\theta > 0\ \exists(h_{n_k})_{k\in\mathbf{N}} \text{ subsequence } \exists(b^l)_{l\in\mathbf{N}} \in B(\ell_1) \text{ s.t.} \\ \inf\left\{\langle T(h_{n_k}), b^l\rangle; l < k\right\} \ > \ \sup\left\{\langle T(h_{n_k}), b^l\rangle; l > k\right\} + \theta \end{array}$$

Since (f_α) is a τ'-Cauchy net and D is σ^*-compact

$$(9)\qquad \exists\alpha_0 \in J\forall\alpha,\beta \geq \alpha_0 : \ \sup_{g\in D} |\langle f_\alpha - f_\beta, g\rangle| \ < \ \frac{\theta}{8}.$$

The definition of the vector measure m reveals

$$(10)\qquad \forall k \in \mathbf{N}\exists\alpha(k) \geq \alpha_0 : \ \ ||h_{n_k} - \mathbf{E}_{\pi_{n_k}}(f_{\alpha(k)})|| \ < \ \frac{\theta}{8}.$$

The martingale $(\mathbf{E}_{\pi_{n_k}}(f_{\alpha_0}); k \in \mathbf{N})$ is relatively weakly compact since it converges in norm. Thus by **[7,Theorem 1]**:

$$(11)\qquad \inf\left\{\langle T(\mathbf{E}_{\pi_{n_k}}(f_{\alpha_0})), b^l\rangle; l < k\right\} \ \leq \ \sup\left\{\langle T(\mathbf{E}_{\pi_{n_k}}(f_{\alpha_0})), b^l\rangle; l > k\right\}.$$

Finally using all the above facts one computes for $k, l \in \mathbf{N}$:

$$\begin{array}{rcl} |\langle T(h_{n_k}) - T(\mathbf{E}_{\pi_{n_k}}(f_{\alpha_0})), b^l\rangle| & = & |\langle h_{n_k} - \mathbf{E}_{\pi_{n_k}}(f_{\alpha_0}), \sum_{j=1}^{\infty} b^l_j g_j\rangle| \\ & \leq & |\langle h_{n_k} - \mathbf{E}_{\pi_{n_k}}(f_{\alpha(k)}), \sum_{j=1}^{\infty} b^l_j g_j\rangle| \\ & + & |\langle \mathbf{E}_{\pi_{n_k}}(f_{\alpha(k)}) - \mathbf{E}_{\pi_{n_k}}(f_{\alpha_0}), \sum_{j=1}^{\infty} b^l_j g_j\rangle| \\ & < & \frac{\theta}{8} + |\langle f_{\alpha(k)} - f_{\alpha_0}, \sum_{j=1}^{n_k-1} b^l_j g_j\rangle| \\ \text{by} & & (10) \\ & < & \frac{\theta}{8} + \frac{\theta}{8} \\ \text{according to} & & (9) \\ & = & \frac{\theta}{4} \end{array}$$

Under consideration of (8) this forms a contradiction to (11), and hence m has a derivative $f \in L_1(\mu, X)$.
Since $(L_1(\mu, X), \tau')$ has a zero neighborhood basis of σ'-closed sets, it is

$$f_\alpha \to f \text{ for } \alpha \in J \text{ in } \tau' \text{ iff } f_\alpha \to f \text{ for } \alpha \in J \text{ in } \sigma'.$$

Thus with the help of (7) it follows the completeness of $(L_1(\mu, X), \tau')$ by

$$\forall g \in L_\infty(\mu, X^*) \quad \lim_{\alpha \in J} \langle f_\alpha, g\rangle \; = \; \langle F, g\rangle = \langle f, g\rangle.$$

Since $(L_1(\mu, X), \tau')$ is complete as shown above, $(L_1(\mu, X), \sigma')$ has the convex compactness property. Hence by **[12,9-2-14 Theorem]** also the space $(L_\infty(\mu, X^*), \tau^*)$ is complete.

The following two corollaries can be gained from the above theorem and abstract results in locally convex theory. The first one was already observed by J.Batt and W.Hiermeyer (see **[1,2.6.Corollary]**). The second one is a James type result (see e.g.**[5]**).

3.2.Corollary: The convex resp. absolutely convex hull of a relatively σ'-compact subset is again relatively σ'-compact.

3.3.Corollary: A σ'-closed and bounded subset $K \subset L_1(\mu, X)$ is σ'-compact if and only if every $g \in L_\infty(\mu, X^*)$ attains its suprema on K.

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