Hereditary properties for duals of Bochner L_p–functions-spaces *

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Abstract

For a finite and positive measure space (Ω, Σ, μ) hereditarity results which hold for Bochner L_p-spaces are derived for their dual spaces. In addition results of N.Kalton, G.Pisier and N.Randrianantoanina-E.Saab are given alternative proofs.

1 Introduction and Preliminaries

There exists a long list on hereditary results for Bochner L_p-spaces, but virtually no equivalent statements are given for their dual spaces. The major problem is the lack of a satisfying representation of their dual spaces, repectively, meaning the second dual of the Bochner L_p-spaces. Let us mention some classical hereditary results concerning embedding classical sequence spaces. S.Kwapien showed that for $1 \leq p < \infty$, $c_0 \subset L_p(\mu, X)$, if and only if it embeds in X [12]. J.Mendoza proved that ℓ_∞ can be found as a copy in $L_p(\mu, X)$, $1 \leq p < \infty$ if and only if it is isomorphically embedded in X (see [14]). Furthermore the result that $\ell_1 \subset L_p(\mu, X)$ for $1 < p < \infty$ if and only if ℓ_1 embeds in X, is due to G.Pisier [17]. We will show Pisier's result by different means, which allows the extension to the w^*–measurable case. In [21, p.404,3.2.Th.]) it was shown that some renorming properties of X are inherited by $L_p(\mu, X)$ for $1 \leq p < \infty$ the space

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of $p-$integrable vector-valued functions enjoys M.Talagrand demonstrated in [22, p.717,Th.11] that weak sequential completeness of X passes to the function space $E(X)$, where E is an order continuous Banach lattice with a weak unit not containing c_0. N.Randriannomia and E.Saab obtained that $L_p(\mu, X), 1 < p < \infty$ enjoys the complete continuity property if and only if X does [18, p.1111,Prop.2]. We will present an alternative proof and extend it to the dual of $L_p(\mu, X)$. The results are based on approximation results for operator-valued functions in section 2, which may be independent interest.
In addition we extend results of J.Voigt [23] resp. the author [20] to the w^*-measurable case.
First we will fix some notations and present some basic facts. Throughtout the paper (Ω, Σ, μ) is a positive and finite measure space. X, Y are Banach spaces with is duals X^*, Y^*. $B(X)$ is the unit ball while $S(X)$ denotes the unit sphere. $K(X,Y)$, $W(X,Y)$ resp. $CW(X,Y)$ stands for the compact, weakly compact resp.conditionally weakly compact operators.
Let us collect some basic facts for w^*-measurable functions.
(1.1) Let $f : \Omega \longrightarrow X^*$ be a function. Then f is called w^*- measurable, if $\Omega \ni \omega \longmapsto \langle x, f(\omega)\rangle$ is measurable for all $x \in X$.
Let $1 \leq p < \infty$. Then we define:

$$\mathcal{L}_p^{w^*}(\mu, X^*) \;\; := \;\; \{f : \Omega \longrightarrow X^*; f \text{ is } w^*\text{- measurable and } \overline{\int_\Omega} \|f(\omega)\|^p d\mu(\omega) < \infty\},$$

where

$$\overline{\int_\Omega} \|f(\omega)\|^p d\mu(\omega) \;\; := \;\; \inf\{\int_\Omega g(\omega) d\mu(\omega);\; g \in \mathcal{L}_1(\mu) \text{ and } \|f(\omega)\|^p \leq g(\omega) \text{ on } \Omega\} \;\; [10, VI.6].$$

For $p :\; 1 \leq p \leq \infty$ we always denote the conjugate of p by p'.

Let $1 \leq p < \infty$ and let $f \in \mathcal{L}_p^{w^*}(\mu, X^*)$. We define:
$\|f\|_{p,*} \;:=\; \sup\{\int_\Omega |\langle f(\omega), g(\omega)\rangle| d\mu(\omega); g \;\in\; \mathcal{E}(\Omega, \Sigma, X), \|g\|_{p'} \;\leq\; 1\}$,
which is a seminorm on $\mathcal{L}_p^{w^*}(\mu, X^*)$. If $p = \infty$, then

$$\mathcal{L}_\infty^{w^*}(\mu, X^*) \;\; := \;\; \{f : \Omega \longrightarrow X^*; f \;\text{ is } w^* - \text{measurable and}$$

$$\|f\|_{\infty,*} := \inf\{K > 0; \forall\, x \in B(X) : \\ \mu(\{\omega \in \Omega; |\langle x, f(\omega)\rangle| > K\}) = 0\} < \infty\}.$$

According to the definition of the norm we have $L_p(\mu, X)^* = L_{p'}^{w^*}(\mu, X^*)$, for $1 \le p < \infty$.

Let $f : \Omega \longrightarrow X^*$ be w^*- measurable. We define the norm function

$$\|f(\cdot)\| : \Omega \longrightarrow \mathbb{R},\ \omega \longmapsto \|f(\omega)\|.$$

If X is separable with a dense sequence $(x_n) \subset B(X)$, then every w^*-measurable function $f : \Omega \longrightarrow X^*$ has measurable norm function, since $\|f(\omega)\| = \sup\limits_{n \in \mathbb{N}} |\langle x_n, f(\omega)\rangle|,\ \omega \in \Omega$.

Since we are investigating operator-valued functions to deduce the hereditary properties we have to make the following definitions.

(1.2) Let (Ω, Σ, μ) be a positive measure space. A function $U : \Omega \longrightarrow L(X, Y)$ is called *strongly measurable* if :

$$\forall\, x \in X :\ U(\cdot)(x) : \Omega \longrightarrow Y \text{ is } \mu - \text{measurable.}$$

(1.3) Let (Ω, Σ, μ) be a positive measure space. A function $U : \Omega \longrightarrow L(X, Y^*)$ is w^**−measurable*, if:

$$\forall\, x \in X :\ U(\cdot)(x) \text{ is } w^* - \text{measurable.}$$

(1.4) Let $1 \le p \le \infty$ and let $T : X \longrightarrow L_p(\mu, Y)$ be linear (resp. $T : X \longrightarrow L_p^{w^*}(\mu, Y^*)$). Then $U : \Omega \longrightarrow L(X, Y)$ is a *strongly measurable operator-valued density* of T (resp. $U : \Omega \longrightarrow L(X, Y^*)$ is a w^**−measurable operator-valued density* of T), if

U is strongly measurable (resp. w^*-measurable) and

$\forall\, x \in X :\ U(\cdot)(x) = T(x)$ in $L_p(\mu, Y)$ (resp. in $L_p^{w^*}(\mu, Y^*)$).

(1.5) Let ρ be a lifting of $\mathcal{L}_\infty := \mathcal{L}_\infty(\mu)$.
Let $U : \Omega \longrightarrow L(X, Y^*)$ be a function.

i) If $\sup_{\omega \in \Omega} \|U(\omega)\| < \infty$, then we write $\rho(U) = U$, if $\rho(\langle y, U(\omega)(x)\rangle) = \langle y, U(\omega)x\rangle$ for all $\omega \in \Omega, x \in X, y \in Y$.

ii) If $\sup_{\omega \in \Omega} \|U(\omega)\| \le \infty$, then we write $\rho[U] = U$, if there exists a partition of Ω, $(A_n) \subset \Sigma$, such that for all $n \in N, \omega \in \Omega,\ x \in X,\ y \in Y : \rho(\langle y, U(\omega)(x)\rangle \chi_{A_n}) = \langle y, U(\omega)(x)\rangle \rho(\chi_{A_n})$.

An application of a result of N. Dinculeanu [8, p.269,Th.5] gives the following fundamental theorem for the lifting.

Theorem 1.6 *Let ρ be a lifting of $\mathcal{L}_\infty$.*
Then there exists map $\tau : L_1^{w^*}(\mu, X^*) \longrightarrow \mathcal{L}_1^{w^*}(\mu, X^*)$, which has the following properties.

a) $\tau(f) \in f,\ \rho[\tau(f)] = \tau(f)\ f \in L_1^{w^*}(\mu, X^*)$.

b) $\|\tau(f)(\cdot)\| \in \mathcal{L}_1(\mu)$ *and* $\|f\|_{1,*} = \int_\Omega \|\tau(f)(\omega)\| d\mu(\omega)$ *for* $f \in L_1^{w^*}(\mu, X^*)$.

c) $\forall\ \alpha, \beta \in \mathbb{R}\ \forall f_1, f_2 \in L_1^{w^*}(\mu, X^*)$:

$$\tau(\alpha f_1 + \beta f_2)(\omega) = \alpha\tau(f_1)(\omega) + \beta\tau(f_2)(\omega)\ f.a.a\ \omega \in \Omega.$$

In the sequel we will write f^ρ insted of $\tau(f)$, where ρ is a lifting of $\mathcal{L}_\infty$.
Let us state some well-known facts for Bochner-integrable functions in the w^*−measurable setting. The proofs are straightforward applications of the classical results.

(1.7) *Extended Lebesgue domination theorem:*
Let $(f_n) \subset L_1^{w^*}(\mu, X^*)$. Let $\hat{f} : \Omega \longrightarrow X^*$ be a function and $g \in \mathcal{L}_1(\mu)$, such that

- $f_n^\rho(\omega) \to \hat{f}(\omega)$ f.a.a. $\omega \in \Omega$,
- $\forall\ n \in \mathbb{N}\ : \|f_n^\rho(\omega)\| \le g(\omega)$ f.a.a $\omega \in \Omega$.

(1.8) *Extended Egorov Theorem*
Let $(f_n) \subset L_1^{w^*}(\mu, X^*)$. let $\hat{f} : \Omega \longrightarrow X^*$ be a function, such that $f_n^\rho \to \hat{f}$ pointwise a.e.. Then

a) $\hat{f}$ is w^*-measurable and $\|\hat{f}(\cdot)\|$ is $\mu-$ measurable,

b) $\forall\ \epsilon > 0\ \exists\ \Omega_\epsilon \in \Sigma,\ \mu(\Omega \setminus \Omega_\epsilon) < \epsilon :\ \sup_{\omega\in\Omega_\epsilon} \|f_n^\rho(\omega) - \hat{f}(\omega)\| < \epsilon$.

(1.9) Let $(f_n) \subset L_1^{w^*}(\mu, X^*), f \in L_1^{w^*}(\mu, X^*)$, such that $\|f - f_n\|_{1,*} \to 0$.
Then there is a subsequence (f_{n_k}), such that $f_{n_k}^\rho \to f^\rho$ pointwise a.e..

As an analogon to J.Diestel, W.Ruess and W.Schachermayer [6] we present a result for the weak compactness of w^*-measurable functions. The proof is based on results of M.Talagrand [22, p.720,Th.14, Th.15], thus, we omit the detailed proof.

Theorem 1.10 *:*

a) Let $1 \leq p < \infty$ and let $K \subset L_p^{w^*}(\mu, X^*)$ be bounded in $L_\infty^{w^*}(\mu, X^*)$. Let ρ be a lifting of $\mathcal{L}_\infty$. Then the following conditions are equivalent:

a) K is relatively weakly compact.

b) $\forall\ (f_n) \subset K\ \exists\ (g_n), g_n \in co(f_n, f_{n+1}, \ldots)$, such that $(g_n^\rho(\omega))$ is relatively compact f.a.a. $\omega \in \Omega$.

c) $\forall\ (f_n) \subset K\ \exists\ (g_n), g_n \in co(f_n, f_{n+1}, \ldots)$, such that $(g_n^\rho(\omega))$ is relatively weakly compact f.a.a. $\omega \in \Omega$.

b) If $K \subset L_1^{w^*}(\mu, X^*)$ is relatively weakly compact, then K is uniformly integrable.

2 Approximation of w^*-measurable operator-valued functions

The following basic result in taken from [S1,p.380,Th.2.1].

Theorem 2.1 *: Every bounded operator $T : X \longrightarrow L_\infty^{w^*}(\mu, Y^*)$ admits of a w^*-measurable density $U : \Omega \longrightarrow L(X, Y^*)$.*
In particular, every bounded operator $T : X \longrightarrow L_\infty(\mu, Y)$ has a strongly measurable density $U : \Omega \longrightarrow L(X, Y)$.

Remark 2.2 Let ω_1 be the first uncountable ordinal. Let $(\phi_\alpha)_{0 \leq \alpha < \omega_1}$, $\phi_\alpha : \Omega \longrightarrow \mathbb{R}$ be a family of measurable functions, such that

$$\forall\ \alpha, \beta,\ \alpha < \beta\ \forall\ \omega \in \Omega :\ \phi_\alpha(\omega) \leq \phi_\beta(\omega).$$

Then there exists a countable ordinal β_0, such that

$$\forall\ \beta \geq \beta_0\ :\ \phi_\beta = \phi_{\beta_0} \text{ a.e. on } \Omega.$$

A straightforward consequence of the foregoing remark is the following proposition.

Proposition 2.3 *Let $1 \leq p \leq \infty$ and let ρ be a lifting of $\mathcal{L}_\infty$.*

a) *Let $T : X \longrightarrow L_p(\mu, Y)$ be linear and bounded. T has a strongly measurable density if and only if $T|_{X_0}$ has a strongly measurable density $U : \Omega \longrightarrow L(X_0, Y)$ for all separable subspaces $X_0 \subset X$.*

b) *Let $T : X \longrightarrow L_p^{w^*}(\mu, Y^*)$ be bounded and linear. T has a w^*-measurable density if and only if $T|_{X_0}$ has a w^*-measurable density $U : \Omega \longrightarrow L(X_0, Y^*)$ for all separable subspaces $X_0 \subset X$.*

Theorem 2.4 *: Let ρ be a lifting of $\mathcal{L}_\infty$. Let (Θ, Ψ, ν) be a finite and positive measure space and let $T : L_1(\nu) \longrightarrow L_1^{w^*}(\mu, X^*)$ be bounded. Then there exists a w^*-measurable density $U : \Omega \longrightarrow L(L_\infty, X^*)$ of $T|_{L_\infty}$, such that $\rho[U] = U$.*
If in addition $\ell_\infty \not\subset X^*$, then $U : \Omega \longrightarrow W(L_\infty, X^*)$.

Proof: We assume that (Θ, Ψ, ν) is the Lebesgue measure space. Thus let $(\pi_n)_{n \in \mathbb{N}}$ be the sequence of dyadic partitions of the interval $[0, 1]$. Then we define for $n \in \mathbb{N}$ and $\tau = (\tau_i) \in \{-1, 1\}^{2^n}$: $f_{n,\tau} := \sum_{i=1}^{2^n} \tau_i \chi_{A_i}$, where $A_i \in \pi_n, i = 1, \dots, 2^n$. We define for $n \in \mathbb{N}, K > 0$:

$$A_{K,n} := \{\omega \in \Omega; \exists\ \tau \in \{-1, 1\}^{2^n} \|(T(f_{n,\tau}))^\rho(\omega)\| > K\}.$$

According to 1.2 $(T(f_{n,\tau}))^\rho$ has a measurable norm function. Thus the sets $A_{K,n}$ are measurable.

(1) $\quad \sup\limits_{n \in \mathbb{N}} \mu(A_{K,n}) \overset{K \to \infty}{\to} 0.$

For the proof first we note that $A_{K,n} \subset A_{K,n+1}$ for all $K > 0$. Let $n \in \mathbb{N}, K > 0$. We define $\sigma : A_{K,n} \longrightarrow \{-1, 1\}^{2^n}$, as follows:

> Let $\{\tau_1, \dots, \tau_m\}$ be a counting of the set $\{-1, 1\}^{2^n}$ with $m = 2^{2^n}$. For $1 \leq j \leq m$ we define $A_{K,n,j} := \{\omega \in A_{K,n}; \|T(f_{n,\tau_j})^\rho(\omega)\| > K\}$ and have $A_{K,n} = \bigcup\limits_{j=1}^{m} A_{K,n,j}$ by definition. Let $C_1 := A_{K,n,1}$ and $C_j := A_{K,n.j} \setminus \bigcup\limits_{i=1}^{j-1} C_i$ for $j = 2, \dots, m$. Evidently $(C_j)_{j=1}^m$ is a $\Sigma-$partition of $A_{K,n}$. Now we define
>
> $$\sigma : A_{K,n} \longrightarrow \{-1, 1\}^{2^n},\ \omega \longmapsto \tau_j,$$

where $\omega \in C_j$. The map $\omega \longmapsto T(f_{n\sigma(\omega)})^\rho$ is measurable, since $(C_j) \subset \Sigma$.

Then we estimate

$$\begin{aligned}
\mu(A_{K,n})\ K &\leq \int\limits_{A_{K,n}} \|(T(f_{n,\sigma(\omega)}))^\rho(\omega)\| d\mu(\omega) \\
&= \int\limits_{A_{K,n}} \|T(\sum_{i=1}^{2^n} \sigma(\omega)_i \chi_{A_i})^\rho(\omega)\| d\mu(\omega) \\
&\overset{1.6c)}{=} \int\limits_{A_{K,n}} \|\sum_{i=1}^{2^n} \sigma(\omega)_i T(\chi_{A_i})^\rho(\omega)\| d\mu(\omega) \\
&\leq \int\limits_{A_{K,n}} \sum_{i=1}^{2^n} \|\sigma(\omega)_i T(\chi_{A_i})^\rho(\omega)\| d\mu(\omega) \\
&= \int\limits_{A_{K,n}} \sum_{i=1}^{2^n} \|T(\chi_{A_i})^\rho(\omega)\| d\mu(\omega)\| \\
&= \sum_{i=1}^{2^n} \int\limits_{A_{K,n}} (\|T(\chi_{A_i})^\rho(\omega)\|) d\mu(\omega) \leq \sum_{i=1}^{2^n} \big(\|T\|(\lambda(A_i))\big) \\
&= \|T\|\lambda([0,1]) = \|T\|.
\end{aligned}$$

This finishes the proof for (1).

Since $B(L_\infty)$ is weakly compact in L_1, $T|_{L_\infty}$ is weakly compact, thus uniformly integrable (see 1.10). Hence, for a given $\epsilon > 0$, there is a $\delta > 0$ such that for all $A \in \Sigma, \mu(A) < \delta$:

$$(2) \qquad \sup_{g \in B(L_\infty)} \int\limits_A \|(T(g))^\rho\| d\mu < \epsilon.$$

Hence, by (1) there exists a $K > 0$, such that $\mu\big(\cup_{n\in\mathbb{N}} A_{K,n}\big) \leq \delta$. Again let $\Omega_\epsilon := \Omega \setminus \big(\cup_{n\in\mathbb{N}} A_{K,n}\big)$. We have $\|(T(f_{n,\tau}))^\rho \chi_{\Omega_\epsilon}\|_{\infty,*} \leq K$, $n \in \mathbb{N}$, $\tau \in \{-1,1\}^{2^n}$. Let $g \in B(L_\infty)$. There exists a sequence (g_k), $g_k \in \mathrm{co}\{f_{n,\tau}; n \in \mathbb{N}, \tau \in \{-1,1\}^{2^n}\}$, such that $g_k \to g$ in L_1.

By the $\|\cdot\|_1 - \|\cdot\|_{1,*}-$ continuity of T, $\|T(g_k) - T(g)\|_{1,*} \longrightarrow 0$. By 1.9, there is a subsequence, such that $(T(g_{k_j}))^\rho \rightarrow (T(g))^\rho$ pointwise a.a.. Hence

$$(3) \quad \|(T(g))^\rho\|_{\infty,*} \leq K.$$

So we are able to define

$$\tilde{T} : L_\infty \longrightarrow L_\infty^{w^*}(\mu, X^*), \quad \tilde{T}(g) := T(g)^\rho \chi_{\Omega_\epsilon}.$$

From (2) and (3) we obtain $\|\tilde{T}\| \leq K$ and $\|T - \tilde{T}\|_{L(L_\infty, L_1^{w^*}(\mu, X^*))} < \epsilon$. The existence of an operator-valued density of $\tilde{T}$ follows from 2.1. Since $\epsilon > 0$ is arbitrary, we find a strongly measurable density on $\bigcup_{n\in\mathbb{N}} \Omega_{\frac{1}{n}} = \Omega$ up to a set of measure 0. If (Θ, Ψ, ν) is an arbitrary, finite, separable and positive measure space, then we can split Ω is a purely atomic and a purely non atomic part. According to the Carathéodory representation the purely non-atomic part is measure isomorphic to $([0,1], \mathcal{L}, \lambda)$, which is treated before, while is purely atomic part is done by a similar estimation as in (1).

The general case is done according to 2.3 b): Let $L \subset L_\infty(\nu)$ be separable. Then there is a separable sub-$\sigma-$algebra $\Psi_0 \subset \Psi$, so that $L \subset L_\infty(\nu|_{\Psi_0})$. Thus, according to the separable case above, there is a strongly measurable density $U : \Omega \longrightarrow L(L_1(\nu|_{\Psi_0}), X)$. 2.3 b) finishes the proof to the general case.

[8, p.215, Prop.7] tells us that we may assume $\rho[U] = U$ (see 1.14 for a definition). If $\ell_\infty \not\subset X^*$, by a result of H.Rosenthal (see [7, p.156,Th.11]) we get the w^*-measurable operator-valued density $U : \Omega \longrightarrow W(L_\infty, X^*)$. $\square$

Theorem 2.5 *Let $\ell_\infty \not\subset X^*$ and let $T : L_1 \longrightarrow L_1^{w^*}(\mu, X^*)$ be a bounded operator. Let ρ be a lifting of $\mathcal{L}_\infty$. Then $T(B(L_1))$ is uniformly integrable if and only if for all $\epsilon > 0$ there is a w^*-measurable $U_\epsilon : \Omega \longrightarrow L(L_1, X^*)$, such that $\|T - \overline{U_\epsilon}\|_{L(L_1, L_1^{w^*}(\mu, X^*))} < \epsilon$, $\overline{U_\epsilon} : L_1 \longrightarrow L_\infty^{w^*}(\mu, X^*)$ and $\rho(U_\epsilon) = U_\epsilon$.*

Proof: Let $T(B(L_1))$ be uniformly integrable. Let ρ be a lifting of $\mathcal{L}_\infty$.
According to 2.4 there exists a w^*-measurable function $U : \Omega \longrightarrow W(L_\infty, X^*)$, such that $T|_{L_\infty} = \overline{U}$ and $\rho[U] = U$.

Let $\epsilon > 0$ be given. Since T is bounded and $T(B(L_1))$ is uniformly integrable, there exists an $M > 0$, such that

$$\sup_{f \in B(L_1)} \|T(f)^\rho - T(f)^\rho \chi_{\{\|T(f)^\rho\| \leq M\}}\|_{1,*} < \epsilon.$$

Thus for all $A \in \mathcal{L}$ let $B_A := \{\omega \in \Omega; \|T(\frac{\chi_A}{\lambda(A)})^\rho(\omega)\| \leq M\}$.
Let $(\pi_n)_{n \in \mathbb{N}}$ be the sequence of the dyadic partitions of the interval $[0, 1]$. We define the operator:

$$\begin{aligned} T_n \quad : \quad & L_1 \longrightarrow L_\infty^{w^*}(\mu, X^*) \\ & f \longmapsto \sum_{A \in \pi_n} (\int_A f d\lambda) \overline{U}(\frac{\chi_A}{\lambda(A)}) \chi_{B_A} = \sum_{A \in \pi_n} (\int_A f d\lambda) T(\frac{\chi_A}{\lambda(A)}) \chi_{B_A} \end{aligned}$$

and obtain:

$$\|T_n(f)\|_{\infty,*} \leq \sum_{A \in \pi_n} |\int_A f d\lambda| \|T(\frac{\chi_A}{\lambda(A)}) \chi_{B_A}\|_{\infty,*} \leq \|f\|_1 M.$$

Next we consider for $n, m \in \mathbb{N}, n \leq m$ a set $D \in \sigma(\pi_n)$. We have

$$\begin{aligned} \|T(\frac{\chi_D}{\lambda(D)})^\rho - T_m(\frac{\chi_D}{\lambda(D)})^\rho\|_{1,*} \quad &= \\ \|\sum_{A \in \pi_n, A \subset D} (\frac{\lambda(A)}{\lambda(D)} T(\frac{\chi_A}{\lambda(A)})^\rho - \frac{\lambda(A)}{\lambda(D)} T(\frac{\chi_A}{\lambda(A)})^\rho \chi_{B_A})\|_{1,*} \quad &\leq \\ \sum_{A \in \pi_n, A \subset D} \frac{\lambda(A)}{\lambda(D)} \|T(\frac{\chi_A}{\lambda(A)})^\rho - T(\frac{\chi_A}{\lambda(A)})^\rho \chi_{B_A}\|_{1,*} < \epsilon. & \end{aligned}$$

Hence, for all $f \in \mathrm{aco}(\frac{\chi_A}{\lambda(A)}, A \in \pi_n)$ and $m \geq n$ we have $\|(T(f))^\rho - T_m(f)^\rho\|_{1,*} < \epsilon$. Thus we can conclude that for all $m \geq n$ and for each $f \in B(L_1), f = \mathbb{E}_{\pi_n} f$:

$$\|T(f) - T_m(f)\|_{1,*} < \epsilon.$$

Let $f \in B(L_\infty)$. Then for a.a. $\omega \in \Omega$:

(1) $\qquad (T_n(f))^\rho(\omega) \subset X^*$ is relatively weakly compact.

For the proof of (1) we realize first that for $n \in \mathbb{N}$ and $A \in \pi_n$: $|\int\limits_A f d\lambda| \leq \lambda(A)$.

$$
\begin{aligned}
(T_n(f))^\rho(\omega) &\overset{\rho[U]=U}{=} \sum_{A\in\pi_n} (\int\limits_A f d\lambda) U(\omega)(\frac{\chi_A}{\lambda(A)})\chi_{B_A}(\omega) \\
&= \sum_{A\in\pi_n} N_A(f) U(\omega)(\chi_A)\chi_{B_A}(\omega) \\
&= \sum_{A\in\pi_n,\omega\in B_A} N_A(f) U(\omega)(\chi_A) \\
&= U(\omega)\Big(\sum_{A\in\pi_n,\omega\in B_A} N_A(f)\chi_A\Big) \in U(\omega)\big(B(L_\infty)\big) \\
&\quad \text{since } |N_A(f)| \leq 1.
\end{aligned}
$$

Since $U(\omega)\big(B(L_\infty)\big)$ is weakly compact in X^*, by 1.10, we have for all $f \in L_\infty$ that $(T_n(f))_{n\in\mathbb{N}}$ is relatively weakly compact in $L_1^{w^*}(\mu, X^*)$.

Since L_1 is separable, there is a subsequence T_{n_k}, such that $T_{n_k}(f_j) \overset{k\to\infty}{\to} \hat{T}(f_j)$ weakly in $L_1^{w^*}(\mu, X^*)$ for all $j \in \mathbb{N}$, where $\{f_1, f_2, \ldots\} \subset L_\infty$ is a L_1-dense subset. Thus, $\hat{T}$ is linear on L_∞. The uniform estimate for the sequence (T_n) shows that $\hat{T}$ is a $\|\cdot\|_1 - \|\cdot\|_{1,*}$ continuous operator on a L_1-dense, linear subspace of L_∞, since this holds for all T_{n_k} $(k \in \mathbb{N})$.
Hence there is an extension to a linear operator $\hat{T} : L_1 \longrightarrow L_1^{w^*}(\mu, X^*)$, such that $T_{n_k}(f) \overset{k\to\infty}{\to} \hat{T}(f)$ weakly for all $f \in L_1$. Because $\|T_{n_k}(f)\|_{\infty,*} \leq M$ for all $f \in B(L_1)$ and $k \in \mathbb{N}$, it follows that $\|\hat{T}(f)\|_{\infty,*} \leq M$. Furthermore $\|T_{n_k}(f) - T(f)\|_{1,*} < \epsilon$ for all $f \in B(L_1)$, $f = \mathbb{E}_{\pi_n} f, n \in \mathbb{N}$ and $k \in \mathbb{N}$, such that $n_k \geq n$. Thus we have

$$\hat{T} : L_1 \longrightarrow L_\infty^{w^*}(\mu, X^*), \text{ with } \|\hat{T} - T\|_{L(L_1, L_1^{w^*}(\mu, X^*))} < \epsilon.$$

By 2.1 there exists a map $U_\epsilon : \Omega \longrightarrow L(L_1, X^*)$ with $\hat{T} = \overline{U_\epsilon}$. With [8, p.215,Prop.7] we may assume $\rho(U_\epsilon) = U_\epsilon$.

The uniform integrability is provided by the fact that for all $\epsilon > 0$ the image $\overline{U_\epsilon}(B(L_1)) \subset L_1^{w*}(\mu, X^*)$ is $L_\infty-$bounded, hence uniformly integrable. The approximation gives the assertion and finishes the proof of the theorem. $\square$

We state the same results for the Bochner function spaces, where the proofs are omitted, since they run parallely and turn out to be easier.

An operator $T : X \longrightarrow L_p(\mu, Y)$ is called σ_p^*-*compact* if $T(B(X))$ is relatively $\sigma(L_p(\mu, Y), L_{p'}(\mu, Y^*))-$compact.

We note that the $\sigma(L_p(\mu, Y), L_{p'}(\mu, Y^*))-$topology coincides with the weak topology if and only if Y^* has the RNP w.r.t. (Ω, Σ, μ) (see [7, p.98,Th1]).

Theorem 2.6 *Let K be a compact totally disconnected Hausdorff space. Let $1 \leq p < \infty$ and $T : C(K) \longrightarrow L_p(\mu, Y)$ be bounded. We consider the following properties:*

a) There exists a subspace $Y_0 \subset Y, \ell_\infty \not\subset Y_0$, such that $T(C(K)) \subset L_p(\mu, Y_0)$.

b) T is σ^*-compact.

c) T is weakly compact.

Then a) $\Rightarrow$ b) $\Leftrightarrow$ c). If (Ω, Σ, μ) is separable, then a) and b) are equivalent.
In particular: ℓ_∞ embeds in $L_p(\mu, Y)$ if and only if it embeds in Y.

Proof: a) $\Rightarrow$ b) : Since $\ell_\infty \not\subset Y_0$ and $T(C(K)) \subset L_p(\mu, Y_0)$, we can use a result of J.Batt and W.Hiermeyer [1, p.411,2.1 Th.] and a result of H.Rosenthal [7, p.156,Th.10] to conclude, that T is σ_p^*-compact.

b) $\Rightarrow$ c): Suppose T is not weakly compact, then by a result of H.Rosenthal (see [7, p.156,Th.11]) there exists a copy of $\ell_\infty \subset T(C(K))$. According to b), $T(C(K)) \subset L$ where $L := \overline{\text{span}} M$ with M σ_p^*-compact. But this contradicts a result in [21, p.409,3.4 Cor.], which says that L admits an equivalent locally uniformly convex norm. But this fails for ℓ_∞ [5, p.120,Th.1].

c) $\Rightarrow$ b): Trivial.

b) $\Rightarrow$ a): If (Ω, Σ, μ) is separable, there exists a countable set $\mathcal{C} \subset \Sigma$, such that the generated $\sigma-$algebra of $\mathcal{C}$ is dense in Σ and for all $\epsilon > 0$ the set $\{A \in \mathcal{C}; \mu(A) \geq \epsilon\}$ is finite. Let $K := T(B(C(K))$.

Then the uniform integrability of K [1, p.411,Th.2.1.] implies that $\{A \in \mathcal{C}; \sup_{f \in T(B(C(K))} \|I_A(f)\| \geq \epsilon\}$ is finite for all $\epsilon > 0$. For all $A \in \mathcal{C}$ the set $I_A(K)$ is relatively weakly compact in Y [1, p.411,Th2.1]. We define $D := \bigcup_{A \in \mathcal{C}} I_A(K)$. By the above D is relatively weakly compact in Y. Let $Y_0 := \overline{\text{span}} D$. Then Y_0 is weakly compactly generated and hence, does not contain a copy of ℓ_∞. Since the algebra generated by $\mathcal{C}$ is dense in Σ, we have $T(C(K)) \subset L_1(\mu, Y_0)$. □

The technic presented in 2.4 resp. 2.5 can be transmitted to the strongly measurable case almost literally. Hence we get using 2.6 for 2.7:

Theorem 2.7 *Let (Θ, Ψ, ν) be a finite and positive measure space and $T : L_1(\nu) \longrightarrow L_1(\mu, X)$ be bounded. Then*
$T|_{L_\infty}$ has a strongly measurable density $U : \Omega \longrightarrow L(L_\infty, X)$.
If (Θ, Ψ, ν) is separable, then $U : \Omega \longrightarrow W(L_\infty, X)$.

The previous theorem should be compaired with [11, p.314,Th.5.2].

Theorem 2.8 *Let $T : L_1 \longrightarrow L_1(\mu, X)$ be a bounded operator. Then $T(B(L_1))$ is uniformly integrable if and only if for all $\epsilon > 0$, there is a strongly measurable map $U_\epsilon : \Omega \longrightarrow L(L_1, X)$, such that $\|T - \overline{U_\epsilon}\|_{L(L_1, L_1(\mu, X))} < \epsilon$ and $\overline{U_\epsilon} : L_1 \longrightarrow L_\infty(\mu, X)$.*

3 Hereditary properties for w^*-measurable function spaces

If X has the RNP, then by [7, p.66,Th.8] every bounded, linear operator $T : L_1([0,1]) \longrightarrow X$ is Dunford-Pettis, i.e it maps weakly convergent sequences to norm convergent sequences. Thus the following condition is weaker than the RNP.

(3.1) A convex, bounded and closed set $K \subset X$ has the *complete continuity property (CCP)* if every bounded, linear operator $T : L_1([0,1]) \longrightarrow X$, such that $\left\{T\left(\frac{\chi_A}{\lambda(A)}\right); A \in \mathcal{L}\ \lambda(A) > 0\right\} \subset K$, is a Dunford Pettis operator (see [2, 9] for more details).
A Banach space X has the *CCP*, if $B(X)$ enjoys the *CCP* (see [9, p.59;p.78,Def.4.2]).

The following results are an extension of the result of J.Voigt that the strong integral of compact operators is again compact.

Theorem 3.2 *: Let $\ell_1 \not\subset X$, and let $U : \Omega \longrightarrow K(X, Y^*)$ be w^*-measurable and bounded. Then for $1 \le p < \infty$:*

$$\overline{U} : X \longrightarrow L_p^{w^*}(\mu, Y^*)$$

is compact.
In particular

$$\int_\Omega U : X \longrightarrow Y^*, \quad x \longmapsto \int_\Omega U(\omega)(x) d\mu(\omega)$$

is compact.

Proof: Let $(x_n) \subset B(X)$ be a given sequence. Since $\ell_1 \not\subset X$, we assume w.l.o.g that (x_n) is weakly Cauchy according to the Rosenthal dichotomy. Let us first assume that Y is separable. Then according to the introduction, every $f \in \mathcal{L}_p^{w^*}(\mu, Y^*)$ has a measurable norm-function. Hence $\Omega \ni \omega \longmapsto \|U(\omega)(x)\|$ is measurable for all $x \in X$. Since $U(\omega)$ is compact, $(U(\omega)(x_n))$ converges in norm to some $\hat{f}(\omega) \in Y^*$ for all $\omega \in \Omega$. The convergence implies that $\hat{f} : \Omega \longrightarrow Y^*$ is w^*-measurable. Since U is bounded and $\|U(\omega)(x_n) - \hat{f}(\omega)\|^p \to 0$ for all $\omega \in \Omega$, the classical Lebesgue domination theorem gives the assertion.
We consider now the non-separable case of Y and assume that $(\overline{U}(x_n))$ is not norm-Cauchy. Then there is an $\epsilon > 0$ and a subsequence $(\overline{U}(x_{n_k}))$, such that

$$\forall\ k \in \mathbb{N} :\ \|\overline{U}(x_{n_{k+1}}) - \overline{U}(x_{n_k})\|_{p,*} > \epsilon.$$

For all $k \in \mathbb{N}$ there is a $g_k \in \mathcal{E}(\Omega, \Sigma, Y)$, $\|g_k\|_{p'} \le 1$, such that:

$$(1) \quad \forall\ k \in \mathbb{N} :\ |\langle g_k, \overline{U}(x_{n_{k+1}}) - \overline{U}(x_{n_k})\rangle| > \epsilon.$$

Let $Y_0 \subset Y$ be a separable subspace, such that $g_k \in \mathcal{E}(\Omega, \Sigma, Y_0)$ for all $k \in \mathbb{N}$. Then let us consider the embedding $\imath : Y_0 \hookrightarrow Y$ and its adjoint $\imath^* : Y^* \longrightarrow Y_0^*$. Evidently $\imath^* \circ U : \Omega \longrightarrow K(X, Y_0^*)$ is w^*-measurable and the separable case of Y is applicable. But we have according to (1) $\forall\ k \in \mathbb{N}$:

$$\|\overline{\imath^* \circ U}(x_{n_{k+1}}) - \overline{\imath^* \circ U}(x_{n_k})\|_{p,*} \ge |\langle g_k, \overline{U}(x_{n_{k+1}}) - \overline{U}(x_{n_k})\rangle| > \epsilon,$$

which is a contradiction. $\square$

Corollary 3.3: Let $C \subset X$ be conditionally weakly compact and $U : \Omega \longrightarrow L(X, Y^*)$ be bounded and w^*-measurable, such that $U(\omega)(C)$ is relatively compact in Y^* for $\omega \in \Omega$. Then $\overline{U}(C)$ is relatively compact in $L_p^{w^*}(\mu, Y^*)$.

Proof: Let $K := \overline{\text{aco}}C$. Then K is conditionally weakly compact. Let p_K denote the Minkowski-functional. Let $X_0 := \{x \in X; p_k(x) < \infty\}$. Then (X_0, p_k) is a Banach space not containing a copy of ℓ_1 and $C \subset B(X_0)$ [4, p.237,2 (iii)]. Let $id : X_0 \hookrightarrow X$ be the continuous inclusion. We consider $V : \Omega \longrightarrow L(X_0, Y^*)$, $V(\omega) = U(\omega) \circ id$, $\omega \in \Omega$. Evidently V is w^*- measurable and $V(\omega) \in K(X_0, Y^*)$ for all $\omega \in \Omega$. Hence 3.2 implies that $\overline{V} : X_0 \longrightarrow L_p^{w^*}(\mu, Y^*)$ is compact. Since $\overline{U}(C) \subset \overline{U}(id(B(X_0))) = \overline{V}(B(X_0))$, the proof is done. $\square$

Proposition 3.4 *: Let X^* have the CCP. Then every bounded and uniformly integrable operator $T : L_1 \longrightarrow L_1^{w^*}(\mu, X^*)$ is Dunford-Pettis.*

Proof: X^* enjoys the *CCP*, thus $\ell_\infty \not\subset X^*$, since $c_0 \not\subset X^*$. Because $T(B(L_1))$ is uniformly integrable and X^* does not contain ℓ_∞, according to 2.6 for a given $\epsilon > 0$ we find a w^*- measurable function $U : \Omega \longrightarrow L(L_1, X^*)$, so that $\overline{U} : L_1 \longrightarrow L_\infty^{w^*}(\mu, X^*)$ and $\|T - \overline{U}\| < \epsilon$. Since X^* has the *CCP*, we conclude that $U(\omega)(B(L_\infty))$ is compact for all $\omega \in \Omega$. Since $B(L_\infty) \subset L_1$ is weakly compact, thus conditionally weakly compact, 3.3 is applicable. This reveals that $\overline{U}(B(L_\infty))$ is compact, too. Thus $\overline{U}|_{L_\infty}$ is compact, which demonstrates the compactness of T, since $\epsilon > 0$ is arbitrary. $\square$

The strongly measurable case reads as follows and the proof is the same as before just exchanging 3.3 and 2.6 by [23, p.260,Th.1.3] and 2.8, respectively.

Proposition 3.5 *Let X have the* CCP. *Then every bounded and uniformly integrable operator $T : L_1 \longrightarrow L_1(\mu, X)$ is Dunford-Pettis.*

Lemma 3.6 *Let $1 < p < \infty$, let X^* enjoy the CCP and let $T : L_1 \longrightarrow L_p^{w^*}(\mu, X^*)$ be bounded. Let ρ be a lifting of $\mathcal{L}_\infty$. Then*

$$\forall\ \epsilon > 0\ \exists\ \delta > 0\ \forall\ A \in \Sigma, \mu(A) < \delta\ \forall\ g \in B(L_\infty) :$$

$$\| \|T(g)^\rho(\cdot)\|_{X^*} - \|T(g)^\rho(\cdot)\|_{X^*} \chi_{\Omega \setminus A} \|_{p,*} < \epsilon.$$

Proof: Straightforward.

Theorem 3.7 *Let $1 < p < \infty$ and let X^* enjoy the CCP. Then $L_p^{w^*}(\mu, X^*)$ also has the CCP.*

Proof: Let $T : L_1 \longrightarrow L_p^{w^*}(\mu, X^*)$ be a bounded operator. The embedding $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$ is continuous, and uniformly integrable. Thus, for the moment, we consider $T : L_1 \longrightarrow L_1^{w^*}(\mu, X^*)$ to be uniformly integrable. 3.3 reveals that

$$(1) \quad T(B(L_\infty)) \text{ is compact in } L_1^{w^*}(\mu, X^*).$$

We wish to prove that $(T(g_n))$ is relatively compact in $L_p^{w^*}(\mu, X^*)$ for an arbitrary sequence $(g_n)_{n\in\mathbb{N}} \subset B(L_\infty)$. Since $B(L_\infty)$ is weakly compact in L_1, we may assume that (g_n) converges weakly to some $g \in B(L_\infty)$ in L_1. Hence, $(T(g_n))_{n\in\mathbb{N}}$ converges in norm to $T(g)$ in $L_1^{w^*}(\mu, X^*)$, which gives a subsequence converging pointwise a.e. on Ω in X^* by 1.6. For the sake of simplicity, we assume that $T(g_n) \overset{n\to\infty}{\to} T(g)$ a.e. on Ω. According to the extended Egorov theorem, we find for all $\epsilon > 0$ a set $\Omega_\epsilon \subset \Omega, \mu(\Omega \setminus \Omega_\epsilon) < \epsilon$ such that $T(g_n)(\cdot) \overset{n\to\infty}{\to} T(g)(\cdot)$ uniformly on Ω_ϵ in X. With 3.7 this implies that $T(g_n) \overset{n\to\infty}{\to} T(g)$ in $L_p^{w^*}(\mu, X^*)$. Thus the image of $B(L_\infty)$ in $L_p^{w^*}(\mu, X^*)$ is compact. □

Corollary 3.8: Since all ingredients for the proof to 3.7 have its strongly measurable equivalence (see 2.8 and 3.5), we chave an alternative proof of the result obtained in [18]: For $1 \leq p \leq \infty$, $L_p(\mu, X)$ has the *CCP* if and only if X has it and $1 < p < \infty$.

We turn now to the embedding result of G.Pisier's and will show his result [17] by different means, which allows the extension to the w^*-measurable case. For this purpose we show that in principle on bounded sets in $L_p^{w^*}(\mu, X^*), 1 < p < \infty$, the weak topologies of $L_p^{w^*}(\mu, X^*)$ and $L_1^{w^*}(\mu, X^*)$ coincide.

Lemma 3.9 *Let $1 < p < \infty$.*

a) Let $(f_n) \subset L_p(\mu, X)$ be bounded. Then

$$\begin{aligned} &(f_n) \text{ is weakly Cauchy in } L_p(\mu, X) \quad \Leftrightarrow \\ &(f_n) \text{ is weakly Cauchy in } L_1(\mu, X). \end{aligned}$$

b) Let $(f_n) \subset L_p^{w^*}(\mu, X^*)$ *be bounded. Then*

$$\begin{aligned} &(f_n) \text{ is weakly Cauchy in } L_p^{w^*}(\mu, X^*) \quad \Leftrightarrow \\ &(f_n) \text{ is weakly Cauchy in } L_1^{w^*}(\mu, X^*) \end{aligned}$$

Proof: We give the proof only for b), since the proof for a) is even easier. First we note for $1 < p < \infty$ and a sequence of pairwise disjoint elements $(A_n) \subset \Sigma$, such that $\bigcup_{n \in \mathbb{N}} A_n = \Omega$:

$$(1)\ T : L_{p'}(\mu, X) \longrightarrow \left(\oplus L_{p'}(\mu|_{A_n}, X)\right)_{\ell_{p'}},\ f \longmapsto (f\chi_{A_n})$$

is an isometry.

$\Rightarrow$: This implication follows directly from the continuity of the inclusion map $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$

$\Leftarrow$: For $A \in \Sigma$ and $G \in L_p^{w^*}(\mu, X^*)^*$ let

$$G\chi_A : L_p^{w^*}(\mu, X^*) \longrightarrow \mathbb{R}, \quad f \longmapsto \langle f\chi_A, G\rangle.$$

Then $G\chi_A \in L_p^{w^*}(\mu, X^*)^*$. If we use (1) and 1.6 c), we get for a disjoint sequence $(A_n) \subset \Sigma$, $\bigcup_{n \in \mathbb{N}} A_n = \Omega$:

$$(2)\ T^{**} : L_p^{w^*}(\mu, X^*)^* \longrightarrow \left(\oplus L_p^{w^*}(\mu|_{A_n}, X^*)^*\right)_{\ell_{p'}},\ G \longmapsto (G\chi_{A_n})$$

is an isometry. Let $G \in B(L_p^{w^*}(\mu, X^*)^*)$. Our main aim is to prove

$$(3) \begin{cases} \forall\ \epsilon > 0\ \exists\ \Omega_\epsilon \in \Sigma,\ \mu(\Omega \setminus \Omega_\epsilon) < \epsilon \\ G\chi_{\Omega_\epsilon} \in L_1^{w^*}(\mu, X^*)^* \text{ and } \|G\chi_{\Omega\setminus\Omega_\epsilon}\| < \epsilon \text{ in } L_p^{w^*}(\mu, X^*)^*. \end{cases}$$

We will divide the proof to (3) into 3 steps.

Step 1:

$$\exists\ A \in \Sigma, \mu(A) > 0\ \exists\ n \in \mathbb{N}\ \forall\ B \subset A, \mu(B) > 0 :\ \|G\chi_B\| < n\mu(B)^{\frac{1}{p'}}.$$

Proof to step 1: We suppose the contrary is true. Then

$$\forall\ A \in \Sigma, \mu(A) > 0\ \forall\ n \in \mathbb{N}\ \exists\ B \subset A, \mu(B) > 0 : \|G\chi_B\| \geq n\mu(B)^{\frac{1}{p'}}.$$

Then by applying the exhaustion argument [7, p.70,Lemma 4], we can find a sequence of pairwise disjoint elements $(B_j) \subset \Sigma$, $\bigcup_{j \in \mathbb{N}} B_j = \Omega$, such that for all $j \in \mathbb{N}$: $\|G\chi_{B_j}\| \geq n\mu(B_j)^{\frac{1}{p'}}$. Using (2), we can conclude for $n \in \mathbb{N}$:

$$\|G\| \geq n\mu(\Omega)^{\frac{1}{p'}}.$$

Since $n \in \mathbb{N}$ is arbitrary, we get $G \notin L_p^{w^*}(\mu, X^*)^*$, which is a contradiction and step 1 is proved.

Step 2 With the result of step 1 we can find a sequence $(C_n) \subset \Sigma$, $\mu(C_n) > 0$ and an increasing sequence $(M_n) \subset \mathbb{R}_+$, such that

$$\forall\, n \in \mathbb{N}\, \forall\, B \subset C_n, \mu(B) > 0 : \|G\chi_B\| < M_n\mu(B)^{\frac{1}{p'}} \text{ and } \bigcup_{n\in\mathbb{N}} C_n = \Omega.$$

The sequence can be assumed to be increasing. Thus we can define a pairwise disjoint sequence by induction:

$$A_1 := C_1,\ A_n := C_n \setminus \cup_{j=1}^{n-1} A_j.$$

Then, according to (2), we deduce:

$$\lim_{m\to\infty} \|G\chi_{\bigcup_{j=m}^{\infty} A_j}\| = \lim_{m\to\infty} \|\sum_{j=m}^{\infty} G\chi_{A_j}\| = 0.$$

Hence, $\lim_{n\to\infty} \|G\chi_{\Omega\setminus C_n}\| = 0$. Hence, for a given $\epsilon > 0$ there is an $n \in \mathbb{N}$, such that

$$(4)\ \forall\ B \subset C_n, \mu(B) > 0 : \|G\chi_B\| < M_n\mu(B)^{\frac{1}{p'}} \text{ and } \|G\chi_{\Omega\setminus C_n}\| < \epsilon.$$

Step 3 Let $\Omega_\epsilon := C_n$ according to (4). All we have to show is $G\chi_{\Omega_\epsilon} \in L_1^{w^*}(\mu, X^*)^*$.

Proof: Let ρ be a lifting of $\mathcal{L}_\infty$. Let $f \in B(L_1^{w^*}(\mu, X^*))$ be given. For $m \in \mathbb{N}_0, k \in \mathbb{N}$ and $1 \leq i \leq k$ we define

$$A_{m,i} := \{\omega \in \Omega_\epsilon;\ m + \frac{i-1}{k} \leq \|f^\rho(\omega)\| < m + \frac{i}{k}\}.$$

Then we can compute with (4) and the definition of $A_{m,i}$:

$$\begin{aligned} |G\chi_{A_{m,i}}(f)| &\leq M_n(m + \frac{i}{k})\mu(A_{m,i})^{\frac{1}{p}}\mu(A_{m,i})^{\frac{1}{p'}} \\ &= M_n(m + \frac{i}{k})\mu(A_{m,i}) \leq M_n(\|f\chi_{A_{m,i}}\|_{1,*} + \frac{1}{k}\mu(A_{m,i})). \end{aligned}$$

Since $\Omega_\epsilon = \bigcup_{m\in\mathbb{N}_0}\bigcup_{1=1}^{k} A_{m,i}$ and the $A_{m,i}$ are p.d., we obtain:

$$\begin{aligned} |G\chi_{\Omega_\epsilon}(f)| &\leq \sum_{m=0}^{\infty}\sum_{i=1}^{k} M_n(\|f\chi_{A_{m,i}}\|_{1,*} + \frac{1}{k}\mu(A_{m,i})) \\ &\leq M_n(\|f\|_{1,*} + \frac{1}{k}\mu(\Omega_\epsilon)). \end{aligned}$$

So we have $|G\chi_{\Omega_\epsilon}(f)| \leq M_n$ if $k \to \infty$ and hence $G\chi_{\Omega_\epsilon} \in L_1^{w^*}(\mu, X^*)^*$. The final arguments are devoted to the proof of the lemma. For this purpose we take a bounded sequence $(f_n) \subset L_p^{w^*}(\mu, X^*)$, which is weakly Cauchy in $L_1^{w^*}(\mu, X^*)$. Let $G \in B(L_p^{w^*}(\mu, X^*)^*)$ be given and let $\epsilon > 0$. We may assume that $(f_n) \subset B(L_p^{w^*}(\mu, X^*))$. According to (3) there is a set $\Omega_\epsilon \in \Sigma$, such that

$$G\chi_{\Omega_\epsilon} \in L_1^{w^*}(\mu, X^*)^* \text{ and } \|G\chi_{\Omega\setminus\Omega_\epsilon}\| < \frac{\epsilon}{4} \text{ in } L_p^{w^*}(\mu, X^*)^*.$$

Thus, $(G\chi_{\Omega_\epsilon}(f_n))$ is a Cauchy sequence in $\mathbb{R}$ and we find an $n_0 \in \mathbb{N}$, such that for all $n, m \geq n_0 : |G\chi_{\Omega_\epsilon}(f_n - f_m)| < \frac{\epsilon}{2}$. Finally, for $n, m \geq n_0$ we have

$$|G(f_n - f_m)| \leq |G\chi_{\Omega_\epsilon}(f_n - f_m)| + |G\chi_{\Omega\setminus\Omega_\epsilon}(f_n - f_m)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The proof to the lemma is done. □

For concluding the final nad main result on the embedding (3.11) we need the following w^*-analogon to [20, 20,Th.2.3].

Theorem 3.10 *Let $U : \Omega \longrightarrow CW(X, Y^*)$ be w^*-measurable and bounded, where $CW(X, Y^*)$ is the ideal of conditionally weakly compact operators. Let ρ be a lifting of $\mathcal{L}_\infty$, such that $\rho(U) = U$. Then for $1 \leq p < \infty$:*

$$\overline{U} : X \longrightarrow L_p^{w^*}(\mu, Y^*) \text{ is conditionally weakly compact.}$$

In particular,

$$\begin{aligned} \int_\Omega U : \quad X &\longrightarrow Y^* \\ x &\longmapsto \int_\Omega U(\omega)(x)d\mu(\omega) \end{aligned}$$

is conditionally weakly compact.

Proof: We follow the idea of the proof in [20, Th.2.3] and assume that $\overline{U}(B(X))$ is not conditionally weakly compact. Then by Rosenthal's dichotomy there is a sequence $(\overline{f_n}) \subset \overline{U}(B(X))$ which is equivalent to the ℓ_1-basis. Choose $(x_n)_{n\in\mathbb{N}} \subset B(X)$ such that for all $n \in \mathbb{N}$: $\overline{f_n} = \overline{U}(x_n)$. We define

$$\begin{aligned} f_n : \quad \Omega &\longrightarrow Y^*, \\ \omega &\longmapsto U(\omega)(x_n). \end{aligned}$$

Since U is bounded, $f_n \in \mathcal{L}_\infty^{w^*}(\mu, Y^*)$ and $\sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. By the result of M.Talagrand's (see [22, p.720,Th.14]), there are subsets $\Omega_1, \Omega_2 \in \Sigma$ of Ω and for $n \in \mathbb{N}$ there is $g_n \in \mathrm{co}(f_n, f_{n+1}, ...)$, such that

$$\begin{array}{ll} i) & \mu(\Omega_1 \cup \Omega_2) = \mu(\Omega), \\ ii) & \forall \omega \in \Omega_1 \ \exists j \in \mathbb{N} : \ (g_n^\rho(\omega))_{n\geq j} \sim \ell_1 - \text{basis}, \\ iii) & \forall \omega \in \Omega_2 : \ (g_n^\rho(\omega))_{n\in\mathbb{N}} \ \text{ is a weak Cauchy sequence.} \end{array}$$

The assumption concerning U and Rosenthal's dichotomy imply that $\Omega_1 = \emptyset$. Thus (g_n^ρ) is a weak Cauchy sequence (see e.g. [22, p.720,Th.15]). But this contradicts the assumption that $(\overline{f_n})$ is equivalent to the ℓ_1-basis. $\square$

Theorem 3.11 *Let $1 < p < \infty$. Then*

a) $\ell_1 \subset L_p(\mu, X)$ isomorphic $\Leftrightarrow$ $\ell_1 \subset X$ isomorphic.

b) $\ell_1 \subset L_p^{w^*}(\mu, X^*)$ isomorphic $\Leftrightarrow$ $\ell_1 \subset X^*$ isomorphic.

Proof: The proof to a) is omitted, since it runs in a parallel way to b).

$\Rightarrow$: We assume $\ell_1 \not\subset X^*$, but we have an isomorphic embedding $T : \ell_1 \hookrightarrow L_p^{w^*}(\mu, X^*)$. Since $\ell_1 \subset L_1$ complemented, we may extend T to $\hat{T} : L_1 \longrightarrow L_p^{w^*}(\mu, X^*)$. The embedding $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$ is uniformly integrable, thus $\hat{T} : L_1 \longrightarrow L_1^{w^*}(\mu, X^*)$ is uniformly integrable. Since $\ell_1 \not\subset X^*$, we also have $\ell_\infty \not\subset X^*$. Let ρ be a lifting of $\mathcal{L}_\infty$. Then by 2.6 for a given $\epsilon > 0$, there is a w^*-measurable $U : \Omega \longrightarrow L(L_1, X^*)$, such that $\overline{U} : L_1 \longrightarrow L_\infty^{w^*}(\mu, X^*)$, $\|\hat{T} - \overline{U}\| < \epsilon$ and $\rho(U) = U$. If we apply 3.10, we can demonstrate that $\overline{U}$ is conditionally weakly compact, since $\ell_1 \not\subset X^*$. Hence, $\hat{T}$ is also conditionally weakly compact, since the ideal of conditionally weakly compact operators is closed. Finally, we can conclude using 3.9 that T is also weakly conditionally compact, which contradicts the Rosenthal dichotomy.

$\Leftarrow$: This implication is obvious, since X^* embeds isometrically in $L_p^{w^*}(\mu, X^*)$. □

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