Grothendieck operators on tensor products

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Abstract

We prove that for Banach spaces E,F,G,H and operators $T\in\mathcal{L}(E,G)$, $S\in\mathcal{L}(F,H)$ the tensor product $T\otimes S: E\otimes_\epsilon F\longrightarrow G\otimes_\epsilon H$ is a Grothendieck operator, provided T is a Grothendieck operator and S is compact.

1 Introduction

J.Diestel and B.Faires proved in '76 that for Banach spaces E,F,G,H, for $T\in\mathcal{A}(E,G)$ and compact $S\in\mathcal{L}(F,H)$ the tensor product of T and S defined as $T\otimes S: E\otimes_\epsilon F\longrightarrow G\otimes_\epsilon H$ belongs again to the operator ideal $\mathcal{A}$, provided $\mathcal{A}$ is closed and injective [DF]. For the ideal of weakly compact operators E.Saksman and H.O.Tylli [ST] have obtained similar results for both the projective and injective tensor product.

We show that the same tensor stability result holds also for the non-injective closed operator ideal of Grothendieck operators. We also apply this result to obtain that when E is a Schwartz space and F is Banach space with the Grothendieck property, the completed injective tensor product $E\otimes_\epsilon F$ is a Grothendieck space [Fr2], that is, the weak* and the weak sequential convergence coincide on equicontinuous subsets.

Let us first fix some notations and definitions. E,F,G,H are Banach spaces. $B(E)$ stands for the unit ball, while E^* denotes the topological dual. By an operator T from E into F we mean a bounded linear map. Let us call an operator $T\in\mathcal{L}(E,F)$ *approximable*, if there exists a sequence of finite rank operators $(v_n)\subset\mathcal{L}(E,F)$ such that $\|T-v_n\|\overset{n\to\infty}{\to}0$ (cf. [Jh]). We refer to [Pi] for background information on operator ideals and for background on measure theory to [DU].

Definition 1.1. *Let E,F be Banach spaces. An operator $T\in\mathcal{L}(E,F)$ is called a Grothendieck operator, if every w^*-null sequence (y_n^*) is mapped by the adjoint T^* into a weak null sequence $(T^*(y_n^*))\subset E^*$.*

The ideal of Grothendieck operators $\mathcal{GR}(E,F)$ is not injective, since the inclusion map $\imath:c_0\hookrightarrow\ell_\infty$ is Grothendieck (note that ℓ_∞ is a Grothendieck space, since w^*-null sequences are weakly null in the dual of ℓ_∞). But the identity $id:c_0\longrightarrow c_0$ is *not* Grothendieck, since otherwise it would be weakly compact.

If $\dim F=\infty$ and E contains a copy of c_0, then by a well-known result $E\otimes_\epsilon F$ contains a complemented copy of c_0 (cf. [C, Fr1]). Hence, $E\otimes_\epsilon F$ is not a Grothendieck space. Thus, if $c_0\hookrightarrow E$ and $E\otimes_\epsilon F$ is a Grothendieck space, we see that F has to be finite dimensional.

Let us give a final definition.

Definition 1.2. *A subset $K\subset E$ is called a Grothendieck set, if for all $T\in\mathcal{L}(E,c_0)$ the set $T(K)$ is relatively weakly compact in c_0.*

We have immediately the following result (cf.[DU, p.179]).

Lemma 1.3. *Let E,F be Banach spaces, $T\in\mathcal{L}(E,F)$. Then the following conditions are equivalent:*

(a) $T\in\mathcal{GR}(E,F)$,

(b) $\forall\ S\in\mathcal{L}(F,c_0)$: $S\circ T$ *is weakly compact,*

(c) $\forall\ A\subset E$ *bounded* : $T(A)$ *is a Grothendieck set.*

It follows that the ideal $\mathcal{GR}$ is surjective and closed.

2 Main results

Let $\mathcal{A}$ be a closed operator ideal and α be a tensor norm. We define the class $\mathcal{A}_\alpha$ of all operators $S: E \longrightarrow F$ such that for any pair of Banach spaces E_1, F_1 and any operator $T \in \mathcal{A}(E_1, F_1)$ the map $T \otimes S: E_1 \otimes_\alpha E \longrightarrow F_1 \otimes_\alpha F$ belongs to $\mathcal{A}$ as well. J.Diestel and B.Faires (see [DF], Th. 1 and Th. 2) proved that $\mathcal{A}_\alpha$ is always a closed operator ideal which is injective whenever $\mathcal{A}$ and α are injective. Analogously, it is easily seen that if $\mathcal{A}$ is surjective and α is projective, then $\mathcal{A}_\alpha$ is surjective. Thus we obtain immediately:

Proposition 2.1. *Let E, F, G, H be Banach spaces, $\mathcal{A}$ be a closed operator ideal, α be a tensor norm and $T \in \mathcal{A}(E, G)$.*

(a) If $S \in \mathcal{L}(F, H)$ is approximable, then $T \otimes S: E \otimes_\alpha F \longrightarrow G \otimes_\alpha H$ is again in $\mathcal{A}$.

(b) If α and $\mathcal{A}$ are injective, then (a) holds even for compact S.

(c) If α is projective and $\mathcal{A}$ is surjective, then (a) holds even for compact S.

Proof: The ideal of approximable operators is the smallest closed operator ideal. Similarly, the ideal of compact operators is the smallest surjective (injective) closed operator ideal. □

Since $\mathcal{GR}$ is surjective, we can state the following result.

Corollary 2.2. *Let E, F, G, H be Banach spaces. Then $T \otimes S: E \otimes_\pi F \longrightarrow G \otimes_\pi H$ belongs to $\mathcal{GR}$ whenever T is Grothendieck and S is compact.*

Remark 2.3. *If we apply Theorem 2.2 and Remark 2.9 of [S] the following criterion of weak compactness in the dual space $(E \otimes_\pi F)^* = \mathcal{L}(E, F^*)$ can be obtained: Let $(T_n) \subset \mathcal{L}(E, F^*)$ be a bounded sequence. Then $T_n \overset{n\to\infty}{\rightarrow} 0$ weakly if and only if $\{(< T_n(x), y^{**} >)_{n \in \mathbb{N}};\ x \in B(E), y^{**} \in B(F^{**})\} \subset c_0$ is relatively weakly compact. This result or Theorem 1 in [K] can be used to give a direct proof of the above corollary. It also follows from the proof that $E \otimes_\pi F$ is a Grothendieck space if E is a Grothendieck space, F is reflexive and every operator from E into F^* is compact.*

At this stage we mention that from P. Enflo's famous example [E] it is an easy consequence that there is a Banach space E for which there is a non-approximable but compact operator from E into itself. In [A] F.A. Alexander obtained a similar result for a closed subspace E of l^p when $2<p<\infty$.

The ideal of Grothendieck operators is not injective. Thus our main aim is to improve 2.1 in that case and to obtain injective tensor stability with compact operators. First we reduce the problem to reflexive F and H.

Lemma 2.4. *Let E,F,G,H be Banach spaces, $T\in\mathcal{L}(E,G)$ and $S\in\mathcal{L}(F,H)$ is compact. Then there exist reflexive Banach spaces G_1,H_1 and operators $S_1\in\mathcal{L}(E,G_1), S_2\in\mathcal{L}(G_1,H_1), S_2$ compact, $S_3\in\mathcal{L}(H_1,H)$, such that*

$$T\otimes S=(id_G\otimes S_3)\circ(T\otimes S_2)\circ(id_E\otimes S_1).$$

Proof: Every compact $S\in\mathcal{L}(F,H)$ admits a compact factorization through a reflexive Banach space according to a result of T.Figel and W.Johnson [Fi, Jo] (see also [DU, p.260]). Then the proof is straightforward. □

We write $Bo(B(E^*))$ for the Borel sets on $B(E^*)$ w.r.t. the w^*-topology. If $m:Bo(B(X^*))\longrightarrow F$ is a vector measure of bounded variation, then $||m||$ is the variation norm. Let us recall the representation of the dual of $E\otimes_\epsilon F$, provided F is reflexive.

Definition and Lemma 2.5. *Let E,F be Banach spaces with F reflexive. $\mathcal{PI}(E,F)\subset\mathcal{L}(E,F)$, are the Pietsch-integral operators, defined as:*

$$T\in\mathcal{PI}(E,F)\Leftrightarrow$$

$$\exists\ m:Bo(B(E^*))\longrightarrow F \text{ vector measure of bounded variation}$$

$$\forall\ x\in E:\ T(x)=\int_{B(E^*)}x(x^*)dm(x^*).$$

We equip $\mathcal{PI}(E,F)$ with the integral norm, i.e. $||T||_{PI}:=\inf\{||m||;\forall\ x\in E:T(x)=\int_{B(E^*)}x(x^*)dm(x^*)\}$ (cf. [DFl, p.522]). Then $\mathcal{PI}(E,F^*)$ is isometric isomorphic to $(E\otimes_\epsilon F)^*$ by the identity $T(x\otimes y)=\langle y,\int_{B(E^*)}x(x^*)dm(x^*)\rangle$ (cf. [DFl, p.522]).

Notation: Let E and F be Banach spaces with F reflexive. Let $(z_n^*) \subset B((E \otimes_\epsilon F)^*)$. According to 2.5 for all $n \in \mathbb{N}$ we choose a vector measure $m_n := m_n(z_n^*): Bo(B(X^*)) \longrightarrow F$ of bounded variation, satisfying:

i) $\lim_{n\to\infty} \big| \|m_n\| - \|z_n^*\| \big| = 0$,

ii) $\forall\ e \in E,\ f \in F, n \in \mathbb{N}:\ z_n^*(e \otimes f) = \langle f, \int_{B(E^*)} e(e^*) dm_n(e^*) \rangle$.

Furthermore we define a finite scalar-valued measure $\mu(\cdot) := \mu((z_n^*))(\cdot) := \sum_{n\in\mathbb{N}} 2^{-n} var(m_n(z_n^*), \cdot)$, where var denotes the variation of the corresponding measure. Then m_n is absolutely continuous w.r.t. μ for all $n \in \mathbb{N}$.

We write $L_{H_1} := \{f \in L_1(\mu, H_1^*); \forall\ e \in E: \int_{B(E^*)} e(e^*) f(e^*) d\mu(e^*) = 0\}$ for a subspace $H_1 \subset H$. For a Banach space H we denote by $q_H: L_1(\mu, H^*) \longrightarrow L_1(\mu, H^*)/L_H$ the canonical quotient map. If $\mu = \mu(z_n^*)$ and $H_1 \subset H$, then let $r_{H_1}: L_1(\mu, H_1^*)/L_{H_1} \longrightarrow (E \otimes_\epsilon H_1)^*$ be the canonical injection.

Theorem 2.6. *Let E, F, G, H be Banach spaces. If $T \in \mathcal{GR}(E,G)$ and $S \in L(F,H)$ is compact, then $T \otimes S: E \otimes_\epsilon F \longrightarrow G \otimes_\epsilon H$ is Grothendieck.*

Proof: By Lemma 2.4 we assume that F, H are reflexive. W.l.o.g. let $\|T\|, \|S\| \leq 1$. Let $(x_n^*) \subset B((G \otimes_\epsilon H)^*)$ be w^*−converging to 0. First we consider the map $T \otimes id: E \otimes_\epsilon H \longrightarrow G \otimes_\epsilon H$. For a finite dimensional subspace $H_1 \subset H$, according to 2.1, we have that for $(z_n^*) := (T \otimes id)^*(x_n^*)$ the restriction

$$(1) \quad ((z_n^*)|_{E \otimes H_1}) \overset{n\to\infty}{\to} 0 \text{ weakly.}$$

Consider now $id \otimes S: E \otimes_\epsilon F \longrightarrow E \otimes_\epsilon H$. For $n \in \mathbb{N}$ let $h_n \in L_1(\mu, H^*)$ be the density of m_n with respect to $\mu := \mu(z_n^*)$. To show that $(id \otimes S)^*((z_n^*))$ is weakly null (then we are done), we have to show that for all $\overline{g} \in B((E \otimes_\epsilon F)^{**})$:

$$(2) \quad \langle \overline{g}, (id \otimes S)^*(z_n^*) \rangle = \int_{B(E^*)} \langle q_F^* \circ r_F^*(\overline{g}), S^* \circ h_n \rangle d\mu =$$

$$\int_{B(E^*)} \langle S \circ q_F^* \circ r_F^*(\overline{g}), h_n \rangle d\mu \to 0.$$

The following arguments are devoted to prove this. We define $g := S \circ q_F^* \circ r_F^*(\overline{g})$. Then $g \in L_\infty(\mu, H)$, since H is reflexive. Further, g has compact range, since S is compact. We assume that (2) is not true. Then

$$(3) \quad \exists\ (h_{n_k}) \text{ subsequence } \exists\ \epsilon > 0 \ \inf_{k \in \mathbb{N}} \Big| \int_{B(E^*)} \langle g, h_{n_k} \rangle d\mu \Big| > \epsilon.$$

For the sake of simplicity assume that (h_n) satisfies (3). Since g has compact range, there is an increasing sequence of finite $Bo(B(E^*))$-partition (π_k), such that

$$(4) \quad \|\mathbb{E}_{\pi_k}(g) - g\|_\infty \to 0 \text{ and } \forall\ n \in \mathbb{N}:\ \|\mathbb{E}_{\pi_k}(h_n) - h_n\|_1 \to 0.$$

We define $\Sigma_0 := \sigma(\cup_{k \in \mathbb{N}} \pi_k)$. Since H is reflexive, for all $k \in \mathbb{N}$ the sequence $(\mathbb{E}_{\pi_k}(h_n))$ is relatively weakly compact in $L_1(\mu, H^*)$. Hence, for all $k \in \mathbb{N}$ there is an $m_k \in L_1(\mu, H^*)$, so that $\mathbb{E}_{\pi_k}(h_n) \to m_k$ weakly (for at least going to a subsequence by a diagonalization argument). (π_k) is increasing, thus, (m_k, π_k) is a bounded martingale, which converges in the $L_1(\mu, H^*)$-norm to an $M \in L_1(\mu, H^*)$ (note that the (h_n) are bounded and H^* has the RNP as a reflexive space). We show now that for all $G \in L_\infty(\mu|_{\Sigma_0}, H)$ with compact range:

(5) $\exists$ subsequence (h_{n_j}) such that

$$\forall\ \delta > 0\ \exists\ N \in \mathbb{N}\ \forall\ j \geq N:\ |\langle G, M \rangle - \langle G, h_{n_j} \rangle| < \delta.$$

Proof of (5): G has compact range, thus there exists an increasing sequence of finite $Bo(B(E^*), w^*)$–partitions $(\pi_k(G))$ such that $\pi_k \subset \sigma(\pi_k(G))$ for $k \in \mathbb{N}$ and $\|G - \mathbb{E}_{\pi_k(G)}(G)\| \to 0$ (cf. [DU, p.67,Lemma 1]). Let (h_{n_j}) be a subsequence with $\mathbb{E}_{\pi_k(G)}(h_{n_j}) \to m_k(G) \in L_1(\mu, H^*)$ weakly for all $k \in \mathbb{N}$ (subsequence argument like above). Thus, since $\pi_k \subset \sigma(\pi_k(G))$, it follows $\mathbb{E}_{\pi_k}(\mathbb{E}_{\pi_k(G)}(h_{n_j})) \to m_k$ (m_k as above). Again $(m_k(G))$ is a martingale. Hence, as above, there exists an $M(G) \in L_1(\mu, H^*)$, such that $m_k(G) \to M(G)$. We have

$$M = M(G).$$

Let $A \in \bigcup_{k \in \mathbb{N}} \pi_k$. Then $\int_A M(G) - M d\mu = \lim_{k \to \infty} \int_A m_k(G) - m_k d\mu = 0$, since $(m_k(G))$ and (m_k) are martingales and there is a $k_0 \in \mathbb{N}$, such that $A \in \pi_{k_0} \subset \sigma(\pi_k(G))$. Hence, for all $B \in \Sigma_0$: $\int_B M(G)d\mu = \int_B M d\mu$. Thus to prove (5) we first note that it suffices to demonstrate (5) for all $G = \mathbb{E}_{\pi_k(G)}G$, $(k \in \mathbb{N})$, since G has compact range and M, h_n, $n \in \mathbb{N}$, are measurable w.r.t. Σ_0. But then (5) follows by:

$$\langle \mathbb{E}_{\pi_k(G)}G, M\rangle - \langle \mathbb{E}_{\pi_k(G)}G, h_{n_j}\rangle = \langle \mathbb{E}_{\pi_k(G)}G, m_k(G)\rangle -$$

$$\langle \mathbb{E}_{\pi_k(G)}G, \mathbb{E}_{\pi_k(G)}h_{n_j}\rangle = \langle \mathbb{E}_{\pi_k(G)}G, m_k(G) - \mathbb{E}_{\pi_k(G)}h_{n_j}\rangle.$$

For a finite dimensional subspace $H_1 \subset H$ we consider the canonical restriction operator $rest_{H_1} : L_1(\mu, H^*)/L_{H^*} \longrightarrow L_1(\mu, H_1^*)/L_{H_1^*}$. Then according to (1) we have:

$$(6)\quad \forall\ z^{**} \in (E \otimes_\epsilon H_1)^{**} : \int_{B(E^*)} \langle q_H^* \circ rest_{H_1}^* \circ r_{H_1}^*(z^{**}), h_n\rangle d\mu \to 0.$$

Since $q_H^* \circ rest_{H_1}^* \circ r_{H_1}^*(z^{**})$ has compact range for all $z^{**} \in (E \otimes_\epsilon H_1)^{**}$ (H_1 is finite dimensional), (5) and (6) imply:

$$\forall\ z^{**} \in (E \otimes_\epsilon H_1)^{**} :\ \langle z^{**}, r_{H_1} \circ rest_{H_1} \circ q_H(M)\rangle = 0.$$

Note that since $M \in L_1(\mu|_{\Sigma_0}, H^*)$ we may assume that $q_H^* \circ rest_{H_1}^* \circ r_{H_1}^*(z^{**})$ is measurable w.r.t. Σ_0. Thus

$$(7)\quad \forall\ H_1 \subset H \text{ finite dimensional} \qquad r_{H_1} \circ rest_{H_1} \circ q_H(M) = 0.$$

But (7) implies

$$(8) \qquad r_H \circ q_H(M) = 0.$$

Hence we compute

$$\begin{aligned} 0 &\overset{(8)}{=} \langle (id \otimes S)^{**}(\overline{g}), r_H \circ q_H(M)\rangle = \langle \overline{g}, (id \otimes S)^*(r_H \circ q_H(M))\rangle \\ &= \int_{B(E^*)} \langle q_F^* \circ r_F^*(\overline{g}), S^* \circ M\rangle d\mu = \int_{B(E^*)} \langle S \circ q_F^* \circ r_F^*(\overline{g}), M\rangle d\mu \\ &= \int_{B(E^*)} \langle g, M\rangle d\mu. \end{aligned}$$

Thus, this contradicts (3) and (5), and we are done. □

One of the main results in [DL] states that if E is a Fréchet-Montel space and F is a Banach space which is Grothendieck, then the Fréchet space $E\otimes_\epsilon F$ is a Grothendieck space whenever the bidual F^{**} has the approximation property. In the case $F=C(K)$, where K is a compact Hausdorff space, this result was proved by F.J. Freniche [Fr2]. We shall now apply Theorem 2.6 and an operator ideal approach to obtain a closely related result avoiding the assumption of the approximation property.

Corollary 2.7. *Let E be a Schwartz space and F a Banach space with the Grothendieck property. Then $E\otimes_\epsilon F$ is a Grothendieck space.*

Proof: By a well-known representation of ϵ-tensor products as projective limits $E\otimes_\epsilon F = proj_{U\in\mathcal{U}_E} E_U\otimes_\epsilon F$, where $\mathcal{U}_E$ is a 0-basis in E. A locally convex space X is Grothendieck if and only if every continuous linear map from X into c_0 maps bounded sets into reletively weakly compact ones. Now, each continuous linear map $T: E\otimes_\epsilon F \longrightarrow c_0$ factorizes through $E_U\otimes_\epsilon F$ for some $U\in\mathcal{U}_E$. Since E is a Schwartz space we can apply our main theorem so that for every $U\in\mathcal{U}_E$ there exists a $V\in\mathcal{U}_E$ contained in U such that the canonical map $E_V\otimes_\epsilon F\rightarrow E_U\otimes_\epsilon F$ is a Grothendieck operator. The result follows immediately. $\square$

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