Factorization of completely bounded weakly compact operators

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Abstract

We prove that in the setting of operator spaces the result of Davis, Figiel, Johnson and Pelczyński on factoring weakly compact operators holds accordingly. Though not related directly to the main theorem we add a remark on the description of weakly compact subsets in the dual of noncommutative vector valued L_1.

1 Introduction and preliminaries

In 1974 Davis, Figiel, Johnson and Pelczyński [4] proved that a weakly compact operator between two Banach spaces factors through a reflexive Banach space. In this note we adapt this result to the setting of operator spaces (Th. 2). While on the Banach space level we simply repeat the well known construction of [4], we will use some results of Pisier [10, 11] in order to keep trace of the operator space structures. Pisier adapts the complex interpolation method for Banach spaces in a canonical manner to operator spaces by constructing a (canonical) operator space structure on the usual Banach interpolation space.

On the one hand this interpolation method serves to introduce the direct l_p-sum $l_p(X_i)$ in the operator space sense of a family $(X_i)_{i\in I}$ of operator spaces $X_i \subset \mathrm{B}(H_i)$. (As usual, $l_p(X_i)$ stands for the space of families $(e_i)_{i\in I}$ with $e_i \in X_i$ for all i and with $\|(e_i)\| = (\sum \|e_i\|^p)^{1/p} < \infty$ if $1 \leq p < \infty$ and $\|(e_i)\| = \sup \|e_i\|$ if $p = \infty$. For a normed space X we denote by $\mathrm{B}(X)$ the space of linear, bounded operators from X to X.) On $l_\infty(X_i)$ one defines a canonical operator space structure via its embedding in $\mathrm{B}(\bigoplus_2 H_i)$ which can be described by $\mathrm{M}_n(l_\infty(X_i)) = l_\infty(\mathrm{M}_n(X_i))$. On $l_1(X_i)$ one defines an operator space structure via the embedding $l_1(X_i) \subset (l_\infty(X_i^*))^*$ (because $(l_1(X_i))^* = l_\infty(X_i^*)$ as Banach spaces), where the dual of $l_\infty(X_i^*)$ bears, of course, its standard operator space structure in the sense of [1]. Now both $l_1(X_i)$ and $l_\infty(X_i)$ embed continuously in the topological product ΠX_i, and it is possible to consider $l_p(X_i)$ as the complex interpolation space $(l_1(X_i), l_\infty(X_i))_\theta$ with $\theta = 1/p$, which can be endowed with an operator space structure by $\mathrm{M}_n(l_p(X_i)) = (\mathrm{M}_n(l_1(X_i)), \mathrm{M}_n(l_\infty(X_i))_\theta$, see [10] for details.

On the other hand, the complex interpolation method serves, in a similar manner as just described for the direct l_p-sums, to construct "noncommutative vector valued Schatten spaces" $S_p[X]$: If S_p (respectively S_p^n) denote the Schatten classes of operators on the Hilbert space $H = l_2$ (respectively on $H = l_2^n$) and if X is any operator space then the "X-valued S_∞" and the "X-valued S_1" are defined as operator spaces by

$$S_\infty[X] = S_\infty \otimes_{\min} X, \quad S_1[X] = S_1 \hat{\otimes} X; \tag{1}$$

here S_∞ (=compact operators on H) has its natural operator space structure, S_1 is the dual of S_∞^* and inherits its operator space structure from the standard dual operator space structure of S_∞, i.e. from the embedding $S_1 \subset S_\infty^*$, and $\otimes_{\min}$ and $\hat{\otimes}$ denote the minimal tensor product and the projective operator space tensor product (see [2, 6, 7, 8]).

We cite two results of Pisiers' on these constructions. The operator space structure of any operator space Y can be "computed" by

$$\|(y_{i,j})\|_{\mathrm{M}_n(Y)} = \sup\left\{ \|a \cdot (y_{i,j}) \cdot b\|_{S_p^n[Y]} \mid a, b \in \mathcal{B}_{S_{2p}^n} \right\}, \tag{2}$$

for all $1 \leq p < \infty$, see Lemma 1.7 of [11]; where "$\cdot$" denotes the usual matrix product and $\mathcal{B}_X$ is the unit ball of X for a normed space X.

As to the direct l_p-sums we have

$$S_p^n[l_p(X_i)] = l_p(S_p^n[X_i]) \tag{3}$$

for $1 \leq p < \infty$, see end of §2 of [11].

2 Factorization theorem

As usual for operator spaces X, Y we denote by $\mathrm{CB}(X, Y)$ the space of linear and completely bounded operators from X to Y.

Theorem 2.1 *Weakly compact completely bounded operators factor through reflexive operator spaces.*

More precisely, let X, Y be operator spaces, let $T \in \mathrm{CB}(X, Y)$ be weakly compact. Then there exist a reflexive operator space R and operators $T_1 \in \mathrm{CB}(X, R)$, $T_2 \in \mathrm{CB}(R, Y)$, such that $T = T_2 \circ T_1$.

Proof: On the Banach space level of the proof we adopt the well known construction of R i.e. we will define a sequence of equivalent norms on Y, and R will turn out to be the diagonal of the direct l_2-sum of these renormed Y.

For $m, n \in \mathbb{N}$ we define

$$K_n := \overline{T^{(n)}(\mathcal{B}_{M_n(X)})}, \qquad K_{n,m} = 2^m K_n + 2^{-m}\mathcal{B}_{\mathrm{M}_n(Y)}.$$

($T^{(n)}$ stands for the map $\mathrm{M}_n(X) \ni (x_{i,j}) \mapsto (Tx_{i,j}) \in \mathrm{M}_n(Y)$.) On each $\mathrm{M}_n(Y)$ we use the Minkowski functional with respect to $K_{n,m}$ to define new norms $\|(y_{i,j})\|_{n,m} = \inf\{\lambda > 0 |\ (y_{i,j}) \in \lambda K_{n,m}\}$. The new norms are well defined on $\mathrm{M}_n(Y)$, because for arbitrary $y_{i,j} \in Y$ we have $(y_{i,j}) \in \lambda 2^{-m}\mathcal{B}_{\mathrm{M}_n(Y)} \subset \lambda K_{n,m}$ with $\lambda = n^2 2^m \max_{i,j}(\|y_{i,j}\|)$, whence $\|(y_{i,j})\|_{n,m} < \infty$. We set $Y_m = (Y, \|\cdot\|_m)$ and $\mathrm{M}_n(Y_m) = (\mathrm{M}_n(Y), \|\cdot\|_{n,m})$. Then each Y_m is isomorphic to Y as a Banach space because

$$2^{-m}\mathcal{B}_{\mathrm{M}_n(Y)} \subset K_{n,m} \subset (2^m \|T\|_{\mathrm{cb}} + 1)\mathcal{B}_{\mathrm{M}_n(Y)}. \tag{4}$$

The norms $\|\cdot\|_{n,m}$ on $\mathrm{M}_n(Y_m)$ yield an operator space structure on Y_m as can be checked by Ruan's characterisation [12]. (For the sake of completeness here are the details. Let $(y_{i,j}) \in \mathrm{M}_n(Y_m)$ be arbitrary and let $\lambda > 0$ be such that $(y_{i,j}) \in \lambda K_{n,m}$. Then for $a, b \in \mathrm{M}_n$ we have that

$$\begin{aligned} a \cdot (y_{i,j}) \cdot b &\in \lambda(a \cdot K_{n,m} \cdot b) = \lambda(2^m T^{(n)}(a \cdot \mathcal{B}_{\mathrm{M}_n(X)} \cdot b) + 2^{-m} a \cdot \mathcal{B}_{\mathrm{M}_n(Y)} \cdot b) \\ &\subset \lambda \|a\| \ \|b\| \left(2^m T^{(n)} \mathcal{B}_{\mathrm{M}_n(X)} + 2^{-m} \mathcal{B}_{\mathrm{M}_n(Y)}\right) \\ &= \lambda \|a\| \ \|b\| \, K_{n,m} \end{aligned}$$

whence $\|a \cdot (y_{i,j}) \cdot b\|_{n,m} \leq \|a\| \ \|(y_{i,j})\|_{n,m} \ \|b\|$. In order to prove $\|(y_{i,j}) \oplus (z_{r,s})\|_{n+k,m}$ $= \max(\|(y_{i,j})\|_{n,m}, \|(z_{r,s})\|_{k,m})$ for $(y_{i,j}) \in \mathrm{M}_n(Y_m)$, $(z_{r,s}) \in \mathrm{M}_k(Y_m)$ we first note that "$\leq$" is clear because $(y_{i,j}) \in \lambda K_{n,m}$ and $(z_{r,s}) \in \mu K_{k,m}$ imply $(y_{i,j}) \oplus (z_{r,s}) \in \max(\lambda, \mu) K_{n+k,m}$. For the other inequality we define $a \in \mathrm{M}_{n+k}$ to be the diagonal matrix with 1 in the first n entries and 0 in the last k entries of the diagonal, i.e. $a = \mathrm{id}_n \oplus 0 \in \mathrm{M}_n \oplus \mathrm{M}_k$. Since X and Y are operator spaces we have that $a \cdot K_{n+k,m} \cdot a \subset K_{n,m} \oplus 0$. Let $(y_{i,j}) \in \mathrm{M}_n(Y_m)$ and $(z_{r,s}) \in \mathrm{M}_k(Y_m)$. If $\lambda > 0$ is such that $(y_{i,j}) \oplus (z_{r,s}) \in \lambda K_{n+k,m}$ then $(y_{i,j}) \oplus 0 = a \cdot ((y_{i,j}) \oplus (z_{r,s})) \cdot a \in \lambda a \cdot K_{n+k,m} \cdot a$ whence $(y_{i,j}) \in \lambda K_{n,m}$ whence $\|(y_{i,j})\|_{n,m} \leq \|(y_{i,j}) \oplus (z_{r,s})\|_{n+k,m}$; similarly, $\|(z_{r,s})\|_{k,m} \leq \|(y_{i,j}) \oplus (z_{r,s})\|_{n+k,m}$.) Again by (4), each Y_m is even completely isomorphic to Y.

In the sequel we adopt Pisier's construction of l_p-sums, as described above, in order to get the operator space $l_2(Y_m)$. It is well known that the "diagonal"

$$R = \{(y_m) \in l_2(Y_m) |\ y_m = y_n \in Y \ \forall m, n \in \mathbb{N}\} \subset l_2(Y_m)$$

is a reflexive Banach space and that the operators

$$\begin{array}{rcccrcl} T_1: & X & \to & R & , \quad T_2: & R & \to & Y \\ & x & \mapsto & (Tx) & & (y) & \mapsto & y \end{array}$$

are well defined, continuous and satisfy $T = T_2 T_1$.

It remains to show that T_1 and T_2 are completely bounded. First we use Pisier's results mentionned above. We have that

$$\begin{aligned}
\left\|((y_{i,j}^{(m)}))\right\|_{\mathrm{M}_n(l_2(Y_m))} &\overset{(2)}{=} \sup\left\{\left\|a\cdot((y_{i,j}^{(m)}))\cdot b\right\|_{S_2^n[l_2(Y_m)]} \mid a,b\in\mathcal{B}_{S_4^n}\right\} \\
&\overset{(3)}{=} \sup\left\{\left\|a\cdot((y_{i,j}^{(m)}))\cdot b\right\|_{l_2(S_2^n[Y_m])} \mid a,b\in\mathcal{B}_{S_4^n}\right\} \\
&= \sup\left\{\left(\sum_m \left\|a\cdot(y_{i,j}^{(m)})\cdot b\right\|^2_{S_2^n[Y_m]}\right)^{1/2} \mid a,b\in\mathcal{B}_{S_4^n}\right\} \qquad (5) \\
&\leq \left(\sum_m \sup\left\{\left\|a\cdot(y_{i,j}^{(m)})\cdot b\right\|^2_{S_2^n[Y_m]} \mid a,b\in\mathcal{B}_{S_4^n}\right\}\right)^{1/2} \\
&\overset{(2)}{=} \left(\sum_m \left\|(y_{i,j}^{(m)})\right\|^2_{\mathrm{M}_n(Y_m)}\right)^{1/2} \qquad (6) \\
&= \left\|((y_{i,j}^{(m)}))\right\|_{l_2(\mathrm{M}_n(Y_m))}.
\end{aligned}$$

Let $(x_{i,j}) \in \mathcal{B}_{\mathrm{M}_n(X)}$ and $y_{i,j} = Tx_{i,j}$. Then $(y_{i,j}) \in K_n \subset 2^{-m}K_{n,m} = 2^{-m}\mathcal{B}_{\mathrm{M}_n(Y_m)}$ whence

$$\|(y_{i,j})\|_{\mathrm{M}_n(Y_m)} \leq 2^{-m} \qquad (7)$$

for all $m \in \mathbb{N}$. As to our notation, note that for $((y_{i,j}^{(m)})) = ((y_{i,j}^{(m)})_{m\in\mathbb{N}})_{i,j\leq n} \in \mathrm{M}_n(R)$ we have $y_{i,j}^{(m)} = y_{i,j} \in Y$ for all $m \in \mathbb{N}$. It follows that

$$\left\|T_1^{(n)}((x_{i,j}))\right\| = \|((y_{i,j}))\|_{\mathrm{M}_n(R)} \overset{(6)}{\leq} \sum_m \|(y_{i,j})\|^2_{\mathrm{M}_n(Y_m)} \overset{(7)}{\leq} 1,$$

whence $\|T_1\|_{\mathrm{cb}} \leq 1$.

Now we turn to T_2. Here we have

$$\begin{aligned}\|((y_{i,j}))\|_{\mathrm{M}_n(R)} &\overset{(5)}{=} \sup\left\{\left(\sum_m \|a\cdot(y_{i,j})\cdot b\|^2_{S_2^n[Y_m]}\right)^{1/2} \mid a,b\in\mathcal{B}_{S_4^n}\right\} \\ &\geq \sup\left\{\|a\cdot(y_{i,j})\cdot b\|_{S_2^n[Y_1]} \mid a,b\in\mathcal{B}_{S_4^n}\right\} \\ &\overset{(2)}{=} \|(y_{i,j})\|_{\mathrm{M}_n(Y_1)} \\ &\overset{(4)}{\geq} (2\,\|T\|_{\mathrm{cb}}+1)^{-1}\,\|(y_{i,j})\|_{\mathrm{M}_n(Y)}\end{aligned}$$

whence $\|T_2\| \leq (2\,\|T\|_{\mathrm{cb}}+1)$. This ends the proof. ■

Remark: Theorem 2 remains true if we exchange "weakly compact" by "Asplund" (respectively by "conditionally weakly compact") and the reflexive space R by an Asplund space (respectively by a space not containing a copy of l_1). In the proof one only has to use c_0-sums instead of ℓ_2-sums (cf. [3, Th. 5.3.7] respectively [5, p. 237]).

3 A remark on weak compactness in the dual of non-commutative vector-valued L_1-spaces

A collection $K=(K_n)$ of nonvoid sets $K_n \subset \mathrm{M}_n(X)$, X a linear vector space, is called a absolutely matrix convex set on X if it satisfies the following two conditions for all $n,m,r\in\mathbb{N}$:

$$\alpha^* K_r \beta \subset K_n \text{ for all } \alpha,\beta\in\mathrm{M}_{r,n},\ \|\alpha^*\|,\ \|\beta\|\leq 1 \tag{8}$$

$$K_n \oplus K_m \subset K_{n+m} \tag{9}$$

(c.f. [9]). For the following remarks let $K = (K_n)$ be a matrix convex set on a linear vector space X.

Remark 3.1 *(a)* $(\mathcal{B}_{M_n(X)})$ *is an example of an absolutely matrix convex set for every operator space* X.

(b) Let X, Y *be linear vector spaces, let* $T : X \to Y$ *be linear. The image* $L = (T^{(n)}(K_n))$ *of* K *under* T *is a matrix convex set on* Y.

Let $\mathcal{W}$ be a von Neumann algebra, $\mathcal{W}_*$ its predual and let X be an operator space. Analoguously to (1) Pisier [11] defines the continuous noncommutative X-valued L_1 by

$$\mathrm{L}_1(\mathcal{W}, X) := \mathcal{W}_* \hat{\otimes} X.$$

Then we know from [2, Prop. 5.4] that

$$L_1(\mathcal{W}, X)^* = \mathrm{CB}(X, \mathcal{W}) = \mathrm{CB}(\mathcal{W}_*, X^*).$$

Note that for a measure space (Ω, Σ, μ) and a Banach space X the dual of the Bochner space $\mathrm{L}_1(\mu, X)$ can be identified with the space $\mathrm{L}_\infty(\mu, X^*, X)$ of equivalence classes of w^*-measurable and essentially bounded functions $f : \Omega \to X^*$, which in turn is isometrically isomorphic to $\mathrm{B}(X, \mathrm{L}_\infty(\mu))$. Thus $\mathrm{CB}(X, \mathcal{W})$ seems to be a natural candidate for the noncommutative counterpart of $\mathrm{L}_\infty(\mu, X^*, X)$.

Off the category of operator spaces one may also define noncommutative vector valued L_1-spaces only within the category of Banach spaces by the Banach projective tensor product $\otimes_\pi$: For a Banach space X and a von Neumann algebra $\mathcal{W}$

we define the X-valued noncommutative L_1-space by $\mathrm{L}_1(\mathcal{W},X)_{\mathrm{Ban}} = \mathcal{W}_* \otimes_\pi X$. (It then seems natural to define the noncommutative vector valued L_∞-space by $\mathrm{L}_\infty(\mathcal{W},X)_{\mathrm{Ban}} = \mathcal{W} \otimes_\epsilon X$ where $\otimes_\epsilon$ is the Banach injective tensor product and to define the corresponding L_p-spaces by interpolation; as to our knowledge this has not been treated in the literature. In this setting the dual of $\mathrm{L}_1(\mathcal{W},X)_{\mathrm{Ban}}$ is isometrically isomorphic to $\mathrm{B}(X,\mathcal{W}) = \mathrm{B}(\mathcal{W}_*,X^*)$ and seems therefore to be a natural counterpart of $\mathrm{L}_\infty(\mu,X^*,X)$, too.

Up to now the characterization of weakly compact subsets of $\mathrm{L}_\infty(\mu,X^*,X)$ or of $\mathrm{L}_\infty(\mu,X)$, X a Banach space, has not been achieved in a final satisfactory way; here we would like to generalize what so far has been obtained in [13, Th. 2.4] for weakly compact sets in $\mathrm{L}_\infty(\mu,X^*,X)$ to weakly compact sets in $\mathrm{CB}(X,\mathcal{W})$. But since we do not dispose of a counterpart of Ω in the noncommutative setting, we must imitate the ideas of [13, Th. 2.4.] vaguely by taking the set of pure states of $\mathcal{W}$ instead of Ω. Applying 2.1 it is possible to obtain the following result, using the method established in [13], analogously. To exhibit an alternative way we shall not use the factorization theorem, in contrast to [13].

For $v = (v_{i,j}) \in \mathrm{M}_p(V)$, $w = (w_{k,l}) \in \mathrm{M}_q(W)$, V, W vector spaces, we use the notation of [7], where $v \otimes w$ is an element of $\mathrm{M}_{pq}(V \otimes W)$, which can be described as follows: $v \otimes w$ is a $p \times p$-matrix whose (i,j)-th entry is a $q \times q$-matrix whose (k,l)-th entry is $v_{i,j} \otimes w_{k,l}$. Given a pairing $\langle \cdot,\cdot \rangle : V \times W \to \mathbf{C}$we also use the matrix pairing $\langle\langle \cdot,\cdot \rangle\rangle : \mathrm{M}_p(V) \times \mathrm{M}_q(W) \to \mathrm{M}_{pq}$ defined by $\langle\langle v,w \rangle\rangle = (\langle v_{i,j}, w_{k,l}\rangle)_{i,j\le p;k,l\le q}$. For the notation in (10) below note that $\mathrm{M}_n(c_0) = (\mathrm{M}_n \oplus \mathrm{M}_n \oplus \cdots)_{c_0}$ completely isometrically.

Proposition 3.2 *Let $\mathcal{W}$ be a von Neumann algebra and let E_n denote the extreme points of the set of completely positive contractions in* $\mathrm{CB}(\mathcal{W}, \mathrm{M}_n) = \mathrm{M}_n(\mathcal{W}^*)$. *(Thus E_1 is the set of pure states of $\mathcal{W}$.)*

a) Let X be an operator space, and let $(f_n) \subset \mathrm{L}_1(\mathcal{W}, X)^* = \mathrm{CB}(\mathcal{W}_*, X^*)$ be bounded. Then $f_n \to 0$ weakly if and only if there exists a weakly compact absolutely matrix convex set $K = (K_m)$ on c_0 such that

$$(\langle f_n^{(p)}(\omega), x\rangle)_{n\in\mathbb{N}} \in K_{pq} \tag{10}$$

for all $\omega = (\omega_{i,j}) \in E_p$, $x = (x_{k,l}) \in \mathrm{M}_q(\mathcal{B}_X)$.

b) Let X be a Banach space and let $(f_n) \subset \mathrm{L}_1(\mathcal{W}, X)^*_{\mathrm{Ban}} = \mathrm{B}(\mathcal{W}_*, X^*)$ be bounded. Then $f_n \to 0$ weakly if and only if there exists a weakly compact absolutely convex set $K \subset c_0$ such that

$$(\langle f_n(\omega), x\rangle)_{n\in\mathbb{N}} \in K \tag{11}$$

for all pure states ω of $\mathcal{W}$ and all $x \in \mathcal{B}_X$.

Proof: (a) We define a linear bounded operator $T : l_1 \to \mathrm{L}_1(\mathcal{W}, X)^*$ by $Te_n = f_n$ where (e_n) is the canonical basis of l^1. By elementary Banach space theory we know that the set $\{f_n\}$ is relatively weakly compact if and only if T is weakly compact and we know that $f_n \to 0$ weakly if and only if T is weakly compact and the range of $T^*|_{\mathrm{L}_1(\mathcal{W},X)}$ lies in c_0.

For any $\phi \in \mathcal{B}_{\mathcal{W}^*}$ and $x \in \mathcal{B}_X$ we consider $\phi \otimes x$ as a linear functional on $\mathrm{CB}(\mathcal{W}_*, X^*)$ with $\|\phi \otimes x\| \leq 1$. Identifying $\mathrm{L}_1(\mathcal{W}, X)^*$ with $\mathrm{CB}(\mathcal{W}_*, X^*)$ it makes sense to write

$$\langle T^*(\phi \otimes x), e_n \rangle = \langle \phi \otimes x, Te_n \rangle = \langle f_n(\phi), x \rangle, \tag{12}$$

i.e. $T^*(\phi \otimes x) = (\langle f_n(\phi), x \rangle)_{n \in \mathbb{N}}$.

Suppose now that $f_n \to 0$ weakly. Then T^* is weakly compact and c_0-valued. Let K_m be the norm closure of $T^{*(m)}(\mathcal{B}_{\mathrm{M}_m(\mathrm{L}_1(\mathcal{W},X)^{**})})$. Then $K = (K_m) \subset \mathrm{M}_m(c_0)$ is weakly compact and absolutely matrix convex because $(\mathcal{B}_{\mathrm{M}_m(\mathrm{L}_1(\mathcal{W},X)^{**})})$ is absolutely matrix convex. Let $\omega = (\omega_{i,j}) \in E_p$, $x = (x_{k,l}) \in \mathcal{B}_{\mathrm{M}_q(X)}$. Then $\|\omega \otimes x\|_{\mathrm{L}_1(\mathcal{W},X)^{**}} \leq 1$ and by (12)

$$(\langle f_n^{(p)}(\omega), x \rangle)_{n \in \mathbb{N}} = T^{*\,(pq)}(\omega \otimes x) = (T^*(\omega_{i,j} \otimes x_{k,l}))_{i,j \leq p; k,l \leq q} \in K_{pq}.$$

Conversely, suppose there is a weakly compact absolutely matrix convex set $K = (K_m) \subset \mathrm{M}_m(X^*)$ such that (10) holds.

In order to show that $f_n \to 0$ weakly it is enough to show that $T^*(u) \in 4\,K_1$ for each $u \in \mathcal{W}_* \otimes X$ with $\|u\|_{\mathrm{L}_1(\mathcal{W},X)} < 1$ because then the map $T^*|_{\mathrm{L}_1(\mathcal{W},X)}$ takes its values in c_0 and is weakly compact and so is its adjoint map $(T^*|_{\mathrm{L}_1(\mathcal{W},X)})^* = T$.

By definition [7, (3.1)] $\|u\| = \inf\{\|\alpha\|\,\|\phi\|\,\|x\|\,\|\beta\|\}$ where the infimum is taken over all decompositions $u = \alpha \cdot (\phi \otimes x) \cdot \beta$ with $\alpha \in \mathrm{M}_{1,pq}$, $\phi = (\phi_{i,j}) \in \mathrm{M}_p(\mathcal{W}_*)$, $x = (x_{k,l}) \in \mathrm{M}_q(X)$, $\beta \in \mathrm{M}_{pq,1}$. Thus for $\|u\| < 1$ we may choose α, ϕ, x, β such that all have norm < 1. By Wittstock's theorem a normalized element in $\mathrm{CB}(\mathcal{W}, \mathrm{M}_p) = \mathrm{M}_p(\mathcal{W}^*)$ can be written as the linear combination of four completely positive contractions. Since each K_m is w^*-closed in $\mathrm{M}_m(c_0)^{**} = \mathrm{M}_m(l_\infty)$ it follows from (10) and (12) that $T^{*(pq)}(\mathcal{B}_{\mathrm{M}_p(\mathcal{W}_*)} \otimes \mathcal{B}_{\mathrm{M}_q(X)})$ is contained in $(K_{pq} - K_{pq}) + \imath(K_{pq} - K_{pq})$.

The latter set is contained in $4K_{pq}$ because each K_m is absolutely convex. Then

$$T^*(u) = \alpha \cdot T^{*(pq)}((\phi_{i,j} \otimes x_{k,l}))_{i,j\leq p;k,l\leq q} \cdot \beta \in \alpha \cdot 4K_{pq} \cdot \beta \subset 4K_1$$

since (K_m) is absolutely matrix convex on c_0.

(b) The proof works almost like the one of part (a). We define $T : \ell_1 \to \mathrm{L}_1(\mathcal{W}, X)^*_{\mathrm{Ban}}$ by $e_n \mapsto f_n$. Then (12) holds accordingly. Now if $f_n \to 0$ weakly, T^* is weakly compact and we let K be the norm closure of $T^*(\mathcal{B}_{\mathrm{L}_1(\mathcal{W},X)_{\mathrm{Ban}}})$.

For the converse implication suppose a weakly compact absolutely convex set $K \subset c_0$ satisfies (11). Take $u \in \mathcal{W}_* \otimes X$, $\|u\|_{\mathrm{L}_1(\mathcal{W},X)_{\mathrm{Ban}}} \leq 1$. Then for each $\varepsilon > 0$ there are $\lambda_i \geq 0$, $\phi_i \in \mathcal{B}_{\mathcal{W}_*}$, $x_i \in \mathcal{B}_X$, $i = 1, \ldots, k$, such that $u = \sum \lambda_i \, \phi_i \otimes x_i$ and $\sum \lambda_i = 1 + \varepsilon$. By (11) and (12) we conclude $T^*(\frac{u}{1+\varepsilon}) = T^*(\sum \frac{\lambda_i}{1+\varepsilon} \, \phi_i \otimes x_i) \in (K-K)+\imath(K-K) \subset 4\,\mathrm{aco}(K)$ since by the theorem of Krein Smulian the absolutely convex hull of a weakly compact set is again weakly compact. This ends the proof.

■

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