Quotients of Bochner function spaces

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Abstract

We study when c_0 is contained as a quotient in Bochner function spaces $L_p(\mu, X)$.

1 Introduction

The analysis of the position of classical sequence spaces as copies, complemented copies or quotients of the underlying Bochner L_p−spaces has been standing in the center of interest. Important results in this field are due to J. Bourgain, G. Emmanuele, S. Kwapien, J. Mendoza and G. Pisier [B1, B2, B3, E, Kw, Me1, Me2, Pi]. Actually, the c_0-case is well understood [Me3, Theorem 9], except being "isomorphic to a quotient". In fact, this question appears as Problem 4 in that recent survey of J. Mendoza [Me3]. In this paper, we provide a complete answer to this problem, showing the following result.

Theorem 1.1 *Let (Ω, Σ, μ) be a finite measure space, X be a Banach space and $1 < p < \infty$. Then,*

1. *If μ is purely atomic, then $L_p(\mu, X)$ has a quotient isomorphic to c_0 if and only if X has.*
2. *If μ is not purely atomic, then $L_p(\mu, X)$ has a quotient isomorphic to c_0 if and only if X contains a quotient isomorphic to c_0 or X contains a subspace isomorphic to ℓ_1.*

We notice that there is no general relation between the two options given in statement *2*: c_0 has no copy of ℓ_1 and ℓ_∞ has no c_0-quotient. On the other hand, it is known that every infinite dimensional $L_1(\mu)$-space always has a complemented copy of ℓ_1, so the above problem is trivial for $p = 1$.

For the sake of clearness, we are going to comment some of the terminology and the notation used in the paper. We denote by $\mathcal{L}_1(\mu)$ (resp. $\mathcal{L}_\infty(\mu)$)

the seminormed space of integrable (resp. μ-essentially bounded) scalar-valued functions defined on Ω and by $L_1(\mu)$ resp. $L_\infty(\mu)$ the space of the equiva= lence classes. For $1 \leq p < \infty$, we consider the spaces [IT, VI.6]

$$\mathcal{L}_p^{w^*}(\mu, X^*) := \{f : \Omega \longrightarrow X^*;\ f \text{ is } w^*\text{- measurable and } \overline{\int_\Omega} \|f(\omega)\|^p d\mu(\omega) < \infty\},$$

where, as usual, w^*-measurable means that $\omega \in \Omega \longmapsto \langle x, f(\omega)\rangle$ is measurable, for all $x \in X$, and

$$\overline{\int_\Omega} \|f(\omega)\|^p d\mu(\omega) := \inf\{\int_\Omega g(\omega) d\mu(\omega);\ g \in \mathcal{L}_1(\mu),\ \|f(\omega)\|^p \leq g(\omega) \text{ on } \Omega\}.$$

If p' denotes the conjugate of p, then we define the following s= eminorm on $\mathcal{L}_p^{w^*}(\mu, X^*)$,

$$\|f\|_{p,*} := \sup\Big\{\int_\Omega |\langle f(\omega), g(\omega)\rangle| d\mu(\omega);\ g = \sum_{i=1}^n x_i \chi_{A_i},\ x_i \in X,\ A_i \in= \Sigma,\ \|g\|_{p'} \leq 1\Big\}.$$

When $p = \infty$, a similar procedure can be done and we consider the co= rresponding seminormed space,

$$\mathcal{L}_\infty^{w^*}(\mu, X^*) := \big\{f : \Omega \longrightarrow X^*;\ f \text{ is } w^*\text{-measurable and } \|f\|_{\infty,*} < \infty\big\},$$

with $\|f\|_{\infty,*} := \inf\{K > 0;\ \mu(\{\omega \in \Omega; |\langle x, f(\omega)\rangle| > K\}),$ for all $\|x\| \leq 1\}$.

Now, we define the quotients $L_p^{w^*}(\mu, X^*) = \mathcal{L}_p^{w^*}(\mu, X^*)/\mathcal{N}_p(\mu, X^*)$ $(1 \leq p \leq \infty)$, where $\mathcal{N}_p(\mu, X^*) = \{f \in \mathcal{L}_p^{w^*}(\mu, X^*); \langle x, f(\omega)\rangle = 0\ \mu - a.e.,$ for all $x \in X\}$.

According to [Din, p. 269, Theorem 5], for $1 \leq p < \infty$, every $\phi \in L_p(\mu, X)^*$ defines a X^*-valued vector measure of bounded variation with a w^*-measurable density. Thus, the definition of the seminorm on $\mathcal{L}_{p'}^{w^*}(\mu, X^*)$ shows that $L_p(\mu, X)^*$ and $L_{p'}^{w^*}(\mu, X^*)$ are isometrically isomorphic. As one can expect, the natural inclusion of $L_{p'}(\mu, X^*)$ into $L_{p'}^{w^*}(\mu, X^*)$ is an isometry and the canonical inclusion of $L_p^{w^*}(\mu, X^*)$ into $L_1^{w^*}(\mu, X^*)$ is continuous.

It is worth mentioning that, for every equivalence class of $L_p^{w^*}(\mu, X^*)$ $(1 \leq p \leq \infty)$, we can choose a representative f, such that the associated norm function $\|f(\cdot)\|$ is also measurable (use [Din, p. 269,Theorems 5.1 and 5.2]). When X is separable, things are simpler and it is easy to see that each representative of the class has a norm measurable function.

At a certain point, we also have to use for our investigation w^*-measurable operator-valued functions. A function $U : \Omega \longrightarrow L(X, Y^*)$ is w^*-*measurable,* whenever $U(\cdot)(x)$ is w^*-measurable, for all $x\in X$. If, in addition, $U(\cdot)(x)$ belongs to $\mathcal{L}_p(\mu,X^*)$ $(1\leq p\leq\infty)$, then we define

$$\overline{U}:X\longrightarrow L_p^{w^*}(\mu,Y^*),\ x\in X\longmapsto\overline{U}(x):=U(\cdot)(x).$$

Moreover, if $1\leq p\leq\infty$ and $T:X\longrightarrow L_p^{w^*}(\mu,Y^*)$ is a linear map, then we say that $U:\Omega\longrightarrow L(X,Y^*)$ is a w^*-*measurable operator-valued density* of T, if U is w^*-measurable and $U(\cdot)(x)=T(x)$ in $L_p^{w^*}(\mu,Y^*)$, for all $x\in X$.

Now, we explain one more notations. Let ρ be a lifting of $\mathcal{L}_\infty(\mu)$ and $U:\Omega\longrightarrow L(X,Y^*)$ be a bounded map. Then,we write

> $\rho(U)=U$, whenever $\langle y,U(\cdot)(x)\rangle\in\mathcal{L}_\infty(\mu)$ and $\rho(\langle y,U(\cdot)(x)\rangle)(\omega)=\langle y,U(\omega)x\rangle$, for all $\omega\in\Omega,\ x\in X,\ y\in Y$.

The rest of terminology and notation are more standard and we refer the reader to [Di] and [Din].

2 Isomorphic quotients to c_0 in Bochner spaces

Before to deal with the proof of the theorem, we list some auxiliary results which we state separately, since they might be of independent interest. The first two results concerns with w^*-measurable functions. The first one is a w^*-analogon to [S3, Theorem 2.3], whose proof can be found in [S2, Theorem 3.10] (resp. [S1, Theorem 2.1.21]). The proof to the second can be seen in [S2, Theorem 2.5].

Theorem 2.1 *Let $U:\Omega\longrightarrow CW(X,Y^*)$ be a w^*-measurable bounded operator, where $CW(X,Y^*)$ is the ideal of conditionally weakly compact operators.*

Suppose that there exists a lifting ρ of $\mathcal{L}_\infty(\mu)$, such that $\rho(U)=U$.

Then, if $1\leq p<\infty$, $\overline{U}:X\longrightarrow L_p^{w^*}(\mu,Y^*)$ is conditionally weakly compact. In particular,

$$\int_\Omega U:X\longrightarrow Y^*,\quad x\longmapsto\int_\Omega U(\omega)(x)d\mu(\omega)$$

is conditionally weakly compact.

Theorem 2.2 *Assume that X^* has no copy of ℓ_∞ and let $T:L_1([0,1])\longrightarrow L_1^{w^*}(\mu,X^*)$ be a bounded operator and ρ be a lifting of $\mathcal{L}_\infty(\mu)$. Then, the image of the closed unit ball of $L_1([0,1])$ under T is uniformly integrable if and only if, for every $\epsilon>0$, there is a w^*-measurable function $U_\epsilon:\Omega\longrightarrow L(L_1([0,1]),X^*)$, such that*

$$||T-\overline{U_\epsilon}||_{L(L_1([0,1]),L_1^{w^*}(\mu,X^*))}<\epsilon,\ \overline{U_\epsilon}:L_1\longrightarrow L_\infty^{w^*}(\mu,X^*),\ \rho(U_\epsilon)=U_\epsilon.$$

The next result extends a well-known property of Bochner function spaces to the w^*-measurable setting.

Theorem 2.3 *Let (f_n) be a bounded sequence in $L_p^{w^*}(\mu,X^*)$, $1<p<\infty$. Then, (f_n) is weakly Cauchy in $L_p^{w^*}(\mu,X^*)$ if and only if (f_n) is weakly Cauchy in $L_1^{w^*}(\mu,X^*)$.*

Proof: In what follows, if $A\in\Sigma$, we denote by $\Sigma^+(A)$ the collection of all measurable subsets of A with positive measure. On the other hand, given $A\in\Sigma$ and $G\in L_p^{w^*}(\mu,X^*)^*$, we define

$$G\chi_A : L_p^{w^*}(\mu,X^*) \longrightarrow \mathbf{K},\ f\longmapsto \langle f\chi_A, = G\rangle.$$

It is plain that $G\chi_A\in L_p^{w^*}(\mu,X^*)^*$.

The *only if* part of the result is clear. For the converse we shall prove first the following fact (F):

Given $G\in L_p^{w^*}(\mu,X^*)^*$ and $\epsilon>0$, there is $\Omega_\epsilon\in\Sigma$ such that

$$\mu(\Omega\setminus\Omega_\epsilon)<\epsilon, G\chi_{\Omega_\epsilon}\in L_1^{w^*}(\mu,X^*)^*, \text{ and } \|G\chi_{\Omega\setminus\Omega_\epsilon}\|<\epsilon \text{ in } L_p^{w^*}(\mu,X^*)^*.$$

We will divide the proof of this fact in some steps. The first one is the following:

$$(1)\left\{\begin{array}{l}\text{There is } A\in\Sigma^+(\Omega) \text{ and } n\in\mathbb{N}, \text{ such that, for all}\\ \qquad B\in\Sigma^+(A), \text{ we have that } \|G\chi_B\|<n\mu(B)^{1/p'}.\end{array}\right.$$

Suppose the contrary and fix $n\in\mathbb{N}$. Then, for all $A\in\Sigma^+(\Omega)$, there is $B\in\Sigma^+(A)$ with $\|G\chi_B\|\geq n\mu(B)^{1/p'}$. Now, applying an exhaustion argument (see e.g. [DU, p.70, Lemma 4]), we can find a sequence of pairwise disjoint elements $(B_j)\subset\Sigma$, $\bigcup_{j\in\mathbb{N}}B_j=\Omega$, such that for all $j\in\mathbb{N}$, $\|G\chi_{B_j}\|\geq n\mu(B)^{1/p'}$. On the other hand, given a countable Σ-partition (A_n) of Ω

$$T: L_p(\mu,X)\rightarrow \left(\oplus L_p(A_n,X)=\{f|_{A_n};\ f\in L_p(\mu,X)\right)_p,\ f\longmapsto (f\chi_{A_n})$$

is clearly an isometry. Thus, we have that

$$(2)\qquad T^{**}: L_{p'}^{w^*}(\mu,X^*)^* \longrightarrow \left(\oplus\right)L_{p'}^{w^*}(A_n,X^*)^*\big)_p\},\ F\longmapsto (F\chi_{A_n})$$

is also an isometry. Therefore, we must conclude that $\|G\|\geq n\mu(\Omega)^{1/p'}$. Since n was arbitrary, we get a contradiction.

ZZZZZZZZZ I THINK THAT THE NEXT ARGUMENT/PARAGRAPH MUST BE EXPLAINED WITH MORE DETAILS ZZZZZZZZZZZZZZZZZZZZZZZZZZZZZZZZZ

With the result of the former step, we can find a sequence $(C_n)\subset\Sigma^+(\Omega)$, and an increasing sequence $(M_n)\subset\mathbb{R}_+$, such that for all $n\in\mathbb{N}$ and for all $B\in\Sigma^+(C_n)$, we have that $\|G\chi_B\|<M_n\mu(B)^{1/p'}$ and $\bigcup_{n\in\mathbb{N}}C_n=\Omega$. We can achieve this by the following argument:

First observe that for disjoint sets $A,\tilde{A}\in\Sigma^+(\Omega)$ satisfying (1) with $n,\tilde{n}\in\mathbb{N}$ repectively, the union $A\cup\tilde{A}\in\Sigma^+(\Omega)$ fulfills (1) with $\max(n,\tilde{n})$, by estimating with the help of (2). Now we find a set C_1 and an $M_1\in\mathbb{N}$ satisfying (1). According to the above argument the set can be chosen maximal up to measure zero. For an increasing sequence $(M_n)\subset N$, $M_n\uparrow\infty$ we find by the above observation a sequence of maximal sets (C_n) each satisfying (1) with M_n accordingly. Since the sets are maximal we have $\bigcup_{n\in\mathbb{N}}C_n=\Omega$.

The sequence is increasing because of the maximality of the sets. Thus, we can define a pairwise disjoint sequence by induction

$$A_1:=C_1,\ A_n:=C_n\setminus\cup_{j=1}^{n-1}A_j.$$

Then, using the representation from T^{**}, we deduce that

$$\lim_{m\to\infty}\|G\chi_{\bigcup_{j=m}^{\infty}A_j}\|=\lim_{m\to\infty}\|\sum_{j=m}^{\infty}G\chi_{A_j}\|=0.$$

Hence, $\lim_{n\to\infty}\|G\chi_{\Omega\setminus C_n}\|=0$. Thus, for a given $\epsilon>0$, there is $n\in\mathbb{N}$, such that

$$(3)\text{ for all } B\in\Sigma^+(C_n),\|G\chi_B\|<M_n\mu(B)^{1/p'}\text{ and }\|G\chi_{\Omega\setminus C_n}\|<\epsilon.$$

We abbord the proof of the fact (F). Given $\epsilon>0$, we take the corresponding $\Omega_\epsilon=C_n$ (for some $n\in\mathbb{N}$) shown in (3). Obviously, we only need to show that $G\chi_{\Omega_\epsilon}\in L_1^{w^*}(\mu,X^*)^*$. Then, take an (representative) element f of the closed unit ball of $L_1^{w^*}(\mu,X^*)$ such that $\|f(\cdot)\|$ is measurable. For $m\in\mathbb{N}_0$, $k\in\mathbb{N}$ and $1\leq i\leq k$, we define

$$A_{m,i}:=\{\omega\in\Omega_\epsilon;\ m+\frac{i-1}{k}\leq\|f(\omega)\|<m+\frac{i}{k}\}\in\Sigma.$$

Then, we can compute

$$\begin{aligned}|G\chi_{A_{m,i}}(f)| &\leq M_n(m+\frac{i}{k})\|\chi_{A_{m,i}}\|_p\|\chi_{A_{m,i}}\|_{p'}\\ &= M_n(m+\frac{i}{k})\mu(A_{m,i})\leq M_n(\|f\chi_{A_{m,i}}\|_{1,*}+\frac{1}{k}\mu(A_{m,i})).\end{aligned}$$

Since $\Omega_\epsilon=\bigcup_{m\in\mathbb{N}_0}\bigcup_{i=1}^k A_{m,i}$ and the $A_{m,i}$ are pairwise disjoint, we obtain

$$\begin{aligned}|G\chi_{\Omega_\epsilon}(f)| &\leq \sum_{m=0}^{\infty}\sum_{i==1}^{k}M_n(\|f\chi_{A_{m,i}}\|_{1,*}+\frac{1}{k}\mu(A_{m,i}))\\ &\leq M_n(\|f\|_{1,*}+\frac{1}{k}\mu(\Omega_\epsilon)).\end{aligned}$$

Taking limit in k, we have $|G\chi_{\Omega_\epsilon}(f)|\leq M_n$ and, hence, $G\chi_{\Omega_\epsilon}\in L_1^{w^*}(\mu,X^*)^*$.

Now, we come back to the proof of the *if* implication. For this purpose, take a bounded sequence (f_n) in $L_p^{w^*}(\mu,X^*)$ which is weakly Cauchy in $L_1^{w^*}(\mu,X^*)$. Suppose that G belongs to the closed unit ball of $L_p^{w^*}(\mu,X^*)^*$ and take $\epsilon>0$. We may assume that $\|f_n\|_{p,*}\leq 1$. According to the fact (F), there is a set $\Omega_\epsilon\in\Sigma$, such that

$$G\chi_{\Omega_\epsilon}\in L_1^{w^*}(\mu,X^*)^* \text{ and } \|G\chi_{\Omega\setminus\Omega_\epsilon}\|<\frac{\epsilon}{4} \text{ in } L_p^{w^*}(\mu,X^*)^*.$$

Thus, $(G\chi_{\Omega_\epsilon}(f_n))$ is a Cauchy sequence in the scalar field and we find $n_0\in\mathbb{N}$, such that for all $n,m\geq n_0:\||G\chi_{\Omega_\epsilon}(f_n-f_m)|<\frac{\epsilon}{2}$. Finally, for $n,m\geq n_0$ we have

$$|G(f_n-f_m)|\leq|G\chi_{\Omega_\epsilon}(f_n-f_m)|+|G\chi_{\Omega\setminus\Omega_\epsilon}(f_n-f_m)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

□

Our next result is perphaps known, but we have not been able to find an explicit reference.

Lemma 2.4 *Let X be a Banach space. Then X^* contains a subspace ismorphic to ℓ_1 if and only if X contains a subspace isomorphic to ℓ_1 or X has a quotient isomorphic to c_0.*

Proof ($\Leftarrow$) One case is elementary and, for the other, apply [Di, Ch. XI, Exercise 2].
($\Rightarrow$) Assume that X has no c_0-quotient and that X has no copy of ℓ_1. Let (x_n^*) be a bounded sequence in X^*. By [SCH, p.396, Th.1] (see also [B4]), we know that som= e disjoint convex combinations (y_n^*) of (x_n^*) are weak*-convergent. Now, applying [D1, Lemma] and Rosenthal's ℓ_1-Theorem, we can extract from (y_n^*) a weakly Cauchy subsequence. Therefore, we have shown that there is a regular method of summability, which produces a weakly Cauchy subsequence, when it is applied to (x_n^*). According to [D2, Theorem 2], we deduce that X^* has no copy of ℓ_1 and we have a contradiction. □

We finally pass to give the proof of Theorem 1.1.
Proof of Theorem 1.1: (1) This case follows from [D1, Lemma], having in mind the following well-known two facts: X is isomorphic to a complemented subspace of $L_p(\mu,X)$ and that, if (x_n^*) is a sequence in $\ell_{p'}(X^*)\equiv\ell_p(X)^*$ equivalent to the unit basis of ℓ_1, then there is $k\in\mathbb{N}$ such that $(x_n^*(k))_n\subset X^*$ is also equivalent to the unit basis of ℓ_1.
(2)($\Leftarrow$) We may assume that (Ω,Σ,μ) is not purely atomic. Therefore, $L_1(\mu)$ does not satisfy the Schur property and we can obtain a weakly null sequence $(\phi_k)\subset L_1(\mu)$ such that $\|\phi_k\|_1>\delta>0$, for all $k\in\mathbb{N}$. In fact, using [C, Lemma 4], we can find a subsequence $(\phi_{k_n})_n$ and a uniformly bounded, weakly null sequence $(\varphi_n)_n$ in $L_1(\mu)$ such that $\|\phi_{k_n}-\varphi_n\|\leq\delta/2$, for all $n\in\mathbb{N}$.

Now, let $(x_n^*)\subset X^*$ be a sequence equivalent to the unit ℓ_1-basis and define $f_n:=\varphi_n\otimes x_n^*\in L_\infty(\mu,X^*)\subset L_{p'}(\mu,X^*)$. Using that there are real positive constans α and β such that $\alpha \leq \|\varphi_n\|_1 \leq \beta$, for all $n \in \mathbb{N}$, it is easy to see that (f_n) is equivalent to the unit ℓ_1-basis in $L_1(\mu, X^*)$. Having in mind Rosenthal's ℓ_1-theor= em and the canonical inclusion of $L_{p'}(\mu, X^*)$ into $L_1(\mu, X^*)$, we deduce that (f_n) has a subsequence, which we still denote the same, equivalent to the unit basis of ℓ_1 in $L_{p'}(\mu, X^*)$ and, therefore, in $L_p(\mu, X)^*$ [DU, p. 97].

Since (φ_n) is weakly null in $L_1(\mu)$ and bounded in $L_\infty(\mu)$, it is clear that $(\int_\Omega \varphi_n \xi d\mu)_n$ is a null sequence, for every Σ-simple X-valued function ξ. In fact, a density argument shows that this also happens with every function in $L_1(\mu, X)$.

Now, take $g \in L_p(\mu, X)$ and compute

$$
\begin{aligned}
|\langle g, f_n \rangle| &= |\int_\Omega \varphi_n(\omega) \langle g(\omega), x_n^* \rangle d\mu(\omega)| \\
&= |\langle \int_\Omega \varphi_n(\omega) g(\omega) d\mu(\omega), x_n^* \rangle| \leq \sup_{k \in \mathbb{N}} \|x_k^*\| \|\int_\Omega \varphi_n(\omega) g(\omega) d\mu(\omega)\| \longrightarrow 0,
\end{aligned}
$$

since (x_n^*) is bounded.

To sum up, we have found a weak*-null sequence in $L_p(\mu, X)^*$ equivalent to the unit ℓ_1-basis. To end the proof, apply [D1, Lemma].

(2)($\Rightarrow$) According to Lemma 2.4, it is enough to show that $L_p(\mu, X)^*$ has a copy of ℓ_1. Using again Lemma 2.4, assume that X^* has no ℓ_1-subspace but we have an isomorphic embedding $T : \ell_1 \hookrightarrow L_p(\mu, X)^*$. The canonical inclusion $\imath : L_p^{w^*}(\mu, X^*) \hookrightarrow L_1^{w^*}(\mu, X^*)$ is uniformly integrable, thus, the extension $\hat{T} : L_1[0,1] \longrightarrow L_1^{w^*}(\mu, X^*)$ of T is also uniformly integrable. Since X^* has no copy of ℓ_1, we also have that X^* has no copy of ℓ_∞.

Now, let ρ be a lifting of $\mathcal{L}_\infty$. Then, by Theorem 2.2, given $\epsilon > 0$, there is a w^*-measurable $U : \Omega \longrightarrow L(L_1, X^*)$, such that

$$\overline{U} : L_1[0,1] \longrightarrow L_\infty^{w^*}(\mu, X^*), \ \|\hat{T} - \overline{U}\| < \epsilon \text{ and } \rho(U) = U.$$

Since X^* has no copy of ℓ_1, then $L(L_1, X^*) = CW(L_1, X^*)$. Therefore, we can apply Theorem 2.1 to U and ρ and we obtain that $\overline{U}$ is conditionally weakly compact. Hence, $\hat{T}$ is also conditionally weakly compact, since the ideal of conditionally weakly compact operators is closed. Finally, we can conclude using Theorem 2.3 that T is also weakly conditionally compact, which contradicts the Rosenthal dichotomy. $\square$

Actually we can not do better in Theorem 1.1.(2.) according to the next simple example.

Example 2.5 *a) Let X be a non reflexive Grothendieck space. Then, as a well-known fact, c_0 is not a quotient. Since X^* is not reflexive but weakly sequentially complete, the Rosenthal ℓ_1-dichotomy prevails in conjunction with 1.1 that c_0 is a quotient of all $L_p(\mu, X)$ for a not purely atomic, finite and positive measure space (Ω, Σ, μ) and $1 < p < \infty$.*

a) If there is (we do not know) a Banach space X, such that $c_0 \not\subset X$, c_0 not a quotient of X and $\ell_1 \subset X^*$ (e.g. a non reflexive Grothendieck space not containing c_0), then we have the remarkable example of a Banach space X, such that

i) $c_0 \not\subset X$,

ii) $c_0 \not\subset X$ *as a quotient,*

iii) $\ell_1 \subset X^*$ *and*

iv) $c_0 \subset L_p(\mu, X)$ *as a quotient for all not purely atomic, finite and positive measure spaces* (Ω, Σ, μ) *and* $1 < p < \infty$.

Remark 2.6 *Let us mention that the main result can be transferred into a more general setting: Let E be an order continuous Banach lattice with a weak unit, or equivalently, an ideal function having an order isometric representation $L_\infty(\mu) \hookrightarrow E \hookrightarrow L_1(\mu)$, with (Ω, Σ, μ) a probability space (cf. [LT, p.25,Th.1.b.14]). Then:*

1. *If μ is purely atomic, $E(X)$ has a quotient isomorphic to c_0 if and only if E or X does.*
2. *If μ is not purely atomic, $E(X)$ has a quotient isomorphic to c_0 if and only if E does or X^* contains a subspace isomorphic to ℓ_1.*

We do not stress the details of the proofs, but we mention some basic ingredients neccessary for the proofs, since they run similar to the L_p-case.

Fact 1: E is reflexive or it contains a quotient isomorphic to c_0.

Fact 2: There exists a continuous inclusion of $E(X)^*$ into $L_1^{w^*}(\mu, X^*)$.

We can conclude

- *for the purely atomic case of the Banach lattice E:*

  c_0 is a quotient of $E(X)$ if and only if c_0 is a quotient of E or of X.

- *for the atomless case of the Banach lattice E:*
  - *If X^* has a subspace isomorphic to ℓ_1, then c_0 is isomorphic to a quotient of $E(X)$.*
  - *Let E be reflexive. Then $E(X)^*$ contains a copy of ℓ_1 if and only if X^* does.*

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