INTRODUCTION TO BOHMIAN MECHANICS
SUMMER TERM 2016

Exercise Sheet 5

Exercise 1: ***Spreading of the Wave Packet***
A free wave packet is a superposition of plane waves with some weight function:

$$\psi(x,t) = \int f(k)\, \mathrm{e}^{\mathrm{i}(kx - \hbar k^2 t/2m)}\, \mathrm{d}k.$$

Show that this wave function solves the free Schrödinger equation.

Assume that at time $t = 0$, the wave function was given by a Gaussian:

$$\psi(x, t=0) = N\mathrm{e}^{-x^2/(2\sigma)}.$$

Compute N and discuss the meaning of both N and σ. What are the units of ψ, N and σ?

For this wave function, what is $f(k)$? What is the group velocity of the wave packet for large σ?

Now, explicitly *compute* $\psi(x,t)$ for $t > 0$. *Compare* your result to $\psi(x, t = 0)$ and discuss.

Compute the trajectory of a Bohmian particle guided by ψ with initial position $X_1(t = 0) = 0$, and that of a Bohmian particle initially at some position $X_2(t = 0) = a > 0$.

Exercise 2: $SO(3)$ ***and*** $SU(2)$
Consider the Euclidean scalar product on $\mathbb{R}^3$. $SO(3)$ is defined as the group of transformations which leave the scalar product invariant, *i.e.* $SO(3) = \left\{\sigma : \mathbb{R}^3 \to \mathbb{R}^3 : \langle x, y\rangle = \langle \sigma x, \sigma y\rangle \forall x, y \in \mathbb{R}^3\right\}$.

a) Write $\sigma \in SO(3)$ as a matrix. What conditions do the column (or row) vectors fulfill? How many free entries remain?
b) Why is the number of free entries consistent with the interpretation of these matrices as rotations?
c) Why does any $\sigma \in SO(3)$ have at least on real eigenvalue?
d) Represent $SO(3)$ as the manifold "ball with radius π" as in the lecture. Why is it not simply connected? Why is the doubly closed path null homotopic?

$SU(2)$ is the set of unitary maps on $\mathbb{C}^2$ with unit derminant.

a) What does a generic matrix $\tau \in SU(2)$ look like?
b) How can you represent $SU(2)$ as a simply connected manifold?
c) As was shown in the lecture, $SU(2)$ is a cover of $SO(3)$. Which rotation corresponds to

$$\begin{pmatrix} \cos\alpha & \mathrm{i}\sin\alpha \\ \mathrm{i}\sin\alpha & \cos\alpha \end{pmatrix}?$$

Hint: Recall the representation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \widehat{=} \begin{pmatrix} z & x - \mathrm{i}y \\ x + \mathrm{i}y & -z \end{pmatrix}.$$

Exercise 3: ***Ground States***
Show that the ground state ψ (with energy e) of a self-adjoint Hamiltonian H can always be chosen as a real function. Proceed along the following steps. You may assume that the ground state wave function is normalised to 1 and the Hamiltonian is of the form $H = -\hbar^2/(2m)\Delta + V(x)$, where $x \in \mathbb{R}^d$ for some $d > 0$.

(1) Remind yourself with a simple computation that eigenvalues of self-adjoint operators are always real. The ground state is an eigenfunction of the Hamiltonian, so its energy e (which is just the eigenvalue) is real.
(2) The most general form for $\psi(x)$ is given by $\psi(x) = R(x)\,\mathrm{e}^{\mathrm{i}S(x)}$ with $R \geq 0$ and $S \in \mathbb{R}$. Why?
(3) Explicitly compute $e = \langle\psi, H\psi\rangle$. Keep in mind that $e \in \mathbb{R}$ and that e is the ground-state energy, *i.e.* for any $\varphi \neq \psi$ it holds that $\langle\varphi, H\varphi\rangle \geq e$. This means that you can freely think about ways to minimise the result of $\langle\psi, H\psi\rangle$.