INTRODUCTION TO BOHMIAN MECHANICS
SUMMER TERM 2016

Exercise Sheet 4

Exercise 1: ***Law of Large Numbers***
Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that

$$\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 f\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \mathrm{d}x_1 \cdots \mathrm{d}x_n = f\left(\frac{1}{2}\right).$$

Hint: Use a trick similar to the one used in Exercise 1 of Sheet 3 and the lectures to first prove for the Lebesgue measure λ

$$\lambda\left(\left\{x \in [0,1]^n : \left|\frac{1}{n}\sum_{i=1}^n x_i - \frac{1}{2}\right| > \varepsilon\right\}\right) \leq \frac{1}{12n\varepsilon^2},$$

and remember that f is continuous.

Exercise 2: ***Typicality – A Story by Schrödinger***
James was a prisoner. A prisoner in a peculiar prison, indeed. He had been sentenced to death and stood before the executioner. However, instead of just ending this story very unromantically, James was given a chance to live. He was offered to play the following guessing game:
Out of an urn filled with 91 counters, he would draw one blindly. The counters were made as follows: 81 of them had a black front face and a blue back face. 9 were blue on one and red on the other side. The last counter was red on one side and white on the other. Apart from the colours, the counters were identical. The counter James would draw would be tossed onto a table, so that only one side could be seen. James would then be allowed to open his eyes and look at the counter. He would have the option to guess what the other side of the counter showed or to deny the guess.

(1) If he were to guess correctly, his life would be spared and he would be freed from the prison.
(2) If he guessed wrong, he would have been executed on the spot.
(3) If he had denied to guess, he would not have been executed, but instead would have to remain in prison for the remainder of his life.

After some thought, James closed his eyes, shuffled the counters, picked one and flipped it. As it fell on the table and came to a rest, he opened his eyes and saw a red counter.
Even before James could say anything, the executioner remarked with a grin on his face: "Think first!"
So James thought. He wanted to say that the backside of the counter was blue, of course, since there were 9 counters with a blue and a red side and only one with a red and a white side, so the probability of this one being blue was $9/10$. But it was just $50-50$ that the counter landed with the red side up, and if it hadn't, James would have had to answer black, the ratio of possible counters then being $81/90 = 9/10$. So in truth, he thought, this was not at all a safe bet of 90%. And thus he declined to guess and stayed in prison forever.
What would you do and why? What does this tell us about the fluctuation hypothesis which claims that what we see in the universe is only a statistical fluctuation?
Schrödinger ended the story with the words "Never be afraid of dangers that have gone by! It's those ahead that matter."

Exercise 3: ***Phase-Space Volume***
Consider N identical particles making up an ideal gas in a volume V. Partition phase space into identically large cells α_i with $\lambda(\alpha_i) = \alpha$ for all i.

a) Compute the phase-space volume of microstates realising the macrostate where N_i particles are in α_i.
b) For this gas, what is the ratio of phase-space volumina corresponding to macrostates
   - where all particles are in only one half of the volume and
   - where the particles are uniformly distributed among all cells?

Exercise 4: ***Poincaré Recurrence Theorem***
The Poincaré recurrence theorem states that the set of points in phase space which, following the dynamics, do *not* return to a neighbourhood of themselves, is a set of measure 0. In somewhat more precise language:
Consider a phase space Ω, an invertible flow Φ (think of the Hamiltonian flow, but as a time-one map: $\Phi = \Phi_{t=1}$), and a stationary measure (with respect to Φ) $\mathbb{P}$ on phase space under which $\mathbb{P}(\Omega) < \infty$. For this system $(\Omega, \Phi, \mathbb{P})$, the following holds:
Take a nonempty set $M \subset \Omega$. Then the set $N \subset M$ of points which never return to M has zero $\mathbb{P}$-measure: $\mathbb{P}(N) = 0$.
Hint: That the set N never returns to M means that $\Phi^n(N) \cap M = \emptyset$ for all $n \geq 1$. You may just assume that all sets are $\mathbb{P}$-measurable. Use stationarity and the finite measure of Ω to show that

$$\sum_{n=0}^{\infty} \mathbb{P}(N) < \infty.$$

What does this mean for statistical physics, for instance for a box whose left half is, at time $t = 0$, filled with particles of an ideal gas while its right half is empty?

Exercise 5: ***Self Study***
Read the following parts in "Bohmian Mechanics" or refresh your memory from some other appropriate source:

(1) The symmetries of the Schrödinger equation, pages 131-133 (chapter 7).
(2) Wave mechanics, group velocity, pages 124-127 (chapter 6).