INTRODUCTION TO BOHMIAN MECHANICS
SUMMER TERM 2016

Exercise Sheet 2

Exercise 1: *Lagrangian Formalism*
This exercise deals with the Lagrangian formalism of classical mechanics.
The Lagrange function is a function of coordinates and velocities (thought of as velocities of trajectories passing through those coordinates): $\mathcal{L} = \mathcal{L}(q, \dot{q})$, where $q = (q_1, \ldots, q_N)$ is an element of the configuration space $\mathbb{R}^N$.

a) Integrating the Lagrangian along any curve $\gamma : [t_0, t] \to \mathbb{R}^N$ gives the action of that path:
$$S[\gamma] = \int_{t_0}^{t} \mathcal{L}(\gamma(s)\,, \dot{\gamma}(s))\, \mathrm{d}s.$$
Show, without full rigor, that extremising the action for fixed endpoints of the trajectories gives an equation of motion for the N particles.

b) With the simplified Lagrangian of the one-dimensional harmonic oscillator $\mathcal{L}(q, \dot{q}) = \dot{q}^2/2 - q^2/2$, compute and discuss the actions of the following paths $[0, 2\pi) \to \mathbb{R}$:
- $\gamma_0(t) = 0$
- $\gamma_1(t) = \sin(t)$
- $\gamma_2(t) = \sin(2t)$
- $\gamma_3(t) = t$
- $\gamma_4(t) = \mathrm{e}^t$

Exercise 2: *Phase Space Part II*
We revisit the phase space and study its Hamiltonian structure.

a) Why is the three-particle elastic collision irresolvable? *("Physics comes to an end" does not count as an answer.)*

b) Show that any (positive, scalar) function of the Hamiltonian is a valid choice for a stationary phase-space density.

c) For a positive energy $E > 0$ in the simplified Lagrangian of the one-dimensional harmonic oscillator $L = \dot{q}^2/2 - q^2/2$, compute the Lebesgue measure of the phase-space set $\big\{(q, p) \in \mathbb{R}^2 | H(q, p) \leq E\big\}$, where H is the Hamiltonian function, given as the Legendre transform of L with $\dot{q} \to p$.

Exercise 3: *Symmetry*

a) Show that the gradient transforms like a vector.

b) Recall (or read up on) *Noether's Theorem* and convince yourself that for a free classical particle momentum and energy are conserved.

Exercise 4: *Probability in Physics*
Prove the Stirling formula, following the steps below.

a) Show that the gamma function generalises the factorial in the following sense:
$$\Gamma(n+1) := \int_0^\infty x^n \mathrm{e}^{-x}\, \mathrm{d}x = n!$$

b) Convince yourself that for large n
$$\int_0^\infty \mathrm{e}^{-\frac{(x-n)^2}{2n}}\, \mathrm{d}x \approx \int_{-\infty}^\infty \mathrm{e}^{-\frac{(x-n)^2}{2n}}\, \mathrm{d}x.$$

c) Bring the integrand of the gamma function to the form of a pure exponential function and Taylor expand its exponent around its maximum, dropping all terms higher than second order. Evaluate the so produced integral.