Tannaka Duality for Arbitrary Hopf Algebras

Peter Schauenburg

Fakultät für Mathematik
Ludwig-Maximilians-Universität München

München, 1991

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Introduction

The object of this paper is to present the basics of a generalization, suited to non-commutative, non-cocommutative Hopf algebras, of the classical theory of Tannakian categories for affine algebraic groups. This theory thus replaces the classical theory of Tannaka-Krein duality in the case of quantum groups.

Roughly speaking, Tannaka-Krein theory can be reformulated (and generalized to affine algebraic groups over an arbitrary ground field, see [3]) to be the process of reconstructing a group from the collection of all its finite-dimensional representations. For this reconstruction, we have to take account of some "structures" the category of all representations has. These are the tensor product of two representations, which is mirrored in the multiplication of functions on the group, and the dual (or contragredient) of a representation, which is connected to taking inverses of elements of the group.

The algebra of functions on an affine algebraic group is a commutative Hopf algebra having the representations of the group as comodules. To give a leisurely description of the reverse process of the reconstruction we are up to, we consider a k-algebra A and its (finite dimensional) left modules instead. Here, a nice "classical" case is that of a group algebra kG. If $A = kG$, then A-modules are just linear representations of the group G.

Suppose M and N are two (finite dimensional) left A-modules. Then, in general, their tensor product over k carries no "reasonable" structure of an A-module depending on the two structures given. If $A = kG$ is a group algebra, we get a nice structure defining the action of a group element g on $m \otimes n \in M \otimes N$ by $g(m \otimes n) := gm \otimes gn$. The new action of kG on $M \otimes N$ can also be described in terms of the comultiplication $\Delta : kG \to kG \otimes kG$, which is defined to be the algebra map taking $g \in G$ to $g \otimes g \in kG \otimes kG$: To let an element of kG act on $M \otimes N$, we comultiply it first and then let the resulting tensor act on $M \otimes N$ componentwise. In the same manner, we can define an A-module structure on the tensor product of A-modules M and N if A is any bialgebra. Denoting comultiplication $\Delta : A \to A \otimes A$ by $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$, we let $a(m \otimes n) := \sum a_{(1)}m \otimes a_{(2)}n$. The structure thus defined on the tensor product of modules is reasonable in the following sense: The category of left modules over a bialgebra is a monoidal category.

In the case of a group algebra there is also a canonical action of G on the linear dual V^* of a finite dimensional representation V. We define $g\varphi \in V^*$ for $\varphi \in V^*$ and $g \in G$ by the value it takes when evaluated on an element $x \in V$, which is $\langle g\varphi, x\rangle := \langle \varphi, g^{-1}x\rangle$. If A is an arbitrary Hopf algebra with antipode s, we can do the same thing for the linear dual of a finite dimensional A-module V, defining $\langle a\varphi, x\rangle := \langle \varphi, s(a)x\rangle$. This structure is well behaved in the sense that the evaluation map $V^* \otimes V \to k$ is an A-linear map making V^* a "left dual" of V in the monoidal category of all A-modules, in a sense that has to be made precise.

This paper studies the reverse implications, in the formally dual case of coalgebras and comodules instead of algebras an modules. In [17] and [18], Ulbrich generalizes the result of [3] and [14] in the case of affine algebraic groups to the general case of non-commutative non-cocommutative Hopf algebras. He describes a general construction of a coalgebra C from a category $\mathcal{C}$ equipped with an "underlying" functor to the category of finitely generated projective k-modules, such that the (underlying k-modules) of the objects of $\mathcal{C}$ are comodules over C. If, moreover, the ground ring is a field, $\mathcal{C}$ is an abelian k-linear category, and the underlying functor is k-linear exact and faithful, then $\mathcal{C}$ is actually equivalent to the category of all C-comodules. Properties of the category $\mathcal{C}$ in consideration can be translated to properties of the coalgebra constructed. Ulbrich's main result is that the coalgebra constructed from a rigid monoidal category is a Hopf algebra.

After a short excursion to the theory of monoidal categories in the first chapter we will reconsider Ulbrich's constructions in the second chapter. We will show that his construction yields a left adjoint to the functor assigning to a coalgebra the category of its finite dimensional right comodules together with its underlying functor. We will also redo the reconstructions of some structures of the coalgebra from structures of the category in consideration.

It is possible to derive a coordinatewise description for the coalgebras or bialgebras we have obtained in an abstract manner. In particular, the objects of the category we start from correspond to multiplicative matrices (in the sense of, say, [9]) in the coendomorphism coalgebra. If a set of objects generates a monoidal category by forming tensor products, the entries of the corresponding multiplicative matrices generate the bialgebra constructed from the category as an algebra.

Chapter 1

Monoidal categories

A monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product. This is required to fulfill certain axioms forcing it to behave reasonably similar to the tensor product of bimodules. Strengthened axioms correspond to the restrictions to, say, finitely generated projective modules or modules over a commutative ring (regarded as bimodules). Examples, hovever, also include the category of $\mathbf{Z}/2\mathbf{Z}$-graded vector spaces underlying the notions of supergeometry, and the categories of "tangles" that have become an important tool in knot theory [5].

The general theory of monoidal categories is very well known. The reader will find almost everything on almost every species of monoidal category in [14]. Other references are [11] and [12]; [3] deals mainly with symmetric monoidal categories. The first chapter of this paper shall, therefore, only give a leisurely overview of the subject. We devote a little more attention to dual objects in not necessarily symmetric categories.

1.1 Monoidal categories

This section deals with the least restricted version of "categories with a nice tensor product", covering even bimodules over a not necessarily commutative ring. The only properties required are "associativity" and existence of a unit object. We touch briefly upon the "correct" version of associativity.

Associativity of the tensor product is encoded in a natural isomorphism
$\alpha : X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$, the standard example being of course, in the case of R-R-bimodules, the usual

$$X \otimes (Y \otimes Z) \ni x \otimes (y \otimes z) \mapsto (x \otimes y) \otimes z \in (X \otimes Y) \otimes Z.$$

Once we are provided with an associativity isomorphism for three tensorands as above, we can easily construct isomorphisms between any two multiple tensor products differing only in the arrangement of parentheses — but we can do so in different ways. If, working with the usual tensor product, we almost always just forget about all the isomorphisms, writing things like $X \otimes Y \otimes Z$ without parentheses, we can do so only because all the isomorphisms constructed from α are the same, or because a "coherent" identification of the formal ordered multiple tensor products with only a different arrangement of parentheses is possible.

The associativity constraint for an abstract monoidal category will be required to have an analogous property. Mac Lane has shown that to this end it is sufficient to require commutativity of only one diagram consisting of associativity morphisms for fourfold tensor products (if we also take the unit object into account we need one more diagram).

1.1.1. Definition: *A* **monoidal category** $\mathcal{C} = (\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ *consists of the following data:*
An underlying category $\mathcal{C}_0$*, also denoted by* $\mathcal{C}$ *if no confusion arises; a bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, called tensor product; a natural isomorphism* $\alpha : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$*, called the associativity constraint; an object* I *of* $\mathcal{C}_0$ *equipped with natural isomorphisms* $\lambda : I \otimes X \to X$ *and* $\rho : X \otimes I \to X$*, called the unit object.*
These isomorphisms are required to satisfy the following conditions (expressed by commutative diagrams), called coherence conditions for reasons hopefully to become apparent soon:

$$\begin{tikzcd}
 & A \otimes ((B \otimes C) \otimes D) \arrow[dr, "\alpha"] & \\
A \otimes (B \otimes (C \otimes D)) \arrow[ur, "1 \otimes \alpha"] \arrow[d, "\alpha"'] & & (A \otimes (B \otimes C)) \otimes D \arrow[d, "\alpha \otimes 1"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha"] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

(This is called the pentagon identity)

$$\begin{tikzcd}
X \otimes (I \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \lambda"'] & & (X \otimes I) \otimes Y \arrow[dl, "\rho \otimes 1"] \\
 & X \otimes Y &
\end{tikzcd}$$

$\mathcal{C}$ *is said to be* **strictly** *monoidal, if the morphisms* α*,* λ *and* ρ *are identities.*

An abelian monoidal category is a monoidal category $\mathcal{C}$ *in which* $\mathcal{C}_0$ *is an abelian category, and* $\otimes$ *is biadditive. If, in addition,* $\otimes$ *is exact in each argument, then* $\mathcal{C}$ *is said to be exact abelian monoidal.*

1.1.2. Remark: *We note right away a few consequences of the definition that are not very straightforward though not hard to check:*

The two additional triangles

$$\begin{tikzcd}
I \otimes (X \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "\lambda"'] & & (I \otimes X) \otimes Y \arrow[dl, "\lambda \otimes 1"] \\
& X \otimes Y &
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (Y \otimes I) \arrow[rr, "\alpha"] \arrow[dr, "1 \otimes \rho"'] & & (X \otimes Y) \otimes I \arrow[dl, "\rho"] \\
& X \otimes Y &
\end{tikzcd}$$

also commute, and we have $\lambda_I = \rho_I$.

In the category of R-R-bimodules the unit object is, of course, R equipped with the obvious R-bimodule structure.

The following example is somewhat central for the considerations of this paper.

1.1.3. Example: *Let B be a bialgebra over a commutative ring k. The category ${}_B\mathcal{M}$ of B-modules has a natural structure of monoidal category. The tensor product of two B-modules M and N is defined to be the usual tensor product over k, equipped with the "diagonal structure"*

$$B \otimes M \otimes N \xrightarrow{\Delta\otimes 1\otimes 1} B \otimes B \otimes M \otimes N \xrightarrow{1\otimes\tau\otimes 1} B \otimes M \otimes B \otimes N \to M \otimes N,$$

that is $b(m \otimes n) = \sum_{(b)} b_{(1)}m \otimes b_{(2)}n$. The neutral object is the ground ring with the "trivial" module structure induced by the counit of B: $b\kappa = \varepsilon(b)\kappa$ for $b \in B$ and $\kappa \in k$.

Rather often, little attention is paid to the unit object and the morphisms λ and ρ, as long as one is fairly sure that there actually *are* such things. This is not only justified by the fact that all calculations involving the unit objects will in practice be simpler than the calculations involving the associativity constraint. The following lemma also shows that neutral objects are rather unique, if they exist.

1.1.4. Remark ([14], p.33, Prop. 2.4.1): *If $(\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ and $(\mathcal{C}_0, \otimes, \alpha, U, \ell, r)$ are monoidal categories only differing in the choice of the unit object, then there is a unique isomorphism $\varphi : U \to I$ making the diagrams*

$$\begin{tikzcd}
U \otimes X \arrow[rr, "\varphi \otimes 1"] \arrow[dr, "\ell_X"'] & & I \otimes X \arrow[dl, "\lambda_X"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes U \arrow[rr, "1 \otimes \varphi"] \arrow[dr, "r_X"'] & & X \otimes I \arrow[dl, "\rho_X"] \\
& X &
\end{tikzcd}$$

commute, namely $\varphi = (U \xrightarrow{\rho_U^{-1}} U \otimes I \xrightarrow{\ell_I} I)$.

The main feature of the coherence conditions stated in the definition of a monoidal category is vaguely the following: By means of the associativity constraints and the unit constraints we can build up isomorphisms between any two multiple tensor products that differ only in the arrangement of parentheses and some number of neutral objects inserted in between. The coherence conditions ensure that the different isomorphisms we can build up "formally" from the elementary isomorphisms α, λ and ρ are actually identical[1]. Thus, we can "coherently" identify these multiple tensor products via the isomorphisms, defining something like a multiple tensor product without parentheses. I cite without proof a form of expressing the coherence result taken from [14, 1.1.2, p.16 and 2.2.2, p.29].

1.1.5. Lemma: *There are functors* $\otimes^\Lambda : \mathcal{C}^\Lambda \to \mathcal{C}$ *for each finite totally ordered set* Λ, *and functorial isomorphisms* $\otimes(\otimes^{\Lambda_1} \times \otimes^{\Lambda_2}) \cong \otimes^\Lambda$ *for each decomposition of totally ordered sets* $\Lambda = \Lambda_1 \dot{\cup} \Lambda_2$ *with* $\Lambda_1 < \Lambda_2$, *such that*

(1) $\otimes^\emptyset(\emptyset) = I$

(2) $\otimes^{\{*\}}(X) = X$

(3) The isomorphism $\otimes^{\Lambda \dot{\cup} \emptyset} \cong \otimes(\otimes^\Lambda \times \otimes^\emptyset)$ *is the given* ρ.

(4) The isomorphism $\otimes^{\emptyset \dot{\cup} \Lambda} \cong \otimes(\otimes^\emptyset \times \otimes^\Lambda)$ *is the given* λ.

(5) The isomorphism $\otimes(\mathcal{I}d \times \otimes) \cong \otimes(\otimes \times \mathcal{I}d)$ *composed from the isomorphisms obtained decomposing a three element set* Λ *in two different ways into one element sets, is the given* α.

For any other such family of functors $\diamond^\Lambda$ *with analogous isomorphisms there are uniquely determined natural isomorphisms* $\iota_\Lambda : \otimes^\Lambda \to \diamond^\Lambda$ *satisfying*

$$\begin{array}{ccc} \otimes^{\Lambda_1 \dot{\cup} \Lambda_2} & \xrightarrow{\iota_{\Lambda_1 \dot{\cup} \Lambda_2}} & \diamond^{\Lambda_1 \dot{\cup} \Lambda_2} \\ \wr\| & & \|\wr \\ \otimes(\otimes^{\Lambda_1} \times \otimes_2^{\Lambda}) & \xrightarrow[\iota_{\Lambda_1} \otimes \iota_{\Lambda_2}]{} & \otimes(\diamond^{\Lambda_1} \times \diamond_2^{\Lambda}) \end{array}$$

Naturally, together with monoidal categories comes the corresponding notion of structure preserving functors. Again we do not require a formula like $\mathcal{F}(X \otimes Y) = \mathcal{F}X \otimes \mathcal{F}Y$ to hold but (as in the case of associativity up to an isomorphism) we only require an isomorphism between these expressions, which is again subject to some coherence conditions essentially ensuring a possibility of "identifying" different expressions.

[1] 1.1.2 should be seen as a first step towards this result. Although we prove neither the general theorem nor the special cases in 1.1.2, we have stated the latter seperately anyway, as they have to be treated as preliminary special cases in the proof of the general theorem.

1.1.6. Definition: *Let $\mathcal{C}$ and $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}_0, \tilde{\otimes}, \tilde{\alpha}, \tilde{I}, \tilde{\lambda}, \tilde{\rho})$ be two monoidal categories.*

A weakly monoidal functor $\mathcal{F} = (\mathcal{F}_0, \xi, \xi_0)$ consists of the following data:
An underlying functor $\mathcal{F}_0 : \mathcal{C}_0 \to \tilde{\mathcal{C}}_0$, also denoted by $\mathcal{F}$ if no confusion is likely to arise; a functorial morphism $\xi : \mathcal{F}X \tilde{\otimes} \mathcal{F}Y \to \mathcal{F}(X \otimes Y)$; a morphism $\xi_0 : \tilde{I} \to \mathcal{F}(I)$.

These are required to fulfill the following coherence conditions:

$$\begin{tikzcd}
\mathcal{F}X\tilde{\otimes}(\mathcal{F}Y\tilde{\otimes}\mathcal{F}Z) \arrow[r, "1\otimes\xi"] \arrow[d, "\mathcal{F}\alpha"'] & \mathcal{F}X\tilde{\otimes}\mathcal{F}(Y\otimes Z) \arrow[r, "\xi"] & \mathcal{F}(X\otimes(Y\otimes Z)) \arrow[d, "\tilde{\alpha}"] \\
(\mathcal{F}X\tilde{\otimes}\mathcal{F}Y)\tilde{\otimes}\mathcal{F}Z \arrow[r, "\xi\otimes 1"'] & \mathcal{F}(X\otimes Y)\tilde{\otimes}\mathcal{F}Z \arrow[r, "\xi"'] & \mathcal{F}((X\otimes Y)\otimes Z)
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{I}\tilde{\otimes}\mathcal{F}X \arrow[r, "\xi_0\otimes 1"] \arrow[dr, "\mathcal{F}\lambda"'] & \mathcal{F}I\tilde{\otimes}\mathcal{F}X \arrow[r, "\xi"] & \mathcal{F}(I\otimes X) \arrow[dl, "\tilde{\lambda}"] \\
& \mathcal{F}X &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}X\tilde{\otimes}\tilde{I} \arrow[r, "1\otimes\xi_0"] \arrow[dr, "\mathcal{F}\rho"'] & \mathcal{F}X\tilde{\otimes}\mathcal{F}I \arrow[r, "\xi"] & \mathcal{F}(X\otimes I) \arrow[dl, "\tilde{\rho}"] \\
& \mathcal{F}X &
\end{tikzcd}$$

A monoidal functor is a weakly monoidal functor in which ξ and ξ_0 are isomorphisms. A strictly monoidal functor is one in which ξ and ξ_0 are identities.

A natural transformation $\zeta : \mathcal{F} \to \mathcal{F}'$ between two (weakly) monoidal functors $(\mathcal{F}, \xi, \xi_0)$ and $(\mathcal{F}', \xi', \xi'_0)$ from $\mathcal{C}$ to $\tilde{\mathcal{C}}$ is said to be a monoidal transformation, if the diagrams

$$\begin{tikzcd}
\mathcal{F}X\otimes\mathcal{F}Y \arrow[r, "\xi"] \arrow[d, "\zeta"'] & \mathcal{F}(X\otimes Y) \arrow[d, "\zeta\otimes\zeta"] \\
\mathcal{F}'X\otimes\mathcal{F}'Y \arrow[r, "\xi'"'] & \mathcal{F}'(X\otimes Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
& I \arrow[dd, "\zeta"] \\
\mathcal{F}I \arrow[ur, "\xi_0"] \arrow[dr, "\xi'_0"'] & \\
& \mathcal{F}'I
\end{tikzcd}$$

commute.

It goes without saying that we will rather often just forget about the condition that $\mathcal{F}$ be compatible with the unit objects, implicitly assuming that even $\mathcal{F}(I) = \tilde{I}$, maybe after rechoosing a unit object in $\tilde{\mathcal{C}}$.

The composition of weakly monoidal functors $(\mathcal{F}, \xi, \xi_0)$ and $(\mathcal{G}, \chi, \xi_0)$ is a weakly monoidal functor again, the structure morphisms being

$$\mathcal{G}\mathcal{F}X \otimes \mathcal{G}\mathcal{F}Y \xrightarrow{\chi_{\mathcal{F}X,\mathcal{F}Y}} \mathcal{G}(\mathcal{F}X \otimes \mathcal{F}Y) \xrightarrow{\mathcal{G}(\xi)} \mathcal{G}\mathcal{F}(X \otimes Y)$$

$$I \xrightarrow{\chi_0} \mathcal{G}I \xrightarrow{\mathcal{G}\xi} \mathcal{G}\mathcal{F}I.$$

We note a few facts about adjunctions between monoidal categories:

1.1.7. Definition: *Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be monoidal categories. A monoidal adjunction between a monoidal functor $(\mathcal{F}, \xi, \xi_0) : \mathcal{B} \to \mathcal{C}$ and a weakly monoidal functor $(\mathcal{G}, \vartheta)$: $\mathcal{C} \to \mathcal{B}$ is an ordinary adjunction making $\mathcal{F}$ a left adjoint of $\mathcal{G}$ such that the adjunction morphisms are monoidal morphisms.*

1.1.8. Lemma: *If $(\mathcal{F}, \xi, \xi_0)$ is a monoidal functor and $\mathcal{G}$ is a right adjoint for $\mathcal{F}$, then $\mathcal{G}$ has a unique structure of weakly monoidal functor making the given adjunction a monoidal one.*

PROOF: We only note how to get at the structure we are looking for: In view of the universal property of the counit Φ we can define $\chi : \mathcal{G}X \otimes \mathcal{G}Y \to \mathcal{G}(X \otimes Y)$ by commutativity of

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\Phi"] \arrow[d, "\Phi \otimes \Phi"'] & \mathcal{G}\mathcal{F}(X \otimes Y) \arrow[d, "\mathcal{G}\xi^{-1}"] \\
\mathcal{G}\mathcal{F}X \otimes \mathcal{G}\mathcal{F}Y \arrow[r, "\chi"'] & \mathcal{G}(\mathcal{F}X \otimes \mathcal{G}Y)
\end{tikzcd}$$

By definition, the counit is then a monoidal transformation and χ is uniquely defined by this property. All that is left to check is that the unit is also a monoidal transformation. To check commutativity of the relevant diagram

$$\begin{tikzcd}
X \otimes Y \arrow[r] \arrow[d] & \mathcal{G}\mathcal{F}(X \otimes Y) \arrow[d] \\
\mathcal{G}\mathcal{F}X \otimes \mathcal{G}\mathcal{F}Y \arrow[r] & \mathcal{G}(\mathcal{F}X \otimes \mathcal{F}Y)
\end{tikzcd}$$

it is sufficient to check it after applying $\mathcal{F}$ and attaching the counit Ψ to the bottom right corner. I will not spell out the tedious proof. $\square$

1.1.9. Remark: *Using monoidal functors we can state the coherence theorem on monoidal categories more directly:*

Every monoidal category is monoidally equivalent to a strict monoidal one.

Here some attention should be paid to the notion of monoidal equivalence. It is an easy consequence of the preceding remark that if a monoidal functor is an equivalence of categories then any equivalence quasi-inverse can be made a monoidal functor which is a left adjoint such that the adjunction is a monoidal one.

Thus we can, without any serious danger, do all the calculations in a general monoidal category as if it was strictly monoidal.

The opposite category of a monoidal category admits a natural structure of monoidal category (thus defining contravariant monoidal functors); also, we can form a monoidal category with the tensor product reversed:

1.1.10. Definition ([14]): *Let* $\mathcal{C} = (\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ *be a monoidal category. The monoidal category symmetric to* $\mathcal{C}$ *is* $\mathcal{C}^{\mathrm{sym}} = (\mathcal{C}_0, \otimes^{\mathrm{sym}}, \alpha^{\mathrm{sym}}, I, \lambda^{\mathrm{sym}}, \rho^{\mathrm{sym}})$*, defined by*

$$\otimes^{\mathrm{sym}} : \mathcal{C} \times \mathcal{C} \ni (X, Y) \longmapsto Y \otimes X \in \mathcal{C}.$$

$$(X \otimes^{\mathrm{sym}} (Y \otimes^{\mathrm{sym}} Z) \xrightarrow{\alpha^{\mathrm{sym}}} (X \otimes^{\mathrm{sym}} Y) \otimes^{\mathrm{sym}} Z) = ((Z \otimes Y) \otimes X \xrightarrow{\alpha^{-1}} Z \otimes (Y \otimes X)).$$

$$(I \otimes^{\mathrm{sym}} X \xrightarrow{\lambda^{\mathrm{sym}}} I) = (X \otimes I \xrightarrow{\rho} I)$$

$$(X \otimes^{\mathrm{sym}} I \xrightarrow{\rho^{\mathrm{sym}}} I) = (I \otimes X \xrightarrow{\lambda} I)$$

It is seen to be a monoidal category in a straightforward manner.

The dual category ${\mathcal{C}_0}^{\mathrm{op}}$ *becomes a monoidal category* $\mathcal{C}^{\mathrm{op}} = ({\mathcal{C}_0}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \alpha^{\mathrm{op}}, I, \lambda^{\mathrm{op}}, \rho^{\mathrm{op}})$ *as well by*

$$X \otimes^{\mathrm{op}} Y = X \otimes Y$$

$$\alpha^{\mathrm{op}} = \alpha^{-1}, \quad \lambda^{\mathrm{op}} = \lambda^{-1}, \quad \rho^{\mathrm{op}} = \rho^{-1}$$

If $\mathcal{C}$ is abelian monoidal, then so are $\mathcal{C}^{\mathrm{sym}}$ and $\mathcal{C}^{\mathrm{op}}$.

We have $\mathcal{C}^{\mathrm{sym\ sym}} = \mathcal{C} = \mathcal{C}^{\mathrm{op\ op}}$, $\mathcal{C}^{\mathrm{op\ sym}} = \mathcal{C}^{\mathrm{sym\ op}}$ as monoidal categories.

1.2 Dual objects and inner hom-functors

A closed monoidal category is defined to have objects behaving similar to the modules of homomorphisms of k-modules in the category of k-modules (over a commutative ring k):

1.2.1. Definition: *Let $\mathcal{C}$ be as above, $X \in \operatorname{Ob} C$. A right adjoint $\underline{\hom}(X, -)$ of the functor $- \otimes X : \mathcal{C} \to \mathcal{C}$ is called a (right)* **inner hom-functor**.

If each $X \in \mathcal{C}$ admits such an inner hom-functor, the monoidal category is said to be (right) closed.

Similarly, a left inner hom-functor is a right adjoint for $X \otimes -$, and we have the notion of left closed category.

Of course, left inner hom-functors in $\mathcal{C}$ are right inner hom-functors in $\mathcal{C}^{\mathrm{sym}}$.

1.2.2. Examples:

(1) *Let R be a ring, $\mathcal{C}$ the monoidal category of R-R-bimodules. Then the R-R-bimodules $\underline{\hom}_{-R}(X,Y)$ of all right R-linear maps and $\underline{\hom}_{R-}(X,Y)$ of all left R-linear maps define right, respectively left inner hom-functors.*

(2) *Let, in particular, k be a commutative ring and consider the category $\underline{Mod}_k$ of k-modules. Then the modules $\operatorname{Hom}_k(M,N)$ define inner hom-functors.*

(3) *Let H be a k-Hopf algebra with antipode s. Then for X an H-module, the modules $\operatorname{Hom}_k(X, -)$ again define an inner hom-functor if we define a module structure on $\operatorname{Hom}_k(X,Y)$ by*

$$\forall h \in H\ \forall f \in \operatorname{Hom}_k(X,Y)\ \forall x \in X \quad (hf)(x) = h_{(1)} f(s(h_{(2)})x)$$

Proof: The first and second example are standard (and the motivating example for the notion of a closed monoidal category). Unit and counit of the adjunction (for the right inner hom-functors) are

$$A_Y : \operatorname{Hom}_k(X,Y) \otimes X \ni f \otimes x \mapsto f(x) \in Y$$

$$E_Y : Y \to \operatorname{Hom}_k(X, Y \otimes X) \qquad E_Y(y)(x) = y \otimes x$$

For the third example we have to check linearity of the adjunction morphisms, which agree with the ones for the underlying k-modules.

Now indeed for $h \in H, f \in \operatorname{Hom}_k(X,Y), x \in X, y \in Y$ we have:

$$\begin{aligned} A_Y(h(f \otimes x)) &= A_Y(h_{(1)} f \otimes h_{(2)} x) \\ &= (h_{(1)} f)(h_{(2)} x) \\ &= h_{(1)} f(s(h_{(2)}) h_{(3)} x) \end{aligned}$$

$$
\begin{aligned}
&= hf(x) = hA_Y(f \otimes x)\\
(hE_Y(y))(x) &= h_{(1)}E_Y(y)(s(h_{(2)})x)\\
&= h_{(1)}(y \otimes s(h_{(2)})x)\\
&= h_{(1)}y \otimes h_{(2)}s(h_{(3)})x\\
&= hy \otimes x = E_Y(hy)(x)
\end{aligned}
$$

□

The following definition captures the nice properties of finitely generated projective modules over a commutative ring in the general context of monoidal categories.

1.2.3. Definition: *Let X be an object of a monoidal category $\mathcal{C}$.*
Let $X^* \in \operatorname{Ob} C, \mathrm{ev} : X^* \otimes X \to I$.

(X^*, ev) *is called a* **(left) dual** *for X if there is a morphism* $\mathrm{db} : I \to X \otimes X^*$ *in $\mathcal{C}$ such that*

$$(X \xrightarrow{\mathrm{db}\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes \mathrm{ev}} X) = 1_X$$

$$(X^* \xrightarrow{1\otimes \mathrm{db}} X^* \otimes X \otimes X^* \xrightarrow{\mathrm{ev}\otimes 1} X^*) = 1_{X^*}$$

$\mathcal{C}$ is said to be **(left) rigid***, if each object of $\mathcal{C}$ admits a left dual.*

The notions of right dual, right rigid are defined in an analogous manner.

Sometimes we will also call an *object* rigid if it admits a left dual.

1.2.4. Example: *A k-module X is finitely generated projective iff there are elements $x_1, \ldots, x_n \in X$ and $x^1, \ldots, x^n \in X^* = \operatorname{Hom}_k(X, k)$ such that*

$$\forall x \in X : \quad \sum_i x^i(x)x_i = x.$$

Then also

$$\forall \varphi \in X^* : \quad \sum_i \varphi(x_i)x^i = \varphi.$$

Thus X^* is a left dual for X in the sense of 1.2.3, iff X is finitely generated projective. In this case, the mapping db *required by the definition is given by* $\mathrm{db}(1_k) = \sum_i x_i \otimes x^i$.

In calculations with this generalization of a "basis and dual basis" to finitely generated projective modules we shall mostly adopt the usual summation convention, writing $\mathrm{db}(1_k) = x_i \otimes x^i$, and we will denote the evaluation by brackets $<,>$, like, say, in $< x^i, x > x_i = x$.

1.2.5. Remark: $(Y, \mathrm{ev} : Y \otimes X \to I)$ *is a left dual for X iff (X, ev) is a right dual for Y.*

Existence of left duals implies existence of inner hom-functors:

1.2.6. Lemma: *Let X be an object of the monoidal category $\mathcal{C}$. If X admits a left dual (X^*, ev), then there is an inner hom-functor $\underline{\mathrm{hom}}(X, -)$, defined by $\underline{\mathrm{hom}}(X, Y) = Y \otimes X^*$.*

PROOF: Define the hom-functor as stated in the lemma. If $\mathrm{db} : I \to X \otimes X^*$ is a morphism as in definition 1.2.3, then define the adjunction morphisms by

$$Y \xrightarrow{1\otimes \mathrm{db}} Y \otimes X \otimes X^* = \mathrm{hom}(X, Y \otimes X)$$

$$\mathrm{hom}(X, Y) \otimes X = Y \otimes X^* \otimes X \xrightarrow{1\otimes \mathrm{ev}} Y.$$

Now the two triangle equations for the adjunction morphisms are direct consequences of the equations in the definition 1.2.3 of a left dual. $\square$

1.2.7. Remark: *A left dual for an object X of $\mathcal{C}$ is a right dual for X in $\mathcal{C}^{\mathrm{sym}}$ and a right dual for X in $\mathcal{C}^{\mathrm{op}}$. In particular, a rigid monoidal category is "left coclosed" in the sense that the functors $X \otimes -$ have left adjoints (co-hom-functors).*

1.2.8. Corollary: *Let $\mathcal{C}$ be a rigid monoidal category. Then for each object X the functor $- \otimes X$ is a left adjoint and thus preserves colimits. The functor $X \otimes -$ is a right adjoint and thus preseves limits. In particular, if $\mathcal{C}$ is also right rigid and k-linear abelian then $\mathcal{C}$ is exact monoidal k-linear abelian.*

To do calculations with dual objects in the sense of 1.2.3 we will make free use of the well known facts on adjoint functors.

1.2.9. Corollary: *If (X^*, ev) is a left dual of X, then the morphism* db *occuring in the definition is unique.*

Indeed, db is the image of $\lambda : I \otimes X \to X$ under the isomorphism

$$\mathcal{M}or(I \otimes X, X) \to \mathcal{M}or(I, X \otimes X^*)$$

In the case of a finitely generated projective k-module the corollary corresponds to the well known fact that the element $x_i \otimes x^i \in M \otimes M^*$ in the example 1.2.4 is unique.

1.2.10. Corollary: *If X admits a left dual, this is unique in the sense that for any two left duals (X^*, ev) and (D, e) for X there is a unique isomorphism $j : D \to X^*$ making the diagrams*

$$\begin{tikzcd} D \otimes X \arrow[rr, "j \otimes 1"] \arrow[dr, "e"'] & & X^* \otimes X \arrow[dl, "\mathrm{ev}"] \\ & I & \end{tikzcd}$$

commute, namely

$$D \xrightarrow{1\otimes \mathrm{db}} D\otimes X\otimes X^* \xrightarrow{e\otimes 1} X^*$$

PROOF: $\underline{\mathrm{hom}}(X, -)$ as a right adjoint is uniquely defined up to a unique functorial isomorphism compatible with the adjunction morphisms. The assertion now follows from 1.2.6. □

1.2.11. Remark ([6], p.100, thm. 3): *If, for each $X \in \mathrm{Ob}\,\mathcal{C}$, there is an inner hom-functor $\underline{\mathrm{hom}}(X, -)$, then there is a uniqe way of making $\underline{\mathrm{hom}}(-, Z)$ a (contravariant) functor for each $Z \in \mathrm{Ob}\,C$ such that $\underline{\mathrm{hom}}(-_1, -_2)$ becomes a bifunctor and the isomorphism*

$$\mathcal{M}or_{\mathcal{C}}(Y\otimes X, Z) \cong \mathcal{M}or_{\mathcal{C}}(Y, \underline{\mathrm{hom}}(X, Z))$$

is also natural in X.

PROOF: (sketch)
We will only show how to get at the necessary morphisms $\underline{\mathrm{hom}}(f, -)$.

Let $A_{XY} : \underline{\mathrm{hom}}(X, Y)\otimes X \to Y$ and $E_{XY} : Y \to \underline{\mathrm{hom}}(X, Y\otimes X)$ be the adjunction morphisms. Let $f : X \to X'$ and $Y \in \mathcal{C}$. Then there is a unique $g =: \underline{\mathrm{hom}}(f, Y)$ satisfying

$$\begin{tikzcd} \underline{\mathrm{hom}}(X', Y) \arrow[d, "g"'] \\ \underline{\mathrm{hom}}(X, Y) \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathrm{hom}}(X', Y)\otimes X \arrow[r, "1\otimes f"] \arrow[d, "g\otimes 1"'] & \underline{\mathrm{hom}}(X', Y)\otimes X' \arrow[d, "A"] \\ \underline{\mathrm{hom}}(X, Y)\otimes X \arrow[r, "A"'] & Y \end{tikzcd}$$

namely

$$\begin{aligned} g = (\underline{\mathrm{hom}}(X', Y) &\xrightarrow{E} \underline{\mathrm{hom}}(X, \underline{\mathrm{hom}}(X', Y)\otimes X') \\ &\xrightarrow{\underline{\mathrm{hom}}(1, 1\otimes f)} \underline{\mathrm{hom}}(X, \underline{\mathrm{hom}}(X', Y)\otimes X) \\ &\xrightarrow{\underline{\mathrm{hom}}(1, A)} \underline{\mathrm{hom}}(X, Y) \quad) \end{aligned}$$

□

1.2.12. Corollary: *Let $\mathcal{C}$ be rigid and choose a dual X^* for each object X in $\mathcal{C}$. Then $(-)^* : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ defines a functor.*

Indeed we see from the preceding remark that for $f : X \to X'$ the morphism f^* will be characterized by

$$\begin{tikzcd} (X')^*\otimes X \arrow[r, "1\otimes f"] \arrow[d, "f^*\otimes 1"'] & (X')^*\otimes X' \arrow[d, "\mathrm{ev}"] \\ X^*\otimes X \arrow[r, "\mathrm{ev}"'] & I \end{tikzcd}$$

and given by

$$f^* = (\ (X')^* \xrightarrow{1\otimes \mathrm{db}} (X')^* \otimes X \otimes X^* \xrightarrow{1\otimes f\otimes 1} (X')^* \otimes X' \otimes X^* \xrightarrow{\mathrm{ev}\otimes 1} X^*)$$

Since we have defined left duals to be, by and large, objects eqipped with morphisms satisfying identities, we certainly expect monoidal functors to preserve them.

1.2.13. Lemma: *Let $(\mathcal{F}, \xi) : \mathcal{C} \to \tilde{\mathcal{C}}$ be a monoidal functor. Let $X \in \mathrm{Ob}\, C$ have (X^*, ev) as a left dual. Then $(\mathcal{F}(X^*), \widetilde{\mathrm{ev}})$ is a left dual for $\mathcal{F}X$, if we define*

$$\widetilde{\mathrm{ev}} = (\mathcal{F}X^* \otimes \mathcal{F}X \xrightarrow{\xi} \mathcal{F}(X^* \otimes X) \xrightarrow{\mathcal{F}(\mathrm{ev})} I)$$

Proof: Define

$$\widetilde{\mathrm{db}} = (\mathcal{F}I \xrightarrow{\mathcal{F}(\mathrm{db})} \mathcal{F}(X \otimes X^*) \xrightarrow{\xi^{-1}} \mathcal{F}X \otimes \mathcal{F}X^*)$$

and check the two identities. $\square$

1.2.14. Corollary: *Let $(\mathcal{F}, \xi)$ be as in 1.2.13, and let both $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be rigid with a chosen functor $(\,—\,)^*$. Then there is a unique functorial isomorphism $u : \mathcal{F}(X^*) \to \mathcal{F}(X)^*$ satisfying*

$$\begin{tikzcd}
\mathcal{F}(X^*) \otimes \mathcal{F}(X) \arrow[r, "u \otimes 1"] \arrow[d, "\xi"'] & \mathcal{F}(X)^* \otimes \mathcal{F}(X) \arrow[d, "\mathrm{ev}"] \\
\mathcal{F}(X^* \otimes X) \arrow[r, "\mathcal{F}(\mathrm{ev})"'] & I
\end{tikzcd}$$

namely

$$\mathcal{F}(X^*) \xrightarrow{1\otimes \mathrm{db}} \mathcal{F}(X^*) \otimes \mathcal{F}(X) \otimes \mathcal{F}(X)^* \xrightarrow{\xi\otimes 1} \mathcal{F}(X^* \otimes X) \otimes \mathcal{F}(X) \xrightarrow{\mathcal{F}(\mathrm{ev})\otimes 1} \mathcal{F}(X)$$

1.2.15. Remark: *Let $\mathcal{C} = (\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ be a monoidal category. Let X and Y be objects of $\mathcal{C}$ having (X^*, ev_X) and (Y^*, ev_Y) as left duals, respectively.*

Then $(Y^* \otimes X^*, ev_{X\otimes Y})$ is a left dual for $X \otimes Y$, with

$$\mathrm{ev}_{X\otimes Y} : Y^* \otimes X^* \otimes X \otimes Y \xrightarrow{1\otimes \mathrm{ev}\otimes 1} Y^* \otimes Y \xrightarrow{\mathrm{ev}} I$$

$$\mathrm{db}_{X\otimes Y} : I \xrightarrow{\mathrm{db}} X \otimes X^* \xrightarrow{1\otimes \mathrm{db}\otimes 1} X \otimes Y \otimes Y^* \otimes X^*$$

Also, if $\mathcal{C}$ is abelian monoidal and X_1, X_2 are objects of $\mathcal{C}$ with left duals $(X_i^*, \mathrm{ev}_{X_i})$, then $(X_1^* \oplus X_2^*, \mathrm{ev}_{X_1 \oplus X_2})$ is a left dual for $X_1 \oplus X_2$.

1.2.16. Corollary: *Let $\mathcal{C} = (\mathcal{C}_0, \otimes, \alpha, I, \lambda, \rho)$ be rigid with chosen left duals. Then there is a unique natural isomorphism $\nu : (X \otimes Y)^* \cong Y^* \otimes X^*$ satisfying*

$$\begin{tikzcd}
Y^* \otimes X^* \otimes X \otimes Y \arrow[r, "\nu \otimes id"] \arrow[d, "1 \otimes \mathrm{ev} \otimes 1"'] & (X \otimes Y)^* \otimes (X \otimes Y) \arrow[d, "\mathrm{ev}"] \\
Y^* \otimes Y \arrow[r, "\mathrm{ev}"'] & I
\end{tikzcd}$$

Moreover, if $\mathcal{C}$ is abelian monoidal, then for $X_1, X_2 \in \mathcal{C}$ the object $(X_1 \oplus X_2)^*$ is a biproduct of X_1 and X_2 with projections and injections dual to the injections and projections of $X_1 \oplus X_2$, respectively.

1.2.17. Corollary: *If $\mathcal{C}$ is rigid, then the functor $(\,—\,)^*$ is a monoidal functor from $\mathcal{C}$ to $\mathcal{C}^{\mathrm{sym\,op}}$. Moreover, if $\mathcal{C}$ is abelian monoidal, then $(\,—\,)^*$ is additive.*

The following definition generalizes the process $f \mapsto f^*$ of dualizing morphisms in a rigid monoidal category. A morphism $S : V \otimes W \to X \otimes Y$ gives a "halfway dualized" morphism $S^\flat : X^* \otimes V \to Y \otimes W^*$.

1.2.18. Lemma and Definition: *Let $\mathcal{C}$ be a monoidal category, let $V, W, X, Y \in$ $\mathrm{Ob}\,\mathcal{C}$.*
Let (X^*, ev_X) and (Y^*, ev_Y) be left duals for X and W, respectively.
The following conditions on morphisms $F : V \otimes W \to X \otimes Y$ and $G : X^* \otimes V \to Y \otimes W^*$ are equivalent:

$$\begin{tikzcd}
X^* \otimes V \otimes W \arrow[r, "1 \otimes F"] \arrow[d, "G \otimes 1"'] & X^* \otimes X \otimes Y \arrow[d, "\mathrm{ev} \otimes 1"] \\
V \otimes W^* \otimes W \arrow[r, "1 \otimes \mathrm{ev}"'] & Y
\end{tikzcd} \tag{1}$$

$$\begin{tikzcd}
V \arrow[r, "1 \otimes \mathrm{db}"] \arrow[d, "\mathrm{db} \otimes 1"'] & V \otimes W \otimes W^* \arrow[d, "F \otimes 1"] \\
X \otimes X^* \otimes V \arrow[r, "1 \otimes G"'] & X \otimes V \otimes W^*
\end{tikzcd} \tag{2}$$

Furthermore, for each $F : V \otimes W \to X \otimes Y$ there is a unique $G : X^* \otimes V \to Y \otimes W^*$ fulfilling one (and thus both) of the two.

We define $F^\flat := G$ and $G^\sharp := F$.

Proof: Remember that X is a right dual for X^*. Thus, lemma 1.2.6 provides us with the following isomorphisms:

(1) $\mathcal{M}or(V\otimes W, X\otimes Y)\cong \mathcal{M}or(X^*\otimes V\otimes W, Y)\cong \mathcal{M}or(X^*\otimes V, Y\otimes W^*)$

(2) $\mathcal{M}or(V\otimes W, X\otimes Y)\cong \mathcal{M}or(V, X\otimes Y\otimes W^*)\cong \mathcal{M}or(X^*\otimes V, Y\otimes W^*)$

To see that the two isomorphism found between $\mathcal{M}or(V\otimes W, X\otimes Y)$ and $\mathcal{M}or(X^*\otimes V, Y\otimes W^*)$ are actually the same, we track an element $F: V\otimes W\to X\otimes Y$. Under the morphisms in (1), it is mapped as

$$\begin{aligned} F &\mapsto (X^*\otimes V\otimes W \xrightarrow{1\otimes F} X^*\otimes X\otimes Y \xrightarrow{\mathrm{ev}\otimes 1} Y) \\ &\mapsto (X^*\otimes V \xrightarrow{1\otimes 1\otimes \mathrm{db}} X^*\otimes V\otimes W\otimes W^* \xrightarrow{1\otimes F\otimes 1} X^*\otimes X\otimes Y\otimes W^* \xrightarrow{\mathrm{ev}\otimes 1\otimes 1} Y\otimes W^*) \end{aligned}$$

Under the morphisms in (2) as

$$\begin{aligned} F &\mapsto (V \xrightarrow{1\otimes \mathrm{db}} V\otimes W\otimes W^* \xrightarrow{F\otimes 1} X\otimes Y\otimes W^*) \\ &\mapsto (X^*\otimes V \xrightarrow{1\otimes 1\otimes \mathrm{db}} X^*\otimes V\otimes W\otimes W^* \xrightarrow{1\otimes F\otimes 1} X^*\otimes X\otimes Y\otimes W^* \xrightarrow{\mathrm{ev}\otimes 1\otimes 1} Y\otimes W^*) \end{aligned}$$

Now we only have to check that both diagrams of the lemma are equivalent to the morphisms $F\in\mathcal{M}or(V\otimes W, X\otimes Y)$ and $G\in\mathcal{M}or(X^*\otimes V, Y\otimes W^*)$ being mapped to each other by this isomorphism:
On the one hand, in (1):

$$\begin{array}{rcccl} \mathcal{M}or(V\otimes W, X\otimes Y) & \cong & \mathcal{M}or(X^*\otimes V\otimes W, Y) & \cong & \mathcal{M}or(X^*\otimes V, Y\otimes W^*) \\ F & \mapsto & (\mathrm{ev}\otimes 1)(1\otimes F) & & \\ & & (1\otimes \mathrm{ev})(G\otimes 1) & \leftarrow\!\shortmid & G \end{array}$$

while on the other hand in (2):

$$\begin{array}{rcccl} \mathcal{M}or(V\otimes W, X\otimes Y) & \cong & \mathcal{M}or(V, X\otimes Y\otimes W^*) & \cong & \mathcal{M}or(X^*\otimes V, Y\otimes W^*) \\ F & \mapsto & (F\otimes 1)(1\otimes \mathrm{db}) & & \\ & & (1\otimes G)(\mathrm{db}\otimes 1) & \leftarrow\!\shortmid & G \end{array}$$

□

Now we state and prove a few formulas for the "operations" $(\text{—})^\flat$ and $(\text{—})^*$.

1.2.19. Remark: *We have seen that if X_1, X_2 and Y_1, Y_2 have X_i^* and Y_i^* as left duals, then we can choose $(X_i\otimes Y_i)^* := Y_i^*\otimes X_i^*$ as a left dual for $X_i\otimes Y_i$. With this choice, we get $(S^\flat)^\flat = S^*$ for a morphism $S: X_1\otimes Y_1\to X_2\otimes Y_2$.*

1.2.20. Lemma: *Let $S: X\otimes Y\to V\otimes W$, $f:\tilde{X}\to X$, $g:\tilde{Y}\to Y$, $h: V\to\tilde{V}$ and $j: X\to\tilde{W}$ be morphisms in $\mathcal{C}$.*
Then (if all the dual objects occurring in the respective statements exist) we have:

$$\begin{aligned} (S(f\otimes 1))^\flat &= S^\flat(1\otimes f) && (1)\\ (S(1\otimes g))^\flat &= (1\otimes g^*)S^\flat && (2)\\ ((h\otimes 1)S)^\flat &= S^\flat(h^*\otimes 1) && (3)\\ ((1\otimes j)S)^\flat &= (j\otimes 1)S^\flat && (4) \end{aligned}$$

The lemma can be reduced to (or, more easily, proved similarly to) the following lemma.

1.2.21. Lemma: *Let C, Z and V be objects of a monoidal category $\mathcal{C}$, that admit left duals. Let $S : A \otimes V \to Z \otimes X$ and $T : B \otimes C \to V \otimes Y$ be morphisms in $\mathcal{C}$. Then the morphisms $S^\flat : Z^* \otimes A \to X \otimes V^*$, $T^\flat : V^* \otimes B \to Y \otimes C^*$ and $((S \otimes 1)(1 \otimes T))^\flat : Z^* \otimes A \otimes B \to X \otimes Y \otimes C^*$ satisfy*

$$((S \otimes 1)(1 \otimes T))^\flat = (1 \otimes T^\flat)(S^\flat \otimes 1)$$

PROOF: We have to show that the outer rectangle of the following diagram commutes:

$$\begin{tikzcd}
Z^* \otimes A \otimes B \otimes C \arrow[r, "1 \otimes 1 \otimes T"] \arrow[d, "S^\flat \otimes 1 \otimes 1"'] & Z^* \otimes A \otimes V \otimes Y \arrow[r, "1 \otimes S \otimes 1"] \arrow[d, "S^\flat \otimes 1 \otimes 1"] & Z^* \otimes Z \otimes X \otimes Y \arrow[dd, "\mathrm{ev} \otimes 1 \otimes 1"] \\
X \otimes V^* \otimes B \otimes C \arrow[r, "1 \otimes 1 \otimes T"'] \arrow[d, "1 \otimes T^\flat \otimes 1"'] & X \otimes V^* \otimes V \otimes Y \arrow[dr, "1 \otimes \mathrm{ev} \otimes 1"'] & \\
X \otimes Y \otimes C^* \otimes C \arrow[rr, "1 \otimes 1 \otimes \mathrm{ev}"'] & & X \otimes Y
\end{tikzcd}$$

Now the small rectangle commutes trivially; the other two quadrangles commute by definition of $S^\flat$, tensored with Y on the right, and by definition of $T^\flat$, tensored with X on the left, respectively. □

1.3 Quasisymmetric monoidal categories

We just give the definition of a quasisymmetric category, as we will need it to be able to talk about bialgebras, a notion that does not make sense in a general monoidal category, involving, as it does, a switching of tensorands in the tensor product of an algebra with itself.

For two objects X, Y in a general monoidal category the tensor products $X \otimes Y$ and $Y \otimes X$ are usually not isomorphic.

The definition of a quasisymmetric monoidal categoriy requires the existence of a "reasonable" isomorphism between these two objects.

1.3.1. Definition: *Let $\mathcal{C} = (\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ be a monoidal category.*
Let $\sigma : X \otimes Y \to Y \otimes X$ be a bifunctorial isomorphism.
$(\mathcal{C}, \sigma)$ is said to be a **quasisymmetric monoidal category***, σ is called a* **quasisymmetry***, if the diagram*

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\sigma"] \arrow[d, "\alpha"'] & (Y \otimes Z) \otimes X & Y \otimes (Z \otimes X) \arrow[l, "\alpha"'] \\
(X \otimes Y) \otimes Z \arrow[r, "\sigma \otimes 1"'] & (Y \otimes X) \otimes Z & Y \otimes (X \otimes Z) \arrow[l, "\alpha"] \arrow[u, "1 \otimes \sigma"']
\end{tikzcd}$$

and the analogous diagram with $(\sigma_{BA})^{-1}$ in place of σ_{AB} commute.

σ is called a symmetry and $(\mathcal{C}, \sigma)$ is called symmetric, if in addition $\sigma^{-1}_{X,Y} = \sigma_{Y,X}$.

1.3.2. Remark: *Taking advantage of the coherence results for monoidal categories we can give the two diagrams the shorthand form*

$$\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X,Y \otimes Z}"] \arrow[dr, "\sigma_{X,Y} \otimes 1"'] & & Y \otimes Z \otimes X \\
& Y \otimes X \otimes Z \arrow[ur, "1 \otimes \sigma_{X,Z}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \otimes Z \arrow[rr, "\sigma_{X \otimes Y,Z}"] \arrow[dr, "1 \otimes \sigma_{Y,Z}"'] & & Z \otimes X \otimes Y \\
& X \otimes Z \otimes Y \arrow[ur, "\sigma_{X,Z} \otimes 1"'] &
\end{tikzcd}$$

Quasisymmetric monoidal categories are the generalization of symmetric ones designed to fit the study of quasitriangular (respectively, coquasitriangular) Hopf algebras, [7, p.71]. While for *symmetric* monoidal categories we have coherence theorems very similar to the ones valid for monoidal categories (all diagrams reasonably composed of the monoidal structure and the symmetry commute), coherence theorems for the quasisymmetric case are bound to be more restricted. Indeed, generally the morphisms $id_{X \otimes Y}$ and $\sigma_{YX}\sigma_{XY}$ fail to agree.

Restricted versions of quasisymmetric monoidal categories have been used, [5], to investigate knot and "tangle" invariants of Jones type.

1.3.3. Example: *Let H be a cocommutative k-bialgebra. Then the category of H-modules is symmetric. The symmetry τ for two H-modules M and N is given by*

$$\tau_{MN}(m \otimes n) = n \otimes m$$

on $m \in M, n \in N$.

The definition 1.3.1 imposes a certain compatibility condition on the quasisymmetry and the associativity constraint of the monoidal category. It is not necessary to seperately require compatibility with the unit object ([14, 2.4.1, p.39] in the symmetric case):

1.3.4. Lemma: *If $(\mathcal{C}, \sigma)$ is quasisymmetric, then σ is compatible with the morphisms λ, ρ associated with the unit object I of $\mathcal{C}$ in the sense that*

$$\begin{tikzcd} X \otimes I \arrow[rr, "\sigma"] \arrow[dr, "\rho"'] & & I \otimes X \arrow[dl, "\lambda"] \\ & X & \end{tikzcd}$$

and the same diagram for σ^{-1} commute.

1.3.5. Definition: *A monoidal functor $(\mathcal{F}, \xi)$ between two quasisymmetric monoidal categories $(\mathcal{C}, \sigma)$ and $(\mathcal{D}, \Phi)$ is said to be compatible with the quasisymmetry or just quasisymmetric, if the diagrams*

$$\begin{tikzcd} \mathcal{F}X \otimes \mathcal{F}Y \arrow[r, "\xi_{XY}"] \arrow[d, "\Phi_{\mathcal{F}X,\mathcal{F}Y}"'] & \mathcal{F}(X \otimes Y) \arrow[d, "\mathcal{F}\sigma_{XY}"] \\ \mathcal{F}Y \otimes \mathcal{F}X \arrow[r, "\xi_{YX}"'] & \mathcal{F}(Y \otimes X) \end{tikzcd}$$

commute.

1.4 Monoids and monoidal functors

The following observations about (weakly) monoidal functors and monoids, which we shall not prove, are intended to sketch the "natural setting" for some of the results we will obtain in section 2.3.

We define monoids (or algebras) in monoidal categories in the obvious and well-known way, see [11]. Dually, we can define comonoids (or coalgebras). Denote by $Mon(\mathcal{C})$ the category of monoids in $\mathcal{C}$, and by $Comon(\mathcal{C})$ the category of comonoids. If the category $\mathcal{C}$ is quasisymmetric, the category $Mon(\mathcal{C})$ is a monoidal category, the tensor product of monoids M, N being the tensor product in $\mathcal{C}$ equipped with the multiplication

$$M \otimes N \otimes M \otimes N \xrightarrow{1 \otimes \sigma \otimes 1} M \otimes M \otimes N \otimes N \xrightarrow{\mu \otimes \mu} M \otimes N.$$

1.4.1. Lemma and Definition: *Let $(\mathcal{G}, \chi, \chi_0) : \mathcal{C} \to \mathcal{B}$ be a weakly monoidal functor. Then there is a functor $\mathcal{G}_m : Mon(\mathcal{C}) \to Mon(\mathcal{B})$, agreeing with $\mathcal{G}$ on the underlying objects of $\mathcal{C}$, and such that the monoidal structure on $\mathcal{G}_m(M)$ is defined by*

$$\mu_{\mathcal{G}M} = (\mathcal{G}M \otimes \mathcal{G}M \xrightarrow{\chi} \mathcal{G}(M \otimes M) \xrightarrow{\mathcal{G}(\mu)} \mathcal{G}M)$$

$$\eta_{\mathcal{G}M} = (I \xrightarrow{\chi_0} \mathcal{G}I \xrightarrow{\mathcal{G}(\eta)} \mathcal{G}M).$$

1.4.2. Lemma: *If $\mathcal{G}$ as in the preceding section is a quasisymmetric weakly monoidal functor between quasisymmetric monoidal categories, then $\mathcal{G}_m$ is a monoidal functor.*

1.4.3. Lemma: *Let $\mathcal{B}, \mathcal{C}$ be monoidal categories. Let $\mathcal{F} : \mathcal{B} \to \mathcal{C}$ be a monoidal functor and let $\mathcal{G} : \mathcal{C} \to \mathcal{B}$ be a weakly monoidal functor right adjoint to $\mathcal{F}$. Then $\mathcal{G}_m$ is right adjoint to $\mathcal{F}_m$.*

In particular, if $\mathcal{C}, \mathcal{B}, \mathcal{F}, \mathcal{G}$ are quasisymmetric, then $\mathcal{G}_m$ and $\mathcal{F}_m$ are monoidally adjoint.

Chapter 2

Coendomorphim coalgebras of monoidal categories

Let $\mathcal{A}$ be a monoidal category. Let $\mathcal{C}$ be an arbitrary category and let $\omega : \mathcal{C} \to \mathcal{A}$ be a functor. We take the viewpoint that ω is a "forgetful" or "underlying" functor from $\mathcal{C}$ to $\mathcal{A}$, that is, that objects of $\mathcal{C}$ are objects of $\mathcal{A}$ having an additional structure (in that they are also objects of $\mathcal{C}$). The goal of this chapter is to find a coalgebra in $\mathcal{A}$ such that objects of $\mathcal{C}$ (or, rather, their images under ω) are comodules over this coalgebra. This can be done under moderate assumptions on the functor ω and the category $\mathcal{A}$, in particular in the case that $\mathcal{A}$ is the category of vector spaces and ω maps into the full subcategory of finite-dimensional vector spaces.

In the best possible case the comodule structures constructed on all the $\omega(X)$ for objects $X \in \mathcal{C}$ constitute an equivalence of $\mathcal{C}$ with the category of comodules over the coalgebra constructed. We prove this under suitable assumptions on $\mathcal{C}$ and ω in the case that $\mathcal{A}$ is the category of vector spaces. In this case we may say that all the structure an object of $\mathcal{A}$ has by actually being the image of an object of $\mathcal{C}$ under ω is completely preserved in the comodule structure constructed. (This rather informal statement remains true, of course, if the functor from $\mathcal{C}$ to a category of comodules is only fully faithful.)

Even if this best possible case does not happen to apply, we should still picture the following constructions along these lines: For a category with an underlying functor to $\mathcal{A}$, that is, loosely speaking, for a collection of objects of $\mathcal{A}$ equipped with some kind of additional structure, we seek a coalgebra over which these objects are comodules, such that as much as possible of the additional information is preserved in those comodule structures. The phrase "as much as possible" will have its formal counterpart in the fact that the coalgebra constructed will be the solution to a universal problem.

For the rest of the chapter we will be concerned with the problem of how additional structures of the category are mirrored in structures and properties of the coalgebra C we have constructed. In particular, we will obtain:

- If $\mathcal{C}$ is monoidal and ω is monoidal, then C is a bialgebra.

- If $\mathcal{C}$ is rigid monoidal and ω is monoidal, then C is a Hopf algebra.

The techniques used are for the most part taken from Ulbrich's papers [17] and [18].

It was natural to develop as much of the theory as possible in the general setting of a (suitable type of) monoidal category instead of the category of k-modules or vector spaces. However, the results of 2.2 are only proved in the original case of the category of vector spaces. In particular it is unclear in the general case under what restrictions the correspondence between coalgebras and categories with underlying functors will be an equivalence.

For all of the second chapter we will make the following
General assumptions: Let $\mathcal{A}$ be a monoidal category whose neutral object is denoted by k. Let $\mathcal{A}_0$ be the full subcategory of $\mathcal{A}$ consisting of all the objects having left duals in the sense of 1.2.3. Of course, if $\mathcal{A}$ is quasisymmetric, then $\mathcal{A}_0$ is rigid. We assume $\mathcal{A}$ to be cocomplete, and we assume the tensor product to be compatible with arbitrary colimits in each argument.

The main example will be $\mathcal{A} = \underline{Vec}$, the category of vector spaces over a fixed field k with $\mathcal{A}_0 = \underline{vec}$, the category of finite dimensional spaces.

We will make free use of the notions algebra and coalgebra in an arbitrary monoidal category $\mathcal{A}$. (These objects are called monoids and comonoids in [11] and [13])

For C a coalgebra in $\mathcal{A}$ let $\mathcal{M}^C$ denote the category of those right C-comodules whose underlying objects in $\mathcal{A}$ actually belong to $\mathcal{A}_0$. In the case $\mathcal{A} = \underline{Vec}$ this is nothing else than the category of finite dimensional right comodules.

In the following we also make use of Sweedler's sigma notation. We will not write the summation symbol, as in $\Delta(c) = c_{(1)} \otimes c_{(2)}$ etc. This notation of course makes sense only if we are dealing with a category $\mathcal{A}$ in which objects are at least modules over a ring (say $\mathcal{A} = \underline{Vec}$ or $\mathcal{A} =$ (modules over a bialgebra).) However, using Yoneda's lemma, one can introduce the well known calculus of "generalized elements" of objects of a general category, see [11] for an example involving monoidal structures. Then the notation above can be defined to make sense, although now $c_{(1)} \otimes c_{(2)}$ will of course in general not be anything like a sum or even finite sum over decomposable tensors. Since I do not want to bother introducing generalized elements for this purpose, the reader may apply occurrences of Sweedler's notation only to the cases where it is *obviously* meaningful.

2.1 Coendomorphism coalgebras

Let *Coalg* be the category of coalgebras in the monoidal category $\mathcal{A}$.

To each category equipped with an "underlying" functor to the category $\mathcal{A}_0$ we will construct a "universal" coalgebra such that the category in question can be embedded in the category of all rigid right comodules over this coalgebra. It will turn out that this construction assigning a coalgebra to a category defines a left adjoint to the functor assigning to a coalgebra the category of its rigid comodules.

Of course, to obtain such a result one has to define a suitable category whose objects are categories equipped with functors to $\mathcal{A}_0$:

2.1.1. Definition: *The category $\mathfrak{C} = \mathfrak{C}(\mathcal{A})$ has objects the pairs $(\mathcal{C}, \omega)$, consisting of a category $\mathcal{C}$ and a functor $\omega : \mathcal{C} \to \mathcal{A}_0$. Morphisms in $\mathfrak{C}$ from $(\mathcal{C}, \omega)$ to $(\mathcal{C}', \omega')$ are equivalence classes $[\mathcal{F}, \zeta]$ of pairs $(\mathcal{F}, \zeta)$ consisting of a functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ and a natural isomorphism $\zeta : \omega \cong \omega'\mathcal{F}$.*
Two such pairs $(\mathcal{F}, \zeta), (\mathcal{G}, \vartheta)$ shall be considered as equivalent iff there is a natural isomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ satisfying

$$\begin{tikzcd} \omega \arrow[r, "\zeta"] \arrow[dr, "\vartheta"'] & \omega'\mathcal{F} \arrow[d, "\omega'\varphi"] \\ & \omega'\mathcal{G} \end{tikzcd}$$

In this situation, We will also write $(\mathcal{F}, \zeta) \overset{\varphi}{\sim} (\mathcal{G}, \vartheta)$.

Composition of morphisms in $\mathfrak{C}$ is defined by

$$[\mathcal{F}, \zeta] \circ [\mathcal{G}, \vartheta] = [\mathcal{F}\mathcal{G}, (\zeta\mathcal{G})\vartheta]$$

(It is easily verified that this is indeed well defined.)

The category $\mathfrak{C}_0$ has the same objects as $\mathfrak{C}$, but morphisms only those functors $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ satisfying $\omega = \omega'\mathcal{F}$.

2.1.2. Remark: *I ignored any set theoretic difficulties in this paper. The above definition (category of all pairs consisting of a category and ...) is obviously set theoretically impossible. It is of course possible to remedy its deficiencies using, say, universes in a standard way.*

The following lemma may serve as a first hint that the definition of $\mathfrak{C}$ is somewhat reasonable.

2.1.3. Lemma: $[\mathcal{F}, \zeta] : (\mathcal{C}, \omega) \to (\mathcal{D}, \nu)$ *is an isomorphism in $\mathfrak{C}$ iff $\mathcal{F}$ is a category equivalence.*

PROOF: Evidently $\mathcal{F}$ is an equivalence if $[\mathcal{F},\zeta]$ is an isomorphism.

Now let $[\mathcal{F},\zeta]$ be a morphism in $\mathfrak{C}$ as above and let $\mathcal{G}$ be a right adjoint of $\mathcal{F}$ such that the adjunction morphisms $\eta : \mathcal{I}d \to \mathcal{G}\mathcal{F}$ and $\varepsilon : \mathcal{F}\mathcal{G} \to \mathcal{I}d$ are isomorphisms.

Then if we set

$$\vartheta = (\nu \xrightarrow{\nu\varepsilon^{-1}} \nu\mathcal{F}\mathcal{G} \xrightarrow{\zeta^{-1}\mathcal{G}} \omega\mathcal{G})$$

the morphism $[\mathcal{G},\vartheta] : (\mathcal{D},\nu) \to (\mathcal{C},\omega)$ becomes an inverse for $[\mathcal{F},\zeta]$ in $\mathfrak{C}$. Indeed

$$\begin{aligned}
(\mathcal{G},\vartheta)(\mathcal{F},\zeta) &= (\mathcal{G}\mathcal{F},\ (\vartheta\mathcal{F})\,\zeta\ ) \\
&= (\mathcal{G}\mathcal{F},\ (\zeta^{-1}\mathcal{G}\mathcal{F})\,(\nu\varepsilon^{-1}\mathcal{F})\,\zeta\ ) \\
&= (\mathcal{G}\mathcal{F},\ (\zeta^{-1}\mathcal{G}\mathcal{F})\,(\nu\mathcal{F}\eta)\,\zeta\ ) && (1) \\
&= (\mathcal{G}\mathcal{F},\omega\eta) && (2) \\
&\overset{\eta^{-1}}{\sim} (\mathcal{I}d,1)
\end{aligned}$$

The equality (1) follows from one of the triangle equations for the adjunction, (2) uses naturality of ζ.

$[\mathcal{F},\zeta] \circ [\mathcal{G},\vartheta] = [\mathcal{I}d,1]$ is proved in an analogous manner. $\square$

The mapping $C \mapsto \mathcal{M}^C$ assigning to a coalgebra C the category of rigid right C-comodules can be considered in the obvious manner as a functor $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}_0$ or $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}$.

We will show that each of these two functors has a left adjoint.

To describe the construction of the universal coalgebra we're after, the notions of dinatural transformations, ends and coends will prove useful. We will recall very briefly the treatment found in [6, p.214-223]

2.1.4. Definition: *A* **diagonally natural** *or* **dinatural** *transformation $\mu : \mathcal{H} \to \mathcal{I}$ between two bifunctors $\mathcal{H},\mathcal{I} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a family of morphisms $\mu_x : \mathcal{H}(x,x) \to \mathcal{I}(x,x)$ satisfying*

$$\begin{tikzcd}
 & \mathcal{H}(x,x) \arrow[r, "\mu_x"] & \mathcal{I}(x,x) \arrow[dr, "\mathcal{I}(1,f)"] & \\
\mathcal{H}(y,x) \arrow[ur, "\mathcal{H}(f,1)"] \arrow[dr, "\mathcal{H}(1,f)"'] & & & \mathcal{I}(x,y) \\
 & \mathcal{H}(y,y) \arrow[r, "\mu_y"'] & \mathcal{I}(y,y) \arrow[ur, "\mathcal{I}(f,1)"'] &
\end{tikzcd}$$

for each morphism $f : x \to y$ in $\mathcal{C}$.

2.1.5. Remark: *As a special case we can consider dinatural transformations between functors that do not depend of both their arguments.*

For example a dinatural transformation between a functor $\mathcal{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ and a functor taking the constant value $d \in \mathcal{D}$ is a family $\mu_x : \mathcal{H}(x,x) \to d$, of morphisms making the diagrams

$$\begin{tikzcd}
\mathcal{H}(y,x) \arrow[r, "\mathcal{H}(1,f)"] \arrow[d, "\mathcal{H}(f,1)"'] & \mathcal{H}(y,y) \arrow[d, "\mu_y"] \\
\mathcal{H}(x,x) \arrow[r, "\mu_x"'] & d
\end{tikzcd}$$

commute for all $f : x \to y$ in $\mathcal{C}$. A dinatural transformation from (or to) a bifunctor to (or from) a constant is called a **wedge**.

The structure morphisms db and ev of the left dual objects in a rigid monoidal category are typical examples for dinatural transformations. Indeed the dinaturality of these morphisms ist just a reformulation of the definition of the functor $(\,—\,)^*$.

2.1.6. Definition: *Let $\mathcal{H} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor.*
An **end** *$E = \int_{c\in\mathcal{C}} \mathcal{H}(c,c)$ of $\mathcal{H}$ is an object E of $\mathcal{D}$ together with a dinatural transformation $\mu : E \to \mathcal{H}$, which is final as a dinatural transformation of a constant functor to $\mathcal{H}$.*

A **coend** *of $\mathcal{H}$ is an object $C = \int^{c\in\mathcal{C}} \mathcal{H}(c,c)$ together with a diagonally natural transformation $\mu : \mathcal{H} \to C$, being initial as a dinatural transformation of $\mathcal{H}$ to a constant.*

Thus an end (or coend) is defined to be a universal wedge (like a (co-)limit is a universal cone).

2.1.7. Remark: *It is easy to see, [6, p.220, Prop. 1], that an end of a functor is a limit of a suitable diagram. Explicitly, an end of $\mathcal{H}$ is a difference kernel of*

$$\prod_{X\in \mathrm{Ob}\,\mathcal{C}} \mathcal{H}(X,X) \underset{F_Z}{\overset{F_Q}{\rightrightarrows}} \prod_{f\in \mathcal{M}or\mathcal{C}} \mathcal{H}(\mathrm{dom}(f), \mathrm{cod}(f))$$

the morphisms F_Q and F_Z being defined by

$$\mathrm{proj}_f F_Q = \mathcal{H}(\mathrm{dom}(f), f)\mathrm{proj}_{\mathrm{cod}(f)}$$

$$\mathrm{proj}_f F_Z = \mathcal{H}(f, \mathrm{cod}(f))\mathrm{proj}_{\mathrm{dom}(f)}.$$

A coend is a colimit of a dual diagram.

In particular, each functor to a (co-)complete category has a (co-)end. Functors preserving (co-)limits also preserve (co-)ends. Contravariant representable functors transform coends into ends.

The notion of an end is designed to comprise as an example the set of all natural transformations between two functors:

Indeed, the set of natural transformations between two functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is

$$\begin{aligned}
&\mathrm{Nat}(\mathcal{F}, \mathcal{G}) \\
&= \left\{ (\varphi_X | X \in \mathcal{C}) \in \prod_{X \in \mathcal{C}} \mathcal{M}or_{\mathcal{D}}(\mathcal{F}X, \mathcal{G}X) \,\middle|\, \begin{array}{l} \forall f : X \to Y \text{ in } \mathcal{C} : \\ \varphi_Y \mathcal{F}(f) = \mathcal{G}(f) \varphi_X \end{array} \right\} \\
&= \left\{ (\varphi_X) \in \prod_{X \in \mathcal{C}} \mathcal{M}or_{\mathcal{D}}(\mathcal{F}X, \mathcal{G}X) \,\middle|\, \begin{array}{l} \forall f : X \to Y \text{ in } \mathcal{C} : \\ \mathcal{M}or(\mathcal{F}f, \mathcal{G}Y)(\varphi_Y) = \mathcal{M}or(\mathcal{F}X, \mathcal{G}f)(\varphi_X) \end{array} \right\} \\
&= \int_{X \in \mathcal{C}} \mathcal{M}or_{\mathcal{D}}(\mathcal{F}X, \mathcal{G}X),
\end{aligned}$$

the standard construction of an end of the bifunctor $\mathcal{M}or_{\mathcal{D}}(\mathcal{F}(-), \mathcal{G}(-))$ in the category of sets.

Thus in particular the endomorphism monoid of a functor, that is to say the monoid consisting of all functorial morphisms of a functor to itself, is an end in the category of sets.

Now it is easy to believe that a coend gives the desired universal coalgebra to be assigned to an object of $\mathfrak{C}$. Indeed the ring of endomorphisms of a k-module is (co-)universal among the algebras acting on that module. Likewise the ring of endomorphisms of a functor $\omega : \mathcal{C} \to \underline{vec}$ will be couniversal among the algebras acting on all $\omega(X)$ in such a way that these operations define a natural transformation. For technical reasons[1] we do all constructions for comodules and coalgebras rather than modules and algebras and will thus construct a kind of "coendomorphism coalgebra".

2.1.8. Definition: *For an object $(\mathcal{C}, \omega) \in \mathfrak{C}$. we call*

$$\underline{\mathrm{coend}}(\omega) = \underline{\mathrm{coend}}(\mathcal{C}, \omega) = \int^{X \in \mathcal{C}} \omega(X)^* \otimes \omega(X)$$

the **coendomorphism coalgebra** *of $\omega = (\mathcal{C}, \omega)$.*

Note that, in view of our general assumptions on $\mathcal{A}$, $\underline{\mathrm{coend}}(\omega)$ always exists.

The following lemma just lists a few equivalent characterizations of the coendomorphism coalgebra.

2.1.9. Lemma: *For $(\mathcal{C}, \omega) \in \mathfrak{C}$ and $C \in \mathcal{A}$ the following statements are equivalent:*

(1) $C \cong \underline{\mathrm{coend}}(\omega)$

(2) The functor $\mathcal{A} \ni M \mapsto \mathrm{Nat}(\omega, \omega \otimes M) \in \underline{Set}$ ist representable with representing object C.

[1] These reasons apply to the main example $\mathcal{A} = \underline{Vec}$: Tensor products preserve colimits, not generally limits; if one knows all the finite dimensional C-comodules, one knows all C-comodules.

(3) There is a natural transformation $\delta^\omega : \omega \to \omega \otimes C$, such that for each $M \in \mathcal{A}$ and each natural transformation $\varphi : \omega \to \omega \otimes M$ there is a unique morphism $f : C \to M$ satisfying $(\omega \otimes f)\delta^\omega = \varphi$:

$$\begin{tikzcd}
\omega \arrow[r, "\delta^\omega"] \arrow[dr, "\varphi"'] & \omega \otimes C \arrow[d, "\omega \otimes f"] & C \arrow[d, "\exists_1 f"] \\
& \omega \otimes M & M
\end{tikzcd}$$

More precisely representations $\psi^\omega : \mathcal{M}or(C, —) \cong \mathrm{Nat}(\omega, \omega \otimes —)$ and arrows $\delta^\omega : \omega \to \omega \otimes C$ as in (2) are in bijective correspondence defined by $\delta^\omega = \psi^\omega_C(1_C)$ and $\psi^\omega_M(f) = (1 \otimes f)\delta^\omega$, respectively.

PROOF:

(1)⇔(2) We have

$$\begin{aligned}
&\mathrm{Nat}(\omega, \omega \otimes M) \\
&\quad = \int_{X \in \mathcal{C}} \mathcal{M}or(\omega(X), \omega(X) \otimes M) \\
&\quad = \int_{X \in \mathcal{C}} \mathcal{M}or(\omega(X)^* \otimes \omega(X), M) \\
&\quad = \mathcal{M}or(\int^{X \in \mathcal{C}} \omega(X)^* \otimes \omega(X), M)
\end{aligned}$$

I will spell out explicitly the bijection thus established between natural transformations $\omega \to \omega \otimes M$ and diagonally natural transformations $\omega^* \otimes \omega \to M$: A natural transformation $\varphi : \omega \to \omega \otimes M$ defines a dinatural transformation with components

$$\omega(X)^* \otimes \omega(X) \xrightarrow{1 \otimes \varphi} \omega(X)^* \otimes \omega(X) \otimes M \xrightarrow{\mathrm{ev} \otimes 1} M$$

On the other hand a dinatural transformation $\kappa : \omega^* \otimes \omega \to M$ defines a natural transformation with components

$$\omega(X) \xrightarrow{\mathrm{db} \otimes 1} \omega(X) \otimes \omega(X)^* \otimes \omega(X) \xrightarrow{1 \otimes \kappa} \omega(X) \otimes M.$$

(2)⇔(3) A functor is representable iff it has a universal element, which in this context is just a δ^ω as in (2). The usual bijective correspondence between representations of a functor and universal elements given in the general case by Yoneda's lemma specializes to the correspondence claimed above.

□

2.1.10. Definition: *For $(\mathcal{C}, \omega) \in \mathfrak{C}$ let $\psi : \mathcal{M}or(C, —) \cong \mathrm{Nat}(\omega, \omega \otimes —)$ be a representation and let $\delta = \delta^\omega : \omega \to \omega \otimes C$ be the corresponding "universal arrow"[2] Define*

[2] Which is of course not a universal arrow in the usual sense of, say, [6, p.55].

$\Delta : C \to C \otimes C$ by $\psi_{C\otimes C}(\Delta) = (\delta \otimes 1)\delta$, that is, by commutativity of

$$
\begin{tikzcd}
\omega \arrow[r, "\delta"] \arrow[d, "\delta"'] & \omega \otimes C \arrow[d, "1 \otimes \Delta"] \\
\omega \otimes C \arrow[r, "\delta \otimes 1"'] & \omega \otimes C \otimes C
\end{tikzcd}
$$

We shall see shortly that C is a coalgebra with comultiplication Δ.

I give an alternative description of the comultiplication of $C = \underline{\mathrm{coend}}(\omega)$: It is the unique morphism Δ satisfying

$$
\begin{tikzcd}
(\omega X)^* \otimes \omega X \arrow[r, "\mu_X"] \arrow[d, "1 \otimes \mathrm{db} \otimes 1"'] & C \arrow[d, "\Delta"] \\
(\omega X)^* \otimes \omega X \otimes (\omega X)^* \otimes \omega X \arrow[r, "\mu_X \otimes \mu_X"'] & C \otimes C
\end{tikzcd}
$$

This is an immediate consequence of the correspondence between dinatural transformations $\omega^* \otimes \omega \to M$ and natural transformations $\omega \to \omega \otimes M$ described in 2.1.9.

The following formula lies at the ground of all calculations with $\underline{\mathrm{coend}}(\omega)$.

2.1.11. Lemma: *Let $\mathcal{C}, \omega, C, \psi, \delta$ and Δ be as above,*
let $f : C \to M$ and $g : C \to N$. Then

$$\psi_{M\otimes N}((f \otimes g)\Delta) = (\psi_M(f) \otimes 1)\psi_N(g)$$

PROOF: The diagram

$$
\begin{tikzcd}
\omega \arrow[rrrr, "\delta"] \arrow[drr, "\delta"] \arrow[dd, "\psi_N(g)"'] & & & & \omega \otimes C \arrow[d, "\omega \otimes \Delta"] \arrow[dll, phantom, "(1)"] \\
& (2) & \omega \otimes C \arrow[rr, "\delta \otimes 1"] \arrow[dll, "\omega \otimes g"'] \arrow[drr, "\psi_M(f) \otimes 1"] & & \omega \otimes C \otimes C \arrow[d, "\omega \otimes f \otimes 1"] \arrow[dl, phantom, "(3)"] \\
\omega \otimes M \arrow[drr, "\psi_M(f) \otimes 1"'] & & (4) & & \omega \otimes M \otimes C \arrow[dll, "\omega \otimes 1 \otimes g"] \\
& & \omega \otimes M \otimes N & &
\end{tikzcd}
$$

commutes at (1) by definition of Δ; at (2) by definition of ψ; at (3) by definition of ψ, tensored with C from the right; at (4) trivially.
Commutativity of the "outer polygon" proves the lemma. $\square$

2.1.12. Theorem: $(\mathcal{C},\omega) \mapsto \underline{\text{coend}}(\omega)$ *defines a left adjoint functor for* $\mathcal{M}^{-} : \underline{Coalg} \to \mathfrak{C}$.

PROOF: Let $(\mathcal{C},\omega) \in \mathfrak{C}$, and let $C = \underline{\text{coend}}(\omega)$ with δ, Δ and ψ as above.

(1) Δ is a coalgebra structure for C, the counit ε being defined by $\psi_k(\varepsilon) = 1_\omega$, that is by commutativity of

$$\begin{tikzcd} \omega \arrow[r, "\delta"] \arrow[dr, "1"'] & \omega \otimes C \arrow[d, "\omega \otimes \varepsilon"] \\ & \omega \end{tikzcd}$$

Indeed Δ is coassociative, for

$$\begin{aligned} \psi_{C\otimes C\otimes C}(\,(\Delta \otimes 1)\Delta\,) &= (\psi_{C\otimes C}(\Delta) \otimes 1)\;\psi_C(1) && \text{by 2.1.11} \\ &= (\,((\delta \otimes 1)\delta) \otimes 1)\;\delta && \text{by def. of } \Delta, \delta \\ &= (\delta \otimes 1 \otimes 1)\,(\delta \otimes 1)\,\delta && \\ &= (\delta \otimes 1 \otimes 1)\,\psi_{C\otimes C}(\Delta) && \text{by def. of } \Delta \\ &= (\psi_C(1) \otimes 1)\,\psi_{C\otimes C}(\Delta) && \text{by def. of } \delta \\ &= \psi_{C\otimes C\otimes C}(\,(1 \otimes \Delta)\Delta\,) && \text{by 2.1.11} \end{aligned}$$

ε is a counit, and thus (C,Δ) is a coalgebra; for instance

$$\begin{aligned} \psi_{C\otimes k}((\varepsilon \otimes 1)\Delta) &= (\psi_k(\varepsilon) \otimes 1)\psi_C(1) \\ &= \psi_C(1) \end{aligned}$$

(2) It is immediate from the definition of the coalgebra structure of C that $\delta : \omega \to \omega \otimes C$ makes each ωX a right C-comodule. The naturality of δ guarantees that for each morphism f in the category $\mathcal{C}$ the underlying morphism ωf is C-colinear. Hence the comodule structures of the ωX define a functor $\mathcal{I}_{\mathcal{C}} : \mathcal{C} \to \mathcal{M}^C$ satisfying $\mathcal{U}_C \mathcal{I}_{\mathcal{C}} = \omega$; in particular they define a morphism in in $\mathfrak{C}$ from $(\mathcal{C},\omega)$ to $(\mathcal{M}^C, \mathcal{U}_C)$.

(3) The morphism $[\mathcal{I}_{\mathcal{C}}, 1]$ described in this way solves the universal problem defined by the functor $\mathcal{M}^{-}$, and thus defines a left adjoint for it.

To prove this let $[\mathcal{T},\tau] : (\mathcal{C},\omega) \to (\mathcal{M}^B, \mathcal{U}_B)$ be a morphism in $\mathfrak{C}$.
Let $\delta^B : \mathcal{U}_B \to \mathcal{U}_B$ denote the natural transformation with components the comodule structure maps.
Via the natural transformation

$$\omega \xrightarrow{\tau} \mathcal{U}_B\mathcal{T} \xrightarrow{\delta^B\mathcal{T}} \mathcal{U}_B\mathcal{T} \otimes B \xrightarrow{\tau^{-1}\otimes 1} \omega \otimes B$$

each ωX for $X \in \mathcal{C}$ then becomes a right B-comodule.

Now our assertion is that there is a unique coalgebra morphism $f : C \to B$ satisfying $[\mathcal{M}^f, 1][\mathcal{I}_{\mathcal{C}}, 1] = [\mathcal{T},\tau]$. This is an equality in $\mathfrak{C}$, of course, requiring the

existence of a morphism $\tilde{\tau} : \mathcal{M}^f\mathcal{I}_{\mathcal{C}} \to \mathcal{T}$ satisfying $\mathcal{U}_B\tilde{\tau} = \tau$, that is, requiring τ to be B-colinear.
Thus we only have to show the existence of a unique coalgebra morphism $f : C \to B$ making

$$\begin{tikzcd}
\omega \arrow[r, "\delta^\omega"] \arrow[d, "\tau"'] & \omega \otimes C \arrow[r, "1 \otimes f"] & \omega \otimes B \arrow[d, "\tau \otimes 1"] \\
\mathcal{U}_B\mathcal{T} \arrow[rr, "\delta^B\mathcal{T}"'] & & \mathcal{U}_B\mathcal{T} \otimes B
\end{tikzcd}$$

commute, and, of course, we have to prove independence of this f of the representant chosen for $[\mathcal{T}, \tau]$.

Now the existence of a unique *morphism in* $\mathcal{A}$ with this property is a direct consequence of the universal property 2.1.9 of C, the commutativity of the diagram being equivalent to $\psi_B(f) = (\tau^{-1} \otimes B)\delta^B\tau$.

Furthermore, the f thus defined is indeed a morphism of coalgebras, since

$$\begin{aligned}
\psi_{B\otimes B}((f \otimes f)\Delta) &= (\psi_B(f) \otimes 1_B)\psi_B(f) \\
&= (\tau^{-1} \otimes 1_B \otimes 1_B)(\delta^B \otimes 1_B)(\tau \otimes 1_B)(\tau^{-1} \otimes 1_B)\delta^B\tau \\
&= (\tau^{-1} \otimes 1_B \otimes 1_B)(\delta^B \otimes 1_B)\delta^B\tau \\
&= (\tau^{-1} \otimes 1_B \otimes 1_B)(\omega \otimes \Delta_B)\delta^B\tau
\end{aligned}$$

(the last step being an axiom for B-comodules), and on the other hand

$$\begin{aligned}
\psi_{B\otimes B}(\Delta_B f) &= (\omega \otimes \Delta_B)\psi_B(f) \\
&= (\omega \otimes \Delta_B)(\tau^{-1} \otimes 1_B)\delta^B\tau
\end{aligned}$$

$\varepsilon_B f = \varepsilon$ can be obtained even more easily.

Finally, let $[\mathcal{T}', \tau'] = [\mathcal{T}, \tau]$ in $\mathfrak{C}$, and let $\varphi : \mathcal{T} \to \mathcal{T}'$ with $(\mathcal{T}, \tau) \overset{\varphi}{\sim} (\mathcal{T}', \tau')$. Then in the diagram

$$\begin{tikzcd}
& \mathcal{U}_B\mathcal{T} \arrow[r, "\delta^B_{\mathcal{T}}"] \arrow[d, "\mathcal{U}_B\varphi"'] & \mathcal{U}_B\mathcal{T} \otimes B \arrow[d, "\mathcal{U}_B\varphi \otimes B"'] & \\
\omega \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & & & \omega \otimes B \arrow[ul, "\tau \otimes B"'] \arrow[dl, "\tau' \otimes B"] \\
& \mathcal{U}_B\mathcal{T}' \arrow[r, "\delta^B_{\mathcal{T}'}"'] & \mathcal{U}_B\mathcal{T}' \otimes B &
\end{tikzcd}$$

the rectangle commutes since φ is a morphism in $\mathcal{M}^B$, the two triangles commute by choice of φ. In particular the morphism f defined above does not depend on the choice of $(\mathcal{T}, \tau)$ or $(\mathcal{T}', \tau')$ as a representant for $[\mathcal{T}, \tau]$.

For future use we note the explicit form of the morphism $f = \underline{\text{coend}}(\mathcal{F},\zeta)$ for a morphism $[\mathcal{F},\zeta] : (\mathcal{C},\omega) \to (\mathcal{D},\nu)$ in $\mathfrak{C}$:
It is the unique morphism in $\mathcal{A}$ making

$$\begin{tikzcd}
\omega \arrow[r, "\delta^\omega"] \arrow[d, "\zeta"'] & \omega \otimes \underline{\text{coend}}(\omega) \arrow[d, "\zeta \otimes f"] \\
\nu\mathcal{F} \arrow[r, "\delta^\nu \mathcal{F}"'] & \nu\mathcal{F} \otimes \underline{\text{coend}}(\nu)
\end{tikzcd},$$

commute, that is, the morphism uniquely defined by $\psi^\omega_{\underline{\text{coend}}(\nu)}(f) = (\zeta \otimes 1)^{-1}(\delta^\nu \mathcal{F})\zeta$. □

2.1.13. Remark: *The theorem just proved shows in particular that the coalgebra morphism assigned to a functor $\mathcal{F}$ and isomorphism ζ only depends on the equivalence class $[\mathcal{F},\zeta]$. In connection with this fact the equivalence relation central in the definition of $\mathfrak{C}$ is mentioned in [14, 2.1.2.3, p.116/117].*

2.1.14. Corollary: $\underline{\text{coend}}$ *also defines a left adjoint for the functor* $\mathcal{M}^{\,-} : \underline{\mathit{Coalg}} \to \mathfrak{C}_0$.

Proof: Indeed in the case that $\tau = id$, the equality $\mathcal{M}^f \mathcal{I}_\mathcal{C} = \mathcal{T}$ obtained in the proof above is valid even in $\mathfrak{C}$. □

Now we have proved that $\underline{\text{coend}}(\omega)$ is a coalgebra and the morphisms $\delta^\omega_X : \omega(X) \to \omega(X) \otimes \underline{\text{coend}}(\omega)$ are comodule structures on $\omega(X)$, we will also make use of Sweedler's notation.

The morphism $f = \underline{\text{coend}}(\mathcal{F},\zeta)$ at the end of the proof of 2.1.12 can then be characterized by

$$\delta^\nu_{\mathcal{F}(X)}\zeta(x) = \zeta(x_{(0)}) \otimes f(x_{(1)}).$$

To be able to do actual calculations with examples of coendomorphism coalgebras or at least compare the definition with alternative ones, we have to get at a presentation of $\underline{\text{coend}}(\omega)$ by generators and relations. For the rest of this section we restrict $\mathcal{A}$ to be the category $\underline{\mathit{Vec}}$ of vector spaces.

2.1.15. Definition: *Let $(\mathcal{C},\omega) \in \mathfrak{C}$. Let $X \in \mathcal{C}$. Let $\varphi \in (\omega V)^*$ and $v \in \omega V$. Denote the image of $\varphi \otimes v$ in $C = \underline{\text{coend}}(\omega)$ by $[\varphi|v]$. Thus we have $[\varphi|v] = \langle \varphi, v_{(0)}\rangle v_{(1)}$.*

Just rewriting the construction of the coendomorphism coalgebra as a coend or a special colimit, we now obtain the following lemma.

2.1.16. Lemma: *The coalgebra $C = \underline{\mathrm{coend}}(\omega)$ is generated as an abelian group by the elements $[\varphi|v]$ from 2.1.15. More precisely, as a vector space over k:*

$$C \cong \bigoplus_{V \in \mathcal{C}} (\omega V)^* \otimes (\omega V)/N$$

where the vector subspace N is generated by all the elements of the form

$$(\omega f)^* \psi \otimes v - \psi \otimes (\omega f) v \qquad V, W \in \mathcal{C}, f : V \to W, v \in \omega V, \psi \in (\omega W)^*$$

The following formulas hold (for $v_i \otimes v^i$ the element "basis and dual basis" in $V \otimes V^*$.)

$$\begin{aligned} \delta_V(v) &= v_i \otimes [v^i|v] \\ \Delta([\varphi|v]) &= [\varphi|v^i] \otimes [v_i|v] \\ \varepsilon([\varphi|v]) &= \langle \varphi, v \rangle \end{aligned}$$

We collect some more facts:

If we choose a basis $\{v_i\}$ in $\omega(V)$ then using the dual basis we can form the elements $\mathcal{T}^i_j := [v^i|v_j] \in C$. The formulas above now read

$$\begin{aligned} \delta(v_j) &= v_i \otimes \mathcal{T}^i_j \\ \Delta(\mathcal{T}^i_j) &= \mathcal{T}^i_k \otimes \mathcal{T}^k_j \\ \varepsilon(\mathcal{T}^i_j) &= \delta^i_j \end{aligned}$$

Thus the elements $\mathcal{T}^i_j$ form a multiplicative matrix in the coalgebra C^{cop} in the sense of [9, p.14].)

Clearly the expression $[\varphi|v]$ is k-bilinear in φ and v.

If we assume $\mathcal{C}$ to be k-linear additive and ω to be k-linear, we can reduce the set of generators:

2.1.17. Definition: *A family of objects in a k-linear additive category is a generating set of objects iff any object of the category is a subobject of a quotient object of a finite biproduct of members of the family.*

2.1.18. Corollary: *Let $(\mathcal{C}, \omega) \in \mathfrak{C}$, assume $\mathcal{C}$ to be k-linear additive and ω to be k-linear.*

Then $C = \underline{\mathrm{coend}}(\omega)$ consists entirely of elements of the form $[\varphi|v]$ from 2.1.15.

If on the other hand $(V_\alpha|\alpha \in N)$ is a generating family of objects in $\mathcal{C}$, and if we choose a basis $\{v_{\alpha,i}\}$in each of the $\omega(V_\alpha)$, then (if we denote the dual bases by $\{v^i_\alpha\}$) C is spanned as a k-vector space by all the elements $[v^i_\alpha|v_{\alpha,j}]$.

The lemma (which we shall not prove) provides us with smaller sets of generators for the vector space C. We can also find a smaller set of relations:

2.1.19. Definition: *Call a set of morphisms in a category a generating set of morphisms iff all the morphisms in the category can be obtained, starting from members of the family (and identities), in a finite number of steps by composing and forming inverses of isomorphisms.*

2.1.20. Lemma: *If F is a generating set of morphisms in $\mathcal{C}$, then the vector space N of relations of C (as defined in 2.1.18) is spanned by the elements*

$$(\omega f)^*\psi \otimes v - \psi \otimes (\omega f)v \quad \textit{for } f \in F$$

2.2 Characterizing the categories of comodules

This section will deal with the question of how precisely the coalgebra constructed in the previous section corresponds to the original category.

In this section $\mathcal{A}$ shall always be the category $\underline{Vec}$ of vector spaces.

First we strongly expect that constructing the coendomorphism coalgebra of a category of comodules over a given coalgebra (and its underlying functor) yields this same coalgebra (up to an isomorphism, of course). Indeed, more precisely, the counits $\underline{\text{coend}}(\mathcal{M}^C) \to C$ of the adjunction from 2.1.12 are isomorphisms. This means that the category $\underline{Coalg}$ can be considered as a reflective subcategory of $\mathfrak{C}$ via the functor $\mathcal{M}^{-}$.

2.2.1. Lemma: *Let C be a k-coalgebra. Let $\mathcal{U} : \mathcal{M}^C \to \underline{vec}$ be the underlying functor. Then for each vector space V*

$$\begin{array}{rcl} \mathrm{Hom}_k(C,V) & \to & \mathrm{Nat}(\mathcal{U}, \mathcal{U} \otimes V) \\ f & \mapsto & (1 \otimes f)\delta^C \end{array}$$

is an isomorphism. In particular, the counit of the adjunction from 2.1.12 is an isomorphism of k-coalgebras.

PROOF: denote the mapping defined above by ψ. The inverse mapping $\overline{\psi} : \mathrm{Nat}(\mathcal{U}, \mathcal{U} \otimes V) \to \mathrm{Hom}_k(C,V)$ can be given explicitly as follows:
Let $\varphi : \mathcal{U} \to \mathcal{U} \otimes V$ be a natural transformation, let $c \in C$. Then there is a finite dimensional sub-comodule $N \subset C$ containing c. We define

$$\overline{\psi}(\varphi)(c) = (\varepsilon|_N \otimes 1)\varphi_N(c).$$

Since C is the union of its finite dimensional subcomodules, and φ, being a natural transformation, is compatible with the inclusions between the subcomodules, this definition is independent of the choice of N.

We have to check that ψ and $\overline{\psi}$ are mutually inverse:

(1) Let $f \in \mathrm{Hom}_k(C,V)$. Let $c \in C$ and let N be a finite dimensional subcomodule of C containing c. Then

$$\begin{array}{rcl} \overline{\psi}\psi(f)(c) & = & (\varepsilon \otimes 1)\psi(f)_N(c) \\ & = & (\varepsilon \otimes f)\delta_N(c) = f(c) \end{array}$$

for since N is a subcomodule in C, we have $(\varepsilon \otimes 1)\delta_N = id_N$. Thus we have proved $\overline{\psi}\psi = id$.

(2) Let $\varphi : \mathcal{U} \to \mathcal{U} \otimes V$ be a natural transformation. Let M be a finite dimensional C-comodule. Then there is a finite dimensional subcoalgebra $E \subset C$ with $\delta_M(M) \subset M \otimes E$. Of course, E is also a subcomodule in C.

φ is functorial, so for each (finite dimensional) vector space W, we have:

$$\varphi_{W\otimes E^\bullet} = 1_W \otimes \varphi_E$$

(For $W \otimes E^\bullet$, the tensor product with the comodule structure coming from the right tensorand, is the direct product of copies of E, the projections being morphisms in $\mathcal{M}^C$.)

δ_M induces a C-*colinear* mapping $\delta_M^E : M \to M \otimes E^\bullet$.
Now we get

$$\begin{aligned}
\psi\overline{\psi}(\varphi)_M &= (1_M \otimes \overline{\psi}(\varphi))\delta_M \\
&= (1_M \otimes \overline{\psi}(\varphi)|_E)\delta_M^E \\
&= (1_M \otimes \varepsilon|_E \otimes 1_C)(1_M \otimes \varphi_E)\delta_M^E \\
&= (1_M \otimes \varepsilon|_E \otimes 1_C)\varphi_{M\otimes E^\bullet}\delta_M^E \\
&= (1_M \otimes \varepsilon|_E \otimes 1_C)(\delta_M^E \otimes 1_V)\varphi_M \\
&= \varphi_M,
\end{aligned}$$

proving $\psi\overline{\psi} = id$.

□

We have proved that the functor $\mathcal{M}^{-}$ has a left adjoint, and that the counit of the adjunction is an isomorphism. This implies in particular, [6, p.88 Thm. 1], that the functor $\mathcal{M}^{-}$ is fully faithful. This corresponds to the assertions in [13, Prop.1+2].

It is clear now that the functor $\mathcal{I}_\mathcal{C}$ is a category equivalence whenever $\mathcal{C}$ is the category of comodules over a given coalgebra. Moreover, we can characterize by "internal properties" those categories in $\mathfrak{C}$ that are isomorphic in $\mathfrak{C}$ to a category of finite dimensional comodules: They are precisely the k-linear abelian categories with faithful exact k-linear functors to the category of finite dimensional vector spaces. The proof also gives an information about "how much bigger", in the general case, the category $\mathcal{M}^{\underline{\mathrm{coend}}(\mathcal{C},\omega)}$ is, compared to the $\mathcal{C}$ we started from: every finite dimensional $\underline{\mathrm{coend}}(\omega)$-comodule is a quotient of a finite limit of comodules occuring as $\mathcal{I}_\mathcal{C}(X)$ for X an object in $\mathcal{C}$.

Thus we want to prove that in the case of a k-linear abelian category $\mathcal{C}$ and a faithful exact functor ω the universal arrow $\mathcal{C} \to \mathcal{M}^{\underline{\mathrm{coend}}(\omega)}$ constructed above is a category equivalence. For this it suffices to prove that $\mathcal{I}_\mathcal{C}$ is fully faithful and essentially surjective. For this in turn it would be sufficient to show that the coalgebra $C = \underline{\mathrm{coend}}(\omega)$ is "contained in" $\mathcal{C}$. For then any other comodule M, being the difference kernel of the pair of arrows

$$M \otimes C \underset{1\otimes\Delta}{\overset{\delta_M\otimes 1}{\rightrightarrows}} M \otimes C \otimes C$$

will also be "contained in" $\mathcal{C}$, and thus $\mathcal{I}_\mathcal{C}$ will be an essentially surjective (and also fully faithful) functor.

However, there is no chance of finding an inverse image under $\mathcal{I}_{\mathcal{C}}$ of the possibly infinite dimensional coalgebra C in the category $\mathcal{C}$. But this is not strictly necessary: C and the pair of morphisms considered can be written as a filtered colimit of objects and pairs of morphisms contained in $\mathcal{C}$. The difference kernel of a *filtered* colimit of pairs of morphisms is the filtered colimit of their difference kernels (which exist in $\mathcal{C}$). The latter colimit can be shown to exist in $\mathcal{C}$ because it is known to be finite dimensional.

The coalgebra C is constructed as the colimit of objects of the form $\omega(X)^* \otimes \mathcal{I}_{\mathcal{C}}(Y)^\bullet$ for $X, Y \in \mathcal{C}$ (the comodule structure coming from the right tensorand). Thus, as a first step, we have to lift tensor products of this kind to the category $\mathcal{C}$. (The following lemma actually proves any k-linear abelian category to be a $\underline{vec}$-category in the sense of [12].)

2.2.2. Lemma and Definition: *Let $\mathcal{C}$ be a k-linear abelian category.*

Then there is a biadditive functor

$$\hat{\otimes} : \underline{vec} \times \mathcal{C} \to \mathcal{C},$$

unique up to isomorphism, satisfying

$$k \hat{\otimes} X = X \tag{1}$$

for all $X \in \mathcal{C}$. There is a unique family of isomorphisms

$$(V \otimes W) \hat{\otimes} X \cong V \hat{\otimes} (W \hat{\otimes} X) \tag{2}$$

which is functorial in $V, W \in \underline{vec}$ and $X \in \mathcal{C}$ and coherent with the monoidal structure of $\underline{vec}$.

Every additive functor $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is compatible with $\hat{\otimes}$ in the sense that

$$\mathcal{F}(V \hat{\otimes} X) \cong V \hat{\otimes} \mathcal{F}(X) \tag{3}$$

functorially in $V \in \underline{vec}$ and $X \in \mathcal{C}$.

PROOF: It suffices to construct $\hat{\otimes}$ on a skeleton for $\underline{vec}$. Let $k^n \hat{\otimes} X = X^n$ be the n-fold biproduct of copies of X. To get around constructing the images of morphisms $k^n \to k^m$ (via their matrix representations), we note that

$$\begin{aligned} \mathrm{Hom}(k^n, \mathcal{M}or(X,Y)) &\cong \mathrm{Hom}(k, \mathcal{M}or(X,Y))^n \\ &\cong \mathrm{Hom}(k, \mathcal{M}or(X,Y)^n) \\ &\cong \mathcal{M}or(X,Y)^n \\ &\cong \mathcal{M}or(X^n, Y) \\ &\cong \mathcal{M}or(k^n \hat{\otimes} X, Y). \end{aligned}$$

Thus $k^n \hat{\otimes} X$ solves a universal problem and extends to a functor in the standard way.

The formulas given are immediate, uniqueness follows directly from (1) and the required biadditivity. As $\hat{\otimes}$ is obtained by just forming biproducts, it is trivially conserved by additive functors. □

For a comodule $M \in \mathcal{M}^C$ and a vector space $V \in \underline{vec}$ the newly defined tensor product $V \hat{\otimes} M \cong V \otimes M^\bullet$ is of course just the usual tensor product carrying the comodule structure on the right tensorand, as expected.

As explained above, any C-comodule is the difference kernel of a filtered colimit of objects having inverse images under $\mathcal{I}_\mathcal{C}$. The filtered colimits being possibly infinite dimensional, we have to transform this into a filtered colimit of difference kernels to find an inverse image for the comodule in consideration.

We only have to recall a few well known facts to prove this possible. Indeed, see for example [6, p.211 Thm. 1], in the category of sets finite limits commute with filtered colimits. As the underlying functor from $\mathcal{M}^C$ to the category of sets creates filtered colimits and finite limits, this leads immediately to

2.2.3. Lemma: *Let C be a k-coalgebra.*
In the category of C-comodules finite limits commute with filtered colimits.

Now we know that an arbitrary C-comodule is a filtered colimit over objects contained in $\mathcal{C}$. We show that if a filtered colimit over C-comodules contained in $\mathcal{C}$ is a finite dimensional comodule, then that colimit exists in $\mathcal{C}$.

2.2.4. Lemma: *Let $\mathcal{C}$ be a k-linear abelian category, and let $\mathcal{F} : \mathcal{C} \to \underline{vec}$ be an exact k-linear faithful functor.*
Then if $\mathcal{D} : \underline{\mathcal{D}} \to \mathcal{C}$ is a filtered diagram such that a (finite dimensional!) colimit for $\mathcal{FD}$ exists, then there is even a colimit for $\mathcal{D}$, which is preserved by $\mathcal{F}$.

PROOF: Let $\iota : \mathcal{FD} \to C$ be a colimit cone.
As an increasing sequence of vector subspaces of C has to become constant, there is some $j \in \underline{\mathcal{D}}$ with $\iota_j(\mathcal{FD}j) = C$

For any $\ell \in \underline{\mathcal{D}}$ and $x \in \mathcal{FD}\ell$ with $\iota_\ell = 0$, there is some $\ell \xrightarrow{f} m$ in $\underline{\mathcal{D}}$ with $\mathcal{FD}(f)(x) = 0$. In particular we can find $\ell \xrightarrow{f_\ell} \hat{\ell}$ in $\mathcal{D}$ with $\mathrm{Ker}(\iota_\ell) \subset \mathrm{Ker}(\mathcal{FD}(f_\ell))$, and thus, since ι_ℓ factors through $\mathcal{FD}(f_\ell)$, even $\mathrm{Ker}(\iota_\ell) = \mathrm{Ker}(\mathcal{FD}(f_\ell))$.
If in particular ι_ℓ is an epimorphism, then the induced mapping $\mathcal{FD}\ell/\mathrm{Ker}(\mathcal{FD}f_\ell) \to C$ is seen to be an isomorphism.

We want to show that $\mathcal{D}j/\mathrm{Ker}(\mathcal{D}f_j) =: \hat{C}$ is a colimit for $\mathcal{D}$. The important part is to construct a reasonable cone $\mathcal{D} \to \hat{C}$, the rest is then straightforward.

First, for any $\ell \in \underline{\mathcal{D}}$ the morphism $\mathcal{FD}\ell \xrightarrow{\iota_\ell} C \cong \mathcal{F}\hat{C}$ is the image under $\mathcal{F}$ of some $\hat{\iota}_\ell : \mathcal{D}\ell \to \hat{C}$.
Since $\underline{\mathcal{D}}$ is filtered we only have to show this in the case that there is a morphism $g : \hat{j} \to \ell$ in $\underline{\mathcal{D}}$. Now in this case $\mathcal{D}(gf_j) : \mathcal{D}j \to \mathcal{D}\ell$ induces a morphism

$$\hat{C} = \mathcal{D}j/\mathrm{Ker}(\mathcal{D}f_j) \xrightarrow{\vartheta} \mathcal{D}\ell/\mathrm{Ker}(\mathcal{D}f_\ell)$$

such that

$$
\begin{tikzcd}
\mathcal{F}\hat{C} \arrow[r, "\text{kan}"] \arrow[dd, "\mathcal{F}\vartheta"'] & \mathcal{F}\mathcal{D}j/\mathrm{Ker}(\mathcal{F}\mathcal{D}f_j) \arrow[dr] & \\
& & C \\
\mathcal{F}(\mathcal{D}\ell/\mathrm{Ker}(\mathcal{D}f_\ell)) \arrow[r, "\text{kan}"'] & \mathcal{F}\mathcal{D}\ell/\mathrm{Ker}(\mathcal{F}\mathcal{D}f_\ell) \arrow[ur] &
\end{tikzcd}
$$

In particular, ϑ is an isomorphism, for $\mathcal{F}\vartheta$ is one, and

$$\mathcal{D}\ell \to \mathcal{D}\ell/\mathrm{Ker}(\mathcal{D}f_\ell) \xrightarrow{\vartheta^{-1}} \hat{C}$$

defines the $\hat{\imath}_\ell$ we were looking for.

It is straightforward to check that $(\hat{\imath} : \mathcal{D} \to \hat{C})$ is a couniversal cone. $\square$

2.2.5. Corollary: *Under the same assumptions, the proof of 2.2.4 shows more precisely:*

A filtered colimit in $\mathcal{C}$ is a quotient of one vertex of the diagram in consideration.

We can immediately strengthen the result of 2.2.4:

2.2.6. Corollary: *Let $\mathcal{C}$ be a k-linear abelian category, let $\mathcal{F} : \mathcal{C} \to \underline{\mathrm{vec}}$ be a faithful exact k-linear functor.*
Then $\mathcal{C}$ has colimits of diagrams whose images under $\mathcal{F}$ have (finite dimensional!) colimits, and $\mathcal{F}$ preserves these.

PROOF: The existence of a colimit for a diagram is equivalent to the existence of a filtered colimit of the colimits of its finite subdiagrams. $\square$

2.2.7. Remark: *Once we know that categories of the type considered above are equivalent to categories of finite dimensional comodules (in a way compatible with the underlying functors), the results of the previous lemmas are of course trivial: The underlying functor from a category of comodules to the category of vector spaces creates colimits (and has a right adjoint). If such a colimit is finite dimensional then it exists of course in the category of finite dimensional comodules.*

Now we have collected the means to prove:

2.2.8. Theorem: *In the pair $(\mathcal{C}, \omega) \in \mathfrak{C}$ let $\mathcal{C}$ be k-linear abelian, and let ω be k-linear, exact and faithful.*
Then the functor $\mathcal{I}_\mathcal{C} : \mathcal{C} \to \mathcal{M}^{\underline{\mathrm{coend}}(\mathcal{C})}$ is a category equivalence.

PROOF: First, by 2.2.2 the diagram as whose colimit $C := \underline{\mathrm{coend}}(\omega)$ is constructed has an inverse image in $\mathcal{C}$.

If $\mathcal{U}$ is a subcategory of $\mathcal{C}$ having only finitely many objects, then clearly $\underline{\mathrm{coend}}(\omega|_\mathcal{U})$ is finite dimensional. Thus in particular there is an object $\hat{\mathcal{U}}$ in $\mathcal{C}$ such that $\mathcal{I}_\mathcal{C}(\hat{\mathcal{U}}) \cong \underline{\mathrm{coend}}(\omega|_\mathcal{U})$. As well, there are an object $\hat{\mathcal{U}}^{(2)}$ in $\mathcal{C}$ such that $\mathcal{I}_\mathcal{C}\hat{\mathcal{U}}^{(2)} \cong \underline{\mathrm{coend}}(\omega|_\mathcal{U}) \otimes \underline{\mathrm{coend}}(\omega|_\mathcal{U})$ and a morphism $\hat{\Delta}_\mathcal{U} : \hat{\mathcal{U}} \to \hat{\mathcal{U}}^{(2)}$ such that

$$\begin{tikzcd}
\mathcal{I}_\mathcal{C}(\hat{\mathcal{U}}) \arrow[r, "\mathcal{I}_\mathcal{C}(\hat{\Delta}_\mathcal{U})"] \arrow[d, equal, "\wr"'] & \mathcal{I}_\mathcal{C}(\hat{\mathcal{U}}^{(2)}) \arrow[d, equal, "\wr"] \\
\underline{\mathrm{coend}}(\omega|_\mathcal{U}) \arrow[r, "\Delta_{\underline{\mathrm{coend}}(\omega|_\mathcal{U})}"'] & \underline{\mathrm{coend}}(\omega|_\mathcal{U}) \otimes \underline{\mathrm{coend}}(\omega|_\mathcal{U})
\end{tikzcd}$$

Now let $(M, M \xrightarrow{\delta_M} M \otimes C)$ be an arbitrary finite dimensional C-comodule. Then there are a subcategory $\mathcal{C}_M$ of $\mathcal{C}$ having only finitely many objects and a morphism $\delta_0 : M \to M \otimes \underline{\mathrm{coend}}(\omega|_{\mathcal{C}_M})$ such that

$$(M \xrightarrow{\delta_0} M \otimes \underline{\mathrm{coend}}(\omega|_{\mathcal{C}_M}) \xrightarrow{1\otimes\mathrm{kan}} M \otimes C) = \delta_M.$$

For each category $\mathcal{U} \supset \mathcal{C}_M$ having only finitely many objects define the morphism $\delta_\mathcal{U}$ as

$$M \xrightarrow{\delta_0} M \otimes \underline{\mathrm{coend}}(\omega|_{\mathcal{C}_M}) \xrightarrow{1\otimes\mathrm{kan}} M \otimes \underline{\mathrm{coend}}(\omega|_\mathcal{U}).$$

Then the pair of morphisms

$$M \otimes C \underset{1 \otimes \Delta}{\overset{\delta_M \otimes 1}{\rightrightarrows}} M \otimes C \otimes C \tag{1}$$

is the filtered colimit of the pairs of morphisms

$$M \otimes \underline{\mathrm{coend}}(\omega|_\mathcal{U}) \underset{1 \otimes \Delta_{\underline{\mathrm{coend}}(\omega|_\mathcal{U})}}{\overset{\delta_\mathcal{U} \otimes 1}{\rightrightarrows}} M \otimes \underline{\mathrm{coend}}(\omega|_\mathcal{U}) \otimes \underline{\mathrm{coend}}(\omega|_\mathcal{U}) \tag{2}$$

for all the categories $\mathcal{U} \supset \mathcal{C}_M$ having only finitely many objects.

Each of the pairs of morphisms (2) has (roughly speaking) an inverse image under $\mathcal{I}_\mathcal{C}$ in $\mathcal{C}$. In particular so do their difference kernels. The filtered colimit of these difference kernels is, by 2.2.3, a difference kernel of (1). Now M is in fact a difference kernel of (1), and we know M to be finite dimensional; thus, by 2.2.4, M has an inverse image under $\mathcal{I}_\mathcal{C}$ in $\mathcal{C}$.

Thus we have proved that $\mathcal{I}_\mathcal{C}$ is essentially surjective. Now all of the construction above (except for the choice of $\mathcal{C}_M$ and δ_0, which does not actually matter) is functorial in M. This proves $\mathcal{I}_\mathcal{C}$ full as well, for as an inverse image of $\mathcal{I}_\mathcal{C}(X)$ for $X \in \mathcal{C}$ we can obviously choose X again.

$\mathcal{I}_\mathcal{C}$ is faithful anyway, ω being so, thus $\mathcal{I}_\mathcal{C}$ is a category equivalence. □

Examining closely the proof of 2.2.8 we deduce

2.2.9. Corollary: *Let $(\mathcal{C}, \omega)$ be arbitrary, $C = \underline{\text{coend}}(\omega)$.*

Then each object in $\mathcal{M}^C$ is the quotient of a subobject of a finite biproduct of objects from the image of the functor $\mathcal{I}_\mathcal{C}$.

PROOF: Indeed, any finite dimensional C-comodule is a filtered colimit of difference kernels of morphisms between finite colimits of finite biproducts (2.2.2) of objects of this kind.

This filtered colimit is, by 2.2.5, just some quotient of a difference kernel of morphisms between finite colimits of finite biproducts of objects from $\mathcal{I}_\mathcal{C}(\mathcal{C})$, thus all in all it has the form we claimed. □

2.2.1 shows a morphism f of coalgebras to be uniquely determined by the functor $\mathcal{M}^f$. We may ask how properties of f translate to properties of $\mathcal{M}^f$. The following simple facts correspond to [3, prop. 2.21, p. 139] in the case of affine group schemes.

2.2.10. Definition: *Let M, N be two objects in a category $\mathcal{C}$. We will call the object M a* **subquotient** *of N if it is a quotient of a subobject of N. Let $\mathcal{C}, \mathcal{C}'$ be two categories, and let $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ be a functor.*

(1) Call $\mathcal{F}$ **closed** *if it is full, and if for each $N \in \mathcal{C}$ and each subquotient M' of $\mathcal{F}(N)$ there is a subquotient M of N such that $\mathcal{F}(M) = M'$.*

(2) Call $\mathcal{F}$ **dense***, if for each object M' in $\mathcal{C}'$ there is an object N in $\mathcal{C}$ such that M' is a subquotient of $\mathcal{F}(N)$.*

Using the notions closed and dense may seem hardly reasonable. They are a little more motivated, however, if we just think of an object as "pretty near" to an other one if it is a subquotient. No deep topological analogies are implied.

2.2.11. Remark: *Let $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ be a faithful functor. Then $\mathcal{F}$ is a category equivalence iff $\mathcal{F}$ is closed and dense. ("only if" holds trivially, and if $\mathcal{F}$ is closed and dense it is immediately seen to be essentially surjective.)*

2.2.12. Lemma: *Let C and C' be two k-coalgebras and let $f : C \to C'$ be a morphism of coalgebras.*

(1) If f is injective, then $\mathcal{M}^f$ is closed.

(2) If f is surjective, then $\mathcal{M}^f$ is dense.

PROOF: The first statement is clear: If $C \subset C'$ is a subcoalgebra, then a C'-subcomodule of a C-comodule is also a C-subcomodule; C'-colinear morphisms between C-comodules are C-colinear as well.

Now let $f : C \to C'$ be a surjection of coalgebras, and let N' be a C'-comodule. Then there is a finite dimensional subcoalgebra $E' \subset C$' such that N' is even an E'-comodule. Moreover, there is a finite dimensional subcoalgebra $E \subset C$ such that $f(E) = E'$.

The morphism $\tilde{f} : E \to E'$ induced by f is an epimorphism of E'-comodules, E being regarded as an E'-comodule via $\tilde{f}$. Thus, E' is a quotient of a C'-comodule which is contained in the image of the functor $\mathcal{M}^f$.
Then the C'-comodule $N \otimes E'$, equipped with the comodule structure of the right tensorand, is also the quotient of a C'-comodule contained in the image of the functor $\mathcal{M}^f$. But since N (being the co-tensor product over E' of N with E') is a subcomodule of this latter comodule, we have proved N to be a subquotient of a comodule contained in the image of $\mathcal{M}^f$. □

2.2.13. Lemma: *Let $C \xrightarrow{f} C'$ be as above.*

(1) If $\mathcal{M}^f$ is closed, then f is injective.

(2) If $\mathcal{M}^f$ is dense, then f is surjective.

PROOF: We factor $f : C \to C'$ into a surjective morphism of coalgebras $g : C \to Z$ and an injective morphism of coalgebras $h : Z \to C'$. Then $\mathcal{M}^f = \mathcal{M}^h\mathcal{M}^g$. From the preceding lemma we know that $\mathcal{M}^g$ is dense and $\mathcal{M}^h$ is closed.

(1) If $\mathcal{M}^f$ is closed, then so is $\mathcal{M}^g$. For if $N \in \mathcal{M}^Z$ is a subquotient of $\mathcal{M}^g(M)$ for some object $M \in \mathcal{M}^C$, then $\mathcal{M}^h(N)$ is a subquotient of $\mathcal{M}^f(M)$. Thus there is a subquotient V of M in $\mathcal{M}^C$ such that $\mathcal{M}^h(N) \cong \mathcal{M}^f(V) = \mathcal{M}^h\mathcal{M}^g(M)$. But then also $\mathcal{M}^g(V) \cong N$.
Thus $\mathcal{M}^g$ is closed and dense, hence a category equivalence. Thus g is an isomorphism, and f is injective as h is.

(2) If $\mathcal{M}^f$ is dense, then so is $\mathcal{M}^h$, because for any $N \in \mathcal{M}^{C'}$ there is a $V \in \mathcal{M}^C$ such that N is a subquotient of $\mathcal{M}^f(V) = \mathcal{M}^h\mathcal{M}^g(V)$, and since $\mathcal{M}^f$ is full and $\mathcal{M}^g$ is faithful, $\mathcal{M}^h$ is full.
Now since $\mathcal{M}^h$ is an equivalence, h is an isomorphism, and since f is surjective, g is so.

□

2.3 The coendomorphism coalgebra of a monoidal category is a bialgebra

The heading can't be all wrong, for it doesn't strictly make sense. We will show:

Let $(\mathcal{C}, \omega)$ be in $\mathfrak{C}$. If then $(\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ is a monoidal category, and (ω, ξ, ξ_0) is a monoidal functor, then $\underline{\text{coend}}(\mathcal{C})$ is a bialgebra.

2.3.1. Remark: *In order that the notion of bialgebra make sense, the category $\mathcal{A}$ shall, from this section on, always be quasisymmetric, the quasisymmetry being denoted by τ. We then define a bialgebra in the obvious way as found in [11] for the case of a symmetric monoidal category. I'm well aware that we should, to be precise, examine critically the facts well known for bialgebras in symmetric categories for the more general case of only quasisymmetric categories: Here, we are missing the coherence results assuring, in the symmetric case, that any reasonable diagram involving only instances of the symmetry (and associativity etc.) commutes. Some remarks on bialgebras in quasisymmetric categories can be found in [2]. For example, the tensor product of two "commutative" bialgebras quite surprisingly need not be commutative as well (since $\sigma^2 \neq id$). However, as far as the following constructions are concerned, no nasty troubles arise. Thus I felt free to do everything in the general case without talking about the basic theory of bialgebras in quasisymmetric categories.*

To prove the results we are up to, we adopt the following strategy: We will see the category $\mathfrak{C}$ to be a monoidal category, the monoidal structure being partly described by the cartesian product of categories. Now objects in $\mathfrak{C}$ which are monoidal categories with the underlying functor being a monoidal one will turn out to be monoids in the monoidal category $\mathfrak{C}$. As $\mathcal{M}^{-}$ is a weakly monoidal functor and $\underline{\text{coend}}$ is a monoidal one, everything follows.

2.3.2. Lemma and Definition: $\mathfrak{C}$ *becomes a monoidal category when equipped with the tensor product, denoted $\times$:*

$$(\mathcal{C}, \omega) \times (\mathcal{D}, \nu) = (\mathcal{C} \times \mathcal{D}, \omega \otimes \nu),$$

where

$$\omega \otimes \nu : \mathcal{C} \times \mathcal{D} \ni (c, d) \mapsto \omega(c) \otimes \nu(d) \in \mathcal{A}_0.$$

Here, the tensor product of two morphisms $[\mathcal{F}, \zeta] : (\mathcal{C}, \omega) \to (\mathcal{C}', \omega')$ and $[\mathcal{G}, \eta] : (\mathcal{D}, \nu) \to (\mathcal{D}', \nu')$ is defined by

$$(\mathcal{F}, \zeta) \times (\mathcal{G}, \eta) = (\mathcal{F} \times \mathcal{G}, \zeta \otimes \eta)$$

PROOF: Obvious. The neutral object is given by the category with only one object (and one identical morphism), together with the constant functor taking the neutral object of $\mathcal{A}$ as its value. $\square$

The category of coalgebras is a monoidal one anyway, if we define the tensor product of two coalgebras C and D in the usual manner as their tensor product in $\mathcal{A}$ equipped with the coalgebra structure

$$C \otimes D \xrightarrow{\Delta\otimes\Delta} C \otimes C \otimes D \otimes D \xrightarrow{1\otimes\tau\otimes 1} C \otimes D \otimes C \otimes D.$$

Now we define

2.3.3. Definition: *$\mathfrak{C}_\otimes$ is the category of monoids (and morphisms of monoids) in the monoidal category $\mathfrak{C}$.*

2.3.4. Lemma: *Let $(\mathcal{C}, \omega) \in \mathfrak{C}$. Then if $\mathcal{C}$ is a monoidal category and the underlying functor ω is a monoidal one, then $(\mathcal{C}, \omega)$ is in $\mathfrak{C}_\otimes$.*

PROOF: Let $(\mathcal{C}, \otimes, \alpha, I, \lambda, \rho)$ be a monoidal category and let (ω, ξ, ξ_0) be a monoidal functor.

The morphism $(\otimes, \xi) : (\mathcal{C}, \omega) \times (\mathcal{C}, \omega) \to (\mathcal{C}, \omega)$ makes $(\mathcal{C}, \omega)$ a monoid in $\mathfrak{C}$: Multiplication is associative, since

$$\begin{aligned}
&(\otimes, \xi)\,((\otimes, \xi) \times (\mathcal{I}d_\mathcal{C}, 1)) \\
&\quad= \ (\otimes, \xi)(\otimes \times \mathcal{I}d_\mathcal{C}, \xi \otimes 1) \\
&\quad= \ (\otimes(\otimes \times \mathcal{I}d_\mathcal{C}), \xi(\xi \otimes 1)) \\
&\quad\overset{\alpha}{\sim} \ (\otimes(\mathcal{I}d_\mathcal{C} \times \otimes), \xi(1 \otimes \xi)) \\
&\quad= \ \cdots = (\otimes, \xi)\,((\mathcal{I}d_\mathcal{C}, 1) \times (\otimes, \xi))
\end{aligned}$$

At the point marked "$\overset{\alpha}{\sim}$" we have used that $\alpha : \otimes(\mathcal{I}d_\mathcal{C} \times \otimes) \cong \otimes(\otimes \times \mathcal{I}d_\mathcal{C})$ is an isomorphism for which by definition of a monoidal functor

$$\begin{tikzcd}
\omega \otimes \omega \otimes \omega \arrow[r, "\xi(1 \otimes \xi)"] \arrow[dr, "\xi(\xi \otimes 1)"'] & \omega \otimes (\otimes \times \mathcal{I}d_\mathcal{C}) \arrow[d, "\omega\alpha"] \\
& \omega \otimes (\mathcal{I}d_\mathcal{C} \times \otimes)
\end{tikzcd}$$

commutes.

The unit of the monoid $(\mathcal{C}, \omega)$ is the morphism $(\{*\}, \mathrm{const}_k) \to (\mathcal{C}, \omega)$ given by the functor $* \mapsto I, 1_* \mapsto 1_I$ and the morphism (natural in $*$) $k \cong \omega(I)$.

$\square$

It may be interesting to note that the inverse implication of the preceding lemma is also correct, provided ω is faithful (which is needed to prove the coherence conditions.)

2.3.5. Lemma: *Let $[\mathcal{F},\zeta] : (\mathcal{C},\omega) \to (\mathcal{C}',\omega')$ be a morphism in $\mathfrak{C}$. Let (ω,ξ,ξ_0), (ω',ξ',ξ_0') and $(\mathcal{F},\Xi)$ be monoidal functors, and let $\zeta : \omega \to \omega'\mathcal{F}$ be a monoidal isomorphism.*

Then $(\mathcal{F},\zeta)$ is a morphism of monoids in $\mathfrak{C}$, that is, a morphism in $\mathfrak{C}_\otimes$.

Proof: If we just forget about the isomorphism making the functors involved morphisms in $\mathfrak{C}$, we have to show $\otimes(\mathcal{F}\times\mathcal{F}) = \mathcal{F}\otimes$ (as an equality in $\mathfrak{C}$).

The monoidal structure of $\omega'\mathcal{F}$ is given by

$$\omega'\mathcal{F}X \otimes \omega'\mathcal{F}Y \xrightarrow{\xi} \omega'(\mathcal{F}X \otimes \mathcal{F}Y) \xrightarrow{\omega'\Xi} \omega'\mathcal{F}(X \otimes Y).$$

Now

$$\begin{aligned}
&(\otimes,\xi)\,((\mathcal{F},\zeta)\times(\mathcal{F},\zeta))\\
&\quad= (\otimes,\xi)(\mathcal{F}\times\mathcal{F},\zeta\otimes\zeta)\\
&\quad= (\mathcal{F}\otimes\mathcal{F},(\xi\mathcal{F}\times\mathcal{F})\zeta\otimes\zeta)
\end{aligned}$$

(where $\mathcal{F}\otimes\mathcal{F}$ has to be read as the functor $\mathcal{C}\times\mathcal{C} \ni (X,Y) \mapsto \mathcal{F}(X)\otimes\mathcal{F}(Y) \in \mathcal{C}$, similar to the definition 2.3.2); on the other hand we have

$$(\mathcal{F},\zeta)(\otimes,\xi) = (\mathcal{F}\otimes,(\zeta\otimes)\xi)$$

These two pairs are the same in $\mathfrak{C}$ since ζ is a monoidal transformation, whence the following diagram commutes:

$$\begin{tikzcd}
\omega\otimes\omega \arrow[r, "\zeta\otimes\zeta"] \arrow[dr, "\xi"'] & \omega'\mathcal{F}\otimes\omega'\mathcal{F} \arrow[r, "\xi'(\mathcal{F}\times\mathcal{F})"] & \omega'(\mathcal{F}\otimes\mathcal{F}) \arrow[d, "\omega'\Xi"] \\
& \omega\otimes \arrow[r, "\zeta\otimes"'] & \omega'\mathcal{F}\otimes
\end{tikzcd}$$

□

We now turn to the way <u>coend</u> and $\mathcal{M}^{-}$ behave on tensor products.

First, note that $\mathcal{M}^{-}$ is a weakly monoidal functor: Indeed, for C and D coalgebras we have a functor

$$\mathcal{X}_{C,D} : \mathcal{M}^C \times \mathcal{M}^D \to \mathcal{M}^{C\otimes D}$$

induced by the tensor product (in $\mathcal{A}$) of C-comodules M and D-comodules N equipped with the $C\otimes D$-comodule structure

$$M\otimes N \to M\otimes C\otimes N\otimes D \xrightarrow{1\otimes\tau\otimes 1} M\otimes N\otimes C\otimes D.$$

Also, we have a functor $\mathcal{X}_0 : (*, \mathrm{const}_I) \to \mathcal{M}^I$ taking I with the trivial I-comodule structure as its value. It is obvious that $(\mathcal{M}^{-}, \mathcal{X}, \mathcal{X}_0)$ is a weakly monoidal functor.

Now monoids in the category of coalgebras are just bialgebras. As a consequence, $\mathcal{M}^{-}$ induces a monoidal functor $\underline{Bialg} \to \mathfrak{C}_{\otimes}$. Explicitly, for H a bialgebra, the category $\mathcal{M}^H$ has a natural structure of monoidal category, the underlying functor being strictly monoidal, if, for any two right H-comodules M, N we define an H-comodule structure on $M \otimes N$ by

$$M \otimes N \longrightarrow M \otimes C \otimes N \otimes C \xrightarrow{1 \otimes \tau \otimes 1} M \otimes N \otimes C \otimes C \xrightarrow{1 \otimes 1 \otimes \mu} M \otimes N \otimes C,$$

that is, in Sweedler's notation, by

$$\delta(m \otimes n) = m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}.$$

The neutral object is the neutral object of $\mathcal{A}$ with the comodule structure induced by $\eta : k \to H$.

On the other hand:

2.3.6. Lemma: $\underline{\text{coend}}$ *is a monoidal functor.*

PROOF: Let $(\mathcal{C}, \omega), (\mathcal{D}, \nu) \in \mathfrak{C}$, let $C = \underline{\mathrm{coend}}(\omega)$ and $D = \underline{\mathrm{coend}}(\nu)$. We have to show that the natural transformation

$$\omega \otimes \nu \xrightarrow{\delta^\omega \otimes \delta^\nu} \omega \otimes C \otimes \nu \otimes D \xrightarrow{1 \otimes \tau \otimes 1} \omega \otimes \nu \otimes C \otimes D$$

is universal in the sense of 2.1.9.

First, we have

$$\begin{aligned}
&\left(\int^{X \in \mathcal{C}} (\omega X)^* \otimes \omega X\right) \otimes \left(\int^{Y \in \mathcal{D}} (\nu Y)^* \otimes \nu Y\right) \\
&= \int^{X \in \mathcal{C}} \int^{Y \in \mathcal{D}} ((\omega X)^* \otimes \omega X \otimes (\nu Y)^* \otimes \nu Y) \\
&= \int^{(X,Y) \in \mathcal{C} \times \mathcal{D}} ((\omega X)^* \otimes \omega X \otimes (\nu Y)^* \otimes \nu Y)
\end{aligned}$$

the first step being a consequence of the tensor product's preserving arbitrary colimits, the second being a kind of "Fubini's theorem" for ends and coends, like stated, say, in [6, chap. IX, sec. 8]. (I will not cite the theorem in more detail, much less give a proof. Its meaning is clear from the standard example for ends: A natural transformation between bifunctors is just a transformation natural in each argument.)

The precise statement of the formula proved is that the tensor product of the universal wedges $\omega^* \otimes \omega \to C$ and $\nu^* \otimes \nu \to D$ defines a dinatural transformation which is an initial one from $\omega^* \otimes \omega \otimes \nu^* \otimes \nu$ to $C \otimes D$.

All that's left to do is transform the "integrand": The isomorphism

$$(\omega X)^* \otimes \omega(X') \otimes (\nu Y)^* \otimes \nu(Y') \xrightarrow{\tau_{\omega^* \otimes \omega, \nu^*} \otimes 1} (\nu Y)^* \otimes (\omega X)^* \otimes \omega(X') \otimes \nu(Y')$$

is natural in all the four objects involved. Thus

$$\nu^* \otimes \omega^* \otimes \omega \otimes \nu \xrightarrow{(\tau_{\omega^*\otimes\omega,\nu})^{-1}\otimes 1} \omega^* \otimes \omega \otimes \nu^* \otimes \nu \to C \otimes D$$

gives a coend for $\nu^* \otimes \omega^* \otimes \omega \otimes \nu$ as well. If we identify $\nu^* \otimes \omega^*$ and $(\omega \otimes \nu)^*$, this yields a "universal arrow" $\omega \otimes \nu \to \omega \otimes \nu \otimes C \otimes D$. We still have to check that this universal arrow has the form claimed. In the diagram

$$\begin{tikzcd}
\omega \otimes \omega^* \otimes \omega \otimes \nu \arrow[r, "1 \otimes \mathrm{db} \otimes 1 \otimes 1 \otimes 1"] \arrow[d, "1 \otimes 1 \otimes 1 \otimes \mathrm{db} \otimes 1"'] & \omega \otimes \nu \otimes \nu^* \otimes \omega^* \otimes \omega \otimes \nu \\
\omega \otimes \omega^* \otimes \omega \otimes \nu \otimes \nu^* \otimes \nu \arrow[r, "1 \otimes \tau_{\omega^*\otimes\omega,\nu} \otimes 1 \otimes 1"'] \arrow[d] & \omega \otimes \nu \otimes \omega^* \otimes \omega \otimes \nu^* \otimes \nu \arrow[u, "1 \otimes 1 \otimes \tau_{\omega^*\otimes\omega,\nu^*}"'] \arrow[d] \\
\omega \otimes C \otimes \nu \otimes D \arrow[r, "1 \otimes \tau_{C,\nu} \otimes 1"'] & \omega \otimes \nu \otimes C \otimes D
\end{tikzcd}$$

the upper part commutes by the equations making τ a quasisymmetry, and by dinaturality of db. The lower part commutes by naturality of τ, applied to the morphism $\omega^* \otimes \omega \to C$.

Multiplying the morphism $\omega \otimes \nu \xrightarrow{\mathrm{db}\otimes 1\otimes 1} \omega \otimes \omega^* \otimes \omega \otimes \nu$ in front of the top left corner of the diagram yields:

- First to the right, then down: The universal arrow $\omega \otimes \nu \to \omega \otimes \nu \otimes C \otimes D$.
- First down, then to the right: The morphism

$$\omega \otimes \nu \xrightarrow{\delta^\omega \otimes \delta^\nu} \omega \otimes C \otimes \nu \otimes D \xrightarrow{1\otimes\tau\otimes 1} \omega \otimes \nu \otimes C \otimes D$$

which was our claim. □

In the proof just given we needed the tensor product in $\mathcal{A}$ to commute with arbitrary colimits. As this is *not* true for limits in our main example $\mathcal{A} = \underline{Vec}$, the proof does not carry over directly to the construction of an endomorphism algebra of ω instead of the coendomorphism coalgebra (which Majid claims in [8], anyway).

I note once more the form of the universal arrow $\delta^{\omega\otimes\nu}$ we have obtained:

$$\delta^{\omega\otimes\nu} = (\omega \otimes \nu \xrightarrow{\delta^\omega \otimes \delta^\nu} \omega \otimes C \otimes \nu \otimes D \xrightarrow{1\otimes\tau\otimes 1} \omega \otimes \nu \otimes C \otimes D)$$

or, in Sweedler's notation:

$$\delta^{\omega\otimes\nu}_{(X,Y)}(x \otimes y) = x_{(0)} \otimes y_{(0)} \otimes x_{(1)} \otimes y_{(1)}$$

for $X, Y \in \mathcal{C}$ and $x \in X, y \in Y$.

We state without giving the (tedious but easy) proof that we have thus constructed the functors $\underline{\text{coend}}$ and $\mathcal{M}^{\overline{}}$ to be monoidally adjoint ones.

A direct consequence of $\underline{\text{coend}}$ being a monoidal functor is

2.3.7. Corollary: *Let $(\mathcal{C},\omega) \in \mathfrak{C}$. If $(\mathcal{C},\otimes,\alpha)$ is a monoidal category and (ω,ξ,ξ_0) is a monoidal functor, then $\underline{\mathrm{coend}}(\omega)$ is a bialgebra.*

We can describe the algebra structure of $H := \underline{\mathrm{coend}}(\omega)$ by 2.3.6 as follows:

$$\begin{tikzcd}
\omega X \otimes \omega Y \arrow[r, "\delta^{\omega\otimes\omega}"] \arrow[d, "\xi"'] & \omega X \otimes \omega Y \otimes C \otimes C \arrow[d, "\xi \otimes \mu"] \\
\omega(X \otimes Y) \arrow[r, "\delta^{\omega}"'] & \omega(X \otimes Y) \otimes C
\end{tikzcd}$$

commutes, or in Sweedler's notation $\delta^\omega(\xi(x \otimes y)) = \xi(x_{(0)} \otimes y_{(0)}) \otimes \mu(x_{(1)} \otimes y_{(1)})$.
The unit $\eta : k \to H$ is defined by the commutativity of

$$\begin{tikzcd}
k \arrow[r, "1"] \arrow[d, "1"'] & k \arrow[d, "1 \otimes \eta"] \\
\omega(I) \arrow[r, "\delta^{\omega}"'] & \omega(I) \otimes C
\end{tikzcd}$$

(where we identified $\omega(I)$ and k), or $\delta^\omega(1_k) = 1_k \otimes 1_C$.

We see that, thus, the comodule structure on tensor products agrees with the monoidal structure defined on the category $\mathcal{M}^C$. In particular the functor $\mathcal{I}_\mathcal{C}$ from 2.1.12 is a monoidal functor. For the isomorphism $\mathcal{I}_\mathcal{C}(X \otimes Y) \cong \mathcal{I}_\mathcal{C}(X) \otimes \mathcal{I}_\mathcal{C}(Y)$ we just have to choose the old morphism ξ, which is colinear by definition of μ.

We could also have deduced this abstractly from our remarks in section 1.4, as well as the following result:

2.3.8. Corollary: *$\underline{\mathrm{coend}}$ defines a left adjoint for the functor $\mathcal{M}^{-} : \underline{Bialg} \to \mathfrak{C}_\otimes$.*

Proof: Let $(\mathcal{C},\omega) \in \mathfrak{C}_\otimes$. We've already defined a bialgebra structure on $H := \underline{\mathrm{coend}}(\omega)$ such that the functor $\mathcal{I}_\mathcal{C} : \mathcal{C} \to \mathcal{M}^H$ from 2.1.12 is (equipped with accessories) a morphism in $\mathfrak{C}_\otimes$.

If B is another bialgebra, and $(\mathcal{T},\tau) : \mathcal{C} \to \mathcal{M}^B$ is a morphism in $\mathfrak{C}_\otimes$, we already know from 2.1.12 that there is a unique morphism $f : H \to B$ of coalgebras satisfying $\mathcal{M}^f \mathcal{I}_\mathcal{C} = \mathcal{T}$. We have to show this to be a morphism of algebras as well, that is, that $f\mu_H = \mu_B(f \otimes f)$. (We omit compatibility with the unit)
For this it suffices, again by 2.1.12 (applied to $\mathcal{C} \times \mathcal{C}$), to check that

$$\mathcal{M}^{f\mu_H}\mathcal{I}_{\mathcal{C}\times\mathcal{C}} = \mathcal{M}^{\mu_B(f\otimes f)}\mathcal{I}_{\mathcal{C}\times\mathcal{C}}$$

in $\mathfrak{C}$. We will show both sides of this equation to be the functors $\mathcal{T}\otimes, \otimes(\mathcal{T} \times \mathcal{T}) : \mathcal{C} \times \mathcal{C} \to \mathcal{M}^B$, which indeed agree by 2.3.5.

We have the following formulae (valid in $\mathfrak{C}$):

$$\mathcal{M}^{\mu_B}\mathcal{X}_{B,B} = \otimes \quad : \mathcal{M}^B \times \mathcal{M}^B \to \mathcal{M}^B$$

$$\mathcal{M}^{f\otimes f}\mathcal{X}_{H,H} = \mathcal{X}_{B,B}(\mathcal{M}^f \times \mathcal{M}^f)$$

$$\mathcal{M}^{\mu_H}\mathcal{I}_{\mathcal{C}\times\mathcal{C}} = \mathcal{I}_{\mathcal{C}}\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{M}^H$$

The first one is valid by definition of the monoidal structure of $\mathcal{M}^B$, the second trivially, the third by definition of μ_H.

We obtain, as desired

$$\begin{aligned} \mathcal{M}^{f\mu_H}\mathcal{I}_{\mathcal{C}\times\mathcal{C}} &= \mathcal{M}^f\mathcal{I}_{\mathcal{C}}\otimes \\ &= \mathcal{T}\otimes \end{aligned}$$

$$\begin{aligned} \mathcal{M}^{\mu_B(f\otimes f)} &= \mathcal{M}^{\mu_B}\mathcal{M}^{f\otimes f}\mathcal{X}_{H,H}(\mathcal{I}_{\mathcal{C}} \times \mathcal{I}_{\mathcal{C}}) \\ &= \mathcal{M}^{\mu_B}\mathcal{X}_{B,B}(\mathcal{M}^f \times \mathcal{M}^f)(\mathcal{I}_{\mathcal{C}} \times \mathcal{I}_{\mathcal{C}}) \\ &= \otimes(\mathcal{T} \times \mathcal{T}) \end{aligned}$$

□

2.3.9. Remark: *Let $(\mathcal{C}, \omega) \in \mathfrak{C}_\otimes$ be such that the category $\mathcal{C}$ and the functor ω both are quasisymmetric. Then it is straightforward to check that $\underline{\mathrm{coend}}(\omega)$ is a commutative bialgebra (in the obvious sense).*

For the rest of the section we turn to calculations with elements of the coendomorphism coalgebra once more, so we let $\mathcal{A} = \underline{Vec}$.

2.3.10. Lemma: *Using the notation from 2.1.15, we have:*

$$\begin{aligned} [\varphi|v][\psi|w] &= [\varphi\otimes\psi|v\otimes w] \\ 1_C &= [1|1] \end{aligned}$$

Where we consider $\varphi\otimes\psi$ as an element of $(\omega(V\otimes W))^*$ via the isomorphism $(\omega V)^*\otimes(\omega W)^* \cong (\omega(V\otimes W))^*$ and $v\otimes w$ as an element of $\omega(V\otimes W)$ via the isomorphism $\omega(V)\otimes\omega(W) \cong \omega(V\otimes W)$.

We have

$$C \;\cong\; T\left(\bigoplus_{V\in\mathcal{C}}(\omega V)^*\otimes\omega V\right)/I$$

(T meaning the tensor algebra), where the ideal I is generated by the elements

$$(\varphi\otimes v)(\psi\otimes w) - (\varphi\otimes\psi)\otimes(v\otimes w) \tag{1}$$

$$(\omega f)^*\psi\otimes v - \psi\otimes(\omega f)v \tag{2}$$

for $V \xrightarrow{f} W$ in $\mathcal{C}$, $v\in\omega V, w\in\omega W, \varphi\in(\omega V)^*, \psi\in(\omega W)^*$.

If we consider the k-linear additive case, we can reduce the sets of generators and relations once more:

2.3.11. Definition: *Call a family of objects of $\mathcal{C}$ a tensor generating set of objects if the finite tensor products of members of this family form a generating set of objects, that is, if all the objects of the category are subobjects of quotient objects of finite biproducts of finite tensor products of members of the family.*

Call a set of morphisms in a monoidal category a monoidal generating set of morphisms if all the morphisms in the category can be obtained from the morphisms in the family and identities in a finite number of steps by forming tensor products, composing, and taking the inverse of isomorphisms.

2.3.12. Lemma: *If (V_α) is a tensor generating set of objects in $\mathcal{C}$, and if we choose a basis $\{v_{\alpha,i}\}$ in each of the ωV_α, then (if we denote the dual bases by $\{v_\alpha^i\}$), the algebra C is generated by the elements $[v_\alpha^i | v_{\alpha,j}]$.*

If F is a tensor generating set of morphisms in $\mathcal{C}$, then the ideal of relations of C is generated by all the elements (1) and (2) from the last lemma, but now only for $f \in F$.

Finally, we can give a "coordinate-wise" description of C in terms of the representing matrices for all the images under ω of the morphisms from a tensor generating set:

2.3.13. Corollary: *Let $(\mathcal{C}, \omega)$ and C be as above. Let (V_α) be a tensor generating family of objects of $\mathcal{C}$, and choose basese in the $\omega(V_\alpha)$ as above. Assume we have a tensor generating set of morphisms in $\mathcal{C}$ that are morphisms between finite tensor products of objects V_α. Then we can express each of these morphisms by its representing matrix with respect to the chosen bases, a morphism $f : V_{\alpha_1} \otimes \cdots \otimes V_{\alpha_n} \to V_{\alpha'_1} \otimes \cdots \otimes V_{\alpha'_m}$, say, by*

$$f(v_{\alpha_1 i_1} \otimes \cdots \otimes v_{\alpha_n i_n}) = f_{i_1 \ldots i_n}^{j_1 \ldots j_m} v_{\alpha'_1 j_1} \otimes \cdots v_{\alpha'_m j_m}$$

Then the algebra $C = \underline{\mathrm{coend}}(\omega)$ is generated by elements $(A_\alpha)_i^j$ subject to the relations

$$(A_{\alpha_1})_{\ell_1}^{k_1} \cdots (A_{\alpha_n})_{\ell_n}^{k_n} f_{i_1 \ldots i_m}^{\ell_1 \ldots \ell_n} = f_{j_1 \ldots j_m}^{k_1 \ldots k_n} (A_{\alpha'_1})_{i_1}^{j_1} \cdots (A_{\alpha'_m})_{i_m}^{j_m}$$

and $(A_\alpha)_i^j$ corresponds to the element $[v^j | v_i]$.

2.4 The coendomorphism coalgebra of a rigid monoidal category is a Hopf algebra

More precisely: We know that if $(\mathcal{C}, \omega) \in \mathfrak{C}_\otimes$, then $\underline{\text{coend}}(\omega)$ is a bialgebra. If, moreover, $\mathcal{C}$ is rigid, then this bialgebra is a Hopf algebra.

First, we recall the reverse implication for the case of ordinary k-Hopf algebras: The category of finitely generated projective comodules over a Hopf algebra H is rigid if we define the comodule structure of the k-linear dual of an H-comodule V by

$$\forall \varphi \in M^* \; \forall v \in V : \quad \langle \varphi_{(0)}, v \rangle \varphi_{(1)} = \langle \varphi, v_{(0)} \rangle s(v_{(1)})$$

Considering $\delta_V : V \to V \otimes H$ as a mapping $I \otimes V \to V \otimes H$, we can form the morphism $\delta_V^\flat : V^* \otimes I \to H \otimes V^*$ according to 1.2.18. Forgetting about the I-tensorand once more, the definition of the comodule structure given above reads

$$\delta_{V^*} = \tau \delta_V^\flat$$

By 1.2.18 this also implies

$$v_i \otimes {v^i}_{(0)} \otimes {v^i}_{(1)} = v_{i(0)} \otimes v^i \otimes s(v_{i(1)})$$

The original definition yields colinearity of the usual evaluation ev, the last formula yields colinearity of db as well:

$$\begin{aligned}
(\text{ev} \otimes id_H)\delta_{V^* \otimes V}(\varphi \otimes v) &= (\text{ev} \otimes 1)(\varphi_{(0)} \otimes v_{(0)} \otimes \varphi_{(1)} v_{(1)}) \\
&= \langle \varphi_{(0)}, v_{(0)} \rangle \varphi_{(1)} v_{(1)} \\
&= \langle \varphi, v_{(0)} \rangle s(v_{(1)}) v_{(2)} \\
&= \langle \varphi, v \rangle 1_H = \delta_k \text{ev}(\varphi \otimes v)
\end{aligned}$$

$$\begin{aligned}
\delta_{V \otimes V^*} \text{db}(1_k) &= \delta_{V \otimes V^*}(v_i \otimes v^i) \\
&= v_{i(0)} \otimes {v^i}_{(0)} \otimes v_{i(1)} {v^i}_{(1)} \\
&= v_{i(0)} \otimes v^i \otimes v_{i(1)} \otimes s(v_{i(2)}) \\
&= v_i \otimes v^i \otimes 1_H \\
&= (\text{db} \otimes 1)\delta_k(1_k)
\end{aligned}$$

In the following, we want to revert this reasoning: A rigid monoidal category has a Hopf algebra as its coendomorphism coalgebra. This result is valid again for the case of a general quasisymmetric monoidal "ground category" $\mathcal{A}$ with the general assumptions made so far. I won't rederive the reverse result above (The category of rigid comodules over a Hopf algebra in $\mathcal{A}$ is rigid.) in the general case.

So we let $(\mathcal{C}, \omega) \in \mathfrak{C}_\otimes$, and we let $\mathcal{C}$ be a rigid monoidal category.

With the functor $\omega^* : \mathcal{C}^{\text{op}} \ni X \mapsto (\omega X)^* \in \mathcal{A}_0$ we make $(\mathcal{C}^{\text{op}}, \omega^*) \in \mathfrak{C}$. We then have $\underline{\text{coend}}(\omega^*) \cong C^{\text{cop}}$.

Indeed (up to the isomorphisms $\omega \cong k \otimes \omega$ and $\omega^* \cong \omega^* \otimes k$) from 1.2.18 we get an isomorphism[3]

$$\mathrm{Nat}(\omega, \omega \otimes M) \ni \varphi \mapsto \varphi^\flat \in \mathrm{Nat}(\omega^*, M \otimes \omega^*)$$

Thus we can define a universal $\delta^{(\omega^*)} : \omega^* \to \omega^* \otimes M$ with the properties in 2.1.9 by $\delta^{(\omega^*)} = \tau\,(\delta^\omega)^\flat$.

Since the formulae 1.2.21 yield (writing $\delta := \delta^\omega$)

$$\begin{aligned}(\Delta \otimes 1)\delta^\flat &= ((1 \otimes \Delta)\delta)^\flat \\ &= ((\delta \otimes 1)\delta)^\flat \\ &= (1 \otimes \delta^\flat)\delta^\flat,\end{aligned}$$

we obtain, defining $\Delta^{(\omega^*)}$ according to 2.1.10, that $\Delta^{(\omega^*)} = \tau\Delta^\omega$ holds: indeed

$$\begin{aligned}(\tau\delta^\flat \otimes 1)\tau\delta^\flat &= (\tau \otimes 1)(1 \otimes \tau)(\tau \otimes 1)(1 \otimes \delta^\flat)\delta^\flat \\ &= (\tau \otimes 1)(1 \otimes \tau)(\tau \otimes 1)(\Delta \otimes 1)\delta^\flat \\ &= (1 \otimes \tau)(\tau \otimes 1)(1 \otimes \tau)(\Delta \otimes 1)\delta^\flat \\ &= (1 \otimes \tau\Delta)\tau\delta^\flat.\end{aligned}$$

2.4.1. Definition: *Let $(\mathcal{C}, \omega) \in \mathfrak{C}$ with $\mathcal{C}$ a rigid monoidal category and ω a monoidal functor. Let $H := \underline{\mathrm{coend}}(\omega)$.*

Denote by $(\,—\,)^\vee$ some chosen left-dual-object functor in $\mathcal{C}$, and by $\omega^\vee$ the functor $X \to \omega(X^\vee)$. Let $j : \omega^* \to \omega^\vee$ be the isomorphism from 1.2.14 and 1.2.10.

Now define the morphism $s : H \to H$ by

$$\begin{tikzcd} \omega^* \arrow[r, "\delta^{(\omega^*)}"] \arrow[d, "j"'] & \omega^* \otimes C \arrow[d, "j \otimes s"] \\ \omega^\vee \arrow[r, "\delta^\vee"'] & \omega^\vee \otimes C \end{tikzcd},$$

2.4.2. Theorem: *Let $(\mathcal{C}, \omega)$ be as above. Then $H = \underline{\mathrm{coend}}(\omega)$ is a Hopf algebra with the morphism s just defined as an antipode.*

PROOF:

(1) For technical reasons we first define a natural morphism $\tilde{\delta} : \omega^\vee \to H \otimes \omega^\vee$ by $\tau\tilde{\delta} = \delta^\vee$, that is, such that the diagram

[3] This isomorphism is, of course, given "object-wise" for families $\varphi_X : \omega(X) \to \omega(X) \otimes M$ of morphisms at first. That it actually is an isomorphism between the sets of *natural* transformations can be proved by a calculation with ends, or using the formulae from 1.2.20.

$$\begin{tikzcd}
\omega^* \arrow[r, "\delta^\flat"] \arrow[d, "j"'] & H \otimes \omega^* \arrow[d, "s \otimes j"] \\
\omega^\vee \arrow[r, "\tilde{\delta}"'] & H \otimes \omega^\vee
\end{tikzcd}$$

commutes. If we define the morphisms $\widetilde{\mathrm{ev}}_X : \omega(X^\vee) \otimes \omega(X) \to k$ und $\widetilde{\mathrm{db}}_X : k \to \omega(X) \otimes \omega(X^\vee)$ as in 1.2.14, then, by definition of s, the diagrams

$$\begin{tikzcd}
\omega^\vee \otimes \omega \arrow[r, "1 \otimes \delta"] \arrow[d, "\tilde{\delta} \otimes 1"'] & \omega^\vee \otimes \omega \otimes H \arrow[d, "\widetilde{\mathrm{ev}} \otimes s"] \\
H \otimes \omega^\vee \otimes \omega \arrow[r, "1 \otimes \widetilde{\mathrm{ev}}"'] & H
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, "\widetilde{\mathrm{db}}"] \arrow[d, "\widetilde{\mathrm{db}}"'] & \omega \otimes \omega^\vee \arrow[d, "(1 \otimes s)\delta \otimes 1"] \\
\omega \otimes \omega^\vee \arrow[r, "\tilde{\delta} \otimes 1"'] & \omega \otimes H \otimes \omega^\vee
\end{tikzcd} .$$

commute.

(2) Since the morphism $\omega(\mathrm{ev}) : \omega(X^\vee \otimes X) \to \omega(I)$ is H-colinear, the diagram

$$\begin{tikzcd}
\omega(X^\vee \otimes X) \arrow[r, "\delta_{X^\vee \otimes X}"] \arrow[d, "\omega(\mathrm{ev}_X)"'] & \omega(X^\vee \otimes X) \otimes H \arrow[d, "\omega(\mathrm{ev}_X) \otimes 1"] \\
\omega(I) \arrow[r, "\delta_I"'] & \omega(I) \otimes H
\end{tikzcd}$$

commutes. Together with the equations defining μ and η, taken from 2.3.7, this yields commutativity of

$$\begin{tikzcd}
\omega(X^\vee) \otimes \omega(X) \arrow[rr, "\delta \otimes \delta"] \arrow[dd, "\widetilde{\mathrm{ev}} \otimes 1"'] & & \omega(X^\vee) \otimes H \otimes \omega(X) \otimes H \arrow[d, "1 \otimes \tau \otimes 1"] \\
& & \omega(X^\vee) \otimes \omega(X) \otimes H \otimes H \arrow[d, "\widetilde{\mathrm{ev}} \otimes 1"] \\
k \arrow[r, "\eta"'] & H & H \otimes H \arrow[l, "\mu"]
\end{tikzcd}$$

Using, in addition, commutativity of

$$\begin{tikzcd}
H \otimes \omega^\vee \otimes \omega \arrow[r, "\tau \otimes 1"] \arrow[d, "1 \otimes \widetilde{\mathrm{ev}}"'] & \omega^\vee \otimes H \otimes \omega \arrow[d, "1 \otimes \tau"] \\
H & \omega^\vee \otimes \omega \otimes H \arrow[l, "\widetilde{\mathrm{ev}} \otimes 1"]
\end{tikzcd},$$

we at last obtain commutativity of

$$\begin{tikzcd}
\omega(X^\vee) \otimes \omega(X) \arrow[rr, "\tilde{\delta} \otimes \delta"] \arrow[d, "\widetilde{\mathrm{ev}}"'] & & H \otimes \omega(X^\vee) \otimes \omega(X) \otimes H \arrow[d, "1 \otimes \widetilde{\mathrm{ev}} \otimes 1"] \\
k \arrow[r, "\eta"'] & H & H \otimes H \arrow[l, "\mu"]
\end{tikzcd}$$

A similar diagram can be derived from colinearity of $\omega(\mathrm{db}) : \omega(k) \to \omega(X \otimes X^\vee)$.

(3) Now we can verify the desired equations, $\mu(1 \otimes s)\Delta = \eta\varepsilon$ and $\mu(s \otimes 1)\Delta = \eta\varepsilon$. In the diagram

$$\begin{tikzcd}
\omega \arrow[r, "\widetilde{\mathrm{db}} \otimes 1"] \arrow[d, "\widetilde{\mathrm{db}} \otimes 1"'] & \omega \otimes \omega^\vee \otimes \omega \arrow[r, "1 \otimes \widetilde{\mathrm{ev}}"] \arrow[dd, "1 \otimes \tilde{\delta} \otimes \delta"] & \omega \arrow[ddd, "1 \otimes \eta"] \\
\omega \otimes \omega^\vee \otimes \omega \arrow[d, "\delta \otimes 1 \otimes \delta"'] & & \\
\omega \otimes H \otimes \omega^\vee \otimes \omega \otimes H \arrow[r, "1 \otimes s \otimes 1^{\otimes 3}"] \arrow[d, "1 \otimes 1 \otimes \widetilde{\mathrm{ev}} \otimes 1"'] & \omega \otimes H \otimes \omega^\vee \otimes \omega \otimes H \arrow[d, "1 \otimes 1 \otimes \widetilde{\mathrm{ev}} \otimes 1"] & \\
\omega \otimes H \otimes H \arrow[r, "1 \otimes s \otimes 1"'] & \omega \otimes H \otimes H \arrow[r, "1 \otimes \mu"'] & \omega \otimes H
\end{tikzcd}$$

the lower left rectangle commutes trivially; the upper left area commutes by (1); the area on the right by (2).

The left vertical edge of the diagram is the morphism $(\delta \otimes 1)\delta = (1 \otimes \Delta)\delta$, thus the route "first down, then to the right" around the diagram is the morphism $(1_\omega \otimes \mu(s \otimes 1)\Delta)\delta$. The route "first to the right, then down" is the morphism $1_\omega \otimes \eta = (1_\omega \otimes \eta\varepsilon)\delta$. Thus, by 2.1.9, we have proved one of the two equations defining an antipode. The other one is obtained in a similar manner.

$\square$

2.4.3. Remark: *Let $\mathcal{C}$ be right rigid as well. Then the functor $(\,-\,)^*$ is an equivalence, and thus the antipode of* $\underline{\mathrm{coend}}(\omega)$ *is an isomorphism by 2.1.3.*

In particular, the category of finite dimensional comodules over a k-Hopf algebra with a non-bijective antipode is left, but not right rigid.

2.4.4. Remark: *It is well known that any morphism of bialgebras between two Hopf algebras is automatically compatible with the antipodes. The corresponding fact in our context is that any monoidal functor between rigid monoidal categories automatically preserves left duals.*

Thus we have actually constructed a pair of adjoint functors $\mathcal{M}^{-}$: $\underline{\mathrm{Hopfalg}} \to \mathfrak{C}_{\otimes *}$ and $\underline{\mathrm{coend}} : \mathfrak{C}_{\otimes *} \to \underline{\mathrm{Hopfalg}}$, where $\underline{\mathrm{Hopfalg}}$ denotes the category of Hopf algebras and morphisms of Hopf algebras (these being the morphisms of bialgebras) and $\mathfrak{C}_{\otimes *}$ denotes the full subcategory of rigid monoidal categories in $\mathfrak{C}_{\otimes}$ (the monoidal functors representing their morphisms also preserve left duals).

2.4.5. Remark: *Let $\mathcal{A} = \underline{\mathrm{Vec}}$. For the elements of* $\underline{\mathrm{coend}}(\omega)$*, in the notation of 2.1.15, we have the formula*

$$s([\varphi|v]) = [\iota v|\varphi],$$

where $\iota : V \to V^{**}$ is the usual isomorphism of vector spaces.

Indeed we have

$$s([\varphi|v]) = \langle \varphi, v_{(0)}\rangle s(v_{(1)}) = \langle \varphi_{(0)}, v\rangle \varphi_{(1)} = \langle \iota v, \varphi_{(0)}\rangle \varphi_{(1)} = [\iota v|\varphi]$$

For the construction of coendomorphism coalgebras and, in the case $\mathcal{A} = \underline{\mathrm{Vec}}$, for the proof that the functor from a given category into a category of comodules is an equivalence, we required two assumptions that are automatically true in the case of a rigid category:

2.4.6. Remarks: *Let $\mathcal{C}$ be a rigid monoidal category and let $\omega : \mathcal{C} \to \mathcal{A}$ be a monoidal functor.*

(1) By 1.2.13 the objects $\omega(X)$ for all $X \in \mathcal{C}$ actually are in $\mathcal{A}_0$. Thus we have $(\mathcal{C}, \omega) \in \mathfrak{C}$ without having to seperately require $\omega : \mathcal{C} \to \mathcal{A}_0$.

(2) Any k-linear exact functor between rigid monoidal categories is faithful, [3, p.121, Prop. 1.19].

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