LMU Munich
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Exercises on Mathematical Statistical Physics II
Sheet 5

Problem 1 (Duhamel)
Let $\phi_0 \in L^2$, V be a multiplication operator (potential) with $\|V\|_\infty < \infty$ and h_0 be selfadjoint on L^2. Consider for any $\varepsilon > 0$ the differential equation

$$i\frac{d}{dt}\phi_t^\varepsilon = -h_0\phi_t^\varepsilon + \varepsilon V \star |\phi_t^\varepsilon|^2\phi_t^\varepsilon$$

with given initial condition $\phi_0^\varepsilon = \phi_0$.
Show that for any time $t > 0$

$$\lim_{\epsilon\to 0} \|\phi_t^\varepsilon - \phi_t^0\|_2 = 0 \ .$$

Hint: Use Duhamel's formula.

Problem 2 (Scattering Length)
Let $f : \mathbb{R}^3 \to \mathbb{C}$ be an L^∞ solution of

$$(-\Delta + V)f \equiv 0$$

for compactly supported multiplication operator $V \in L^\infty$. Normalize f to $\|f\|_\infty = 1$. Let $a := \int V(x)f(x)d^3x$. Show that outside the support of V

$$f = 1 - \frac{a}{|x|} \ .$$

Problem 3 (Stationary Phase)
Let Ψ_t be solution of the free Schr̈dinger equation in one dimension with initial Ψ_0 in Schwartz-space. Show that for any $v \in \mathbb{R}$

$$\lim_{t\to\infty} \left|\sqrt{t}\Psi_t(vt) - \widehat{\Psi}(v)\right| = 0 \ .$$

Hint: Show that the stationary point $k_{stat} = v$ and make an Taylor expansion of $\widehat{\Psi}$ around v up to first order. Show that the zeroth order of the Taylor expansion gives $t^{-1/2}\widehat{\Psi}(v)$. Integrate by parts to get good control of the first order.

The solutions to these exercises will be discussed on Friday, 25.11.2016.