LMU Munich
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Exercises on Mathematical Statistical Physics II
Sheet 1

Problem 1 (Projectors) Let $\varphi(x_j) \in L^2(\mathbb{R}^3, \mathbb{C})$. Show explicitly that $p_i^\varphi := |\varphi(x_i)\rangle\langle\varphi(x_i)|$ and $q_i^\varphi := 1 - p_i^\varphi$ are projectors, i.e. $p \circ p = p$ and $q \circ q = q$.
Which are the eigenvalues of p_i^φ and q_i^φ?

Problem 2 (Scaling) Regard the N-body Schrödinger equation with an interaction potential $V(x_i - x_j)$:

$$i\frac{\partial\psi(\mathbf{x},t)}{\partial t} = \left(-\sum_{i=1}^{N}\Delta_i + \sum_{i\neq j} V(x_i - x_j)\right)\psi(\mathbf{x},t) \tag{1}$$

where $\mathbf{x} = (x_1, x_2, ..., x_N) \in \mathbb{R}^{3N}$ denotes the tuple of the spatial coordinates of the particles.
Perform the following coordinate transformations $\mathbf{y} = N^{-1}\mathbf{x}$ and $\tau = N^{-2}t$. Consider first a general interaction, then the special case of a Coulomb potential $V(x_i - x_j) = (x_i - x_j)^{-1}$.
Of which equation from the lecture does the result remind you?

Problem 3 (Convolution) Let $f * g(x) := \int_{\mathbb{R}^d} f(y)g(x-y)\mathrm{d}^d x$ be the convolution of $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. Show that

a) the convolution defines a commutative algebra (without identity) on these function spaces.

b) $\int_{\mathbf{R}^d}(f*g)(x)\,dx = \left(\int_{\mathbf{R}^d} f(x)\,dx\right)\left(\int_{\mathbf{R}^d} g(x)\,dx\right)$ (Hint: Use Fubini's theorem).

c) $\frac{\mathrm{d}}{\mathrm{d}x}(f*g(x)) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) * g(x) = f(x) * \frac{\mathrm{d}}{\mathrm{d}x}g(x)$ if $f, g \in C^1(\mathbb{R}^d)$.

d) $\mathcal{F}(f*g)(x) = \mathcal{F}(f)(x)\cdot\mathcal{F}(g)(x)$ as well as $\mathcal{F}(f\cdot g)(x) = \mathcal{F}(f)(x)*\mathcal{F}(g)(x)$ with $\mathcal{F}$ denoting the Fourier transform (up to a possible normalization constant).

e) $\|f*g\|_r \leq \|f\|_p \|g\|_q$ for $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. This inequality is known as Young's inequality.

The solutions to these exercises will be discussed on Friday, 28.10. The upcoming exercise sheets and important news regarding the lecture will be posted on `http://www.mathematik.uni-muenchen.de/~schaal/msp2_2016/`.