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Exercises on Mathematical Statistical Physics II
Sheet 8

Problem 1 (Ground state on the bose gas)
In class we showed that for the ground state on the bose gas on a torus of unit volume the number of excitations is small compared to the particles in the zero mode in the limit $N \to \infty$.
Prove this fact now by using the language of second quantization, i.e. show that the number n of excitations (w.r.t. $k = 0$) of the ground state of the Hamiltionian

$$H = \sum_k k^2 a_k^\star a_k + \sum_k \widehat{v}(k) \sum_{j,m} a_j^\star a_{m-k}^\star a_m a_{j-k}$$

restricted to the sector of Fock space with N particles is small compared to the total number of particles, $n \ll N$.

Problem 2 (Potential well)

Let $H = \sum_{x\in\mathbb{Z}} V(x) - \Delta$ where Δ is the discrete Laplacian: $\Delta f(x) = f(x-1) + f(x+1) - 2(f(x))$. Let Ψ^{GS} be the ground state of H. Assume that $V(x) = 0$ for all $|x|$ larger than some x_0 and negative else. Show that the ground state of the system decays exponentially in $|x|$, i.e. that there is a $C > 0$ such that $\left|\Psi^{GS}(x)\right| \le e^{-C|x|}$.
Compare to the decay of the ground state of the interacting gas on the torus in the number of excitations.

Problem 3 (Convergence of the trajectories)

In class we showed that $\frac{d}{dt}\mathbb{E}(|X^t - \overline{X}^t|_\infty) \le C(\mathbb{E}(|X^t - \overline{X}^t|_\infty) + o(1))$. Show, that this implies

$$\lim_{N\to\infty} \mathbb{P}(|X^t - \overline{X}^t|_\infty < \varepsilon) = 1$$

for any $\varepsilon > 0$.

Problem 4 (Bounded Lipschitz distance)

Let $\rho(x) := \begin{cases} \frac{3}{4}(-x^2+1) & \text{if } |x|<1 \\ 0 & \text{else.} \end{cases}$.

Let for some $N \in \mathbb{N}$ and $1 \leq j \leq N$ $x_j \in (-1,1)$ be given by

$$\int_{-1}^{x_1} \rho(x)dx = \frac{1}{N} \text{ and } \int_{x_{j-1}}^{x_j} \rho(x)dx = \frac{1}{N}\ .$$

Let $\rho^{emp}(x) = \frac{1}{N}\sum_{j=1}^{N} \delta(x-x_j)$. Estimate $d_{BL}(\rho - \rho^{emp})$.

The solutions to these exercises will be discussed on Friday, 20.01.2017.