Topologically ordered Quantum Spin Systems

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1) Introduction

- Quantum spin systems are models defined on a lattice or a graph, where each point carries a finite dimensional Hilbert space (a "spin")

  They model interacting matter and can be understood in a non-perturbative manner. In particular, the infinite-volume limit is usually under control, which allows for a fundamental understanding / description of the phases of matter and the transitions between them

  (eg see Dyson-Lieb-Simon's first and only proof of the existence of a phase transition for truly interacting spins)

- In this course, we will focus not on thermal phases, but on ground state phases (i.e. at $T = 0$) where topological properties can be analyzed: "topological phases".

Keywords : * local disorder / absence of local order parameters

* ground state degeneracy
* fractional statistics / anyons.
* string operators / loops
* superselection sectors
* entanglement entropy.

to be discussed in finite volume and in the TD-limit.

- Remark: A closely related topic is that of topological insulators which will not be discussed. (also quantum Hall fluids..)
  The model free fermions whose band structure has a non-trivial topology
  Analogy: ground states ~ Fermi projections.
  Similarity: robustness against local perturbations & anyons.
- Simplest topologically ordered model: Kitaev's "toric code model" which will be extensively discussed here: it is simple enough to allow for explicit computations
- We will see that it is usually preferrable to discuss the QSS in terms of the observables (the Heisenberg picture) rather than the states (the Schrödinger picture)
  ⇝ describe ground state spaces in terms of the operators that leave them invariant (stabilizer codes)

- Plan of the course:

(2w) 2) Quantum spin systems, generalities
algebras, states, dynamics & the Lieb-Robinson bound

(3w) 3) Examples. Heisenberg model, AKLT model
Kitaev's model & Ising gauge theory

(1w) 4) Abelian anyons: generalities

(3w) 5) The TD-limit of Kitaev's model, superselection sectors.

(2w) 6) The (topological) entanglement entropy.

(2w) 7) Perturbation theory: stability of topological order

(1+13 w)

2) Quantum spin systems

- Setup: Γ: countable set of vertices:
typically a lattice ($\mathbb{Z}^d$) or a graph
equipped with a metric d, typically the graph distance: the length of the shortest path of edges between $x, y \in \Gamma$.

For each $x \in \Gamma$: $\mathcal{H}_x$ is a finite-dimensional Hilbert space
typically the same $\mathbb{C}^{2j+1}$ at each x, and carrying the $(2j+1)$-dimensional representation of $su(2)$: a "spin j"

For any finite subset $\Lambda \subset \Gamma$: $\mathcal{H}_\Lambda := \bigotimes_{x \in \Lambda} \mathcal{H}_x$

- Algebra of observables on Λ: $\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda)$ (matrices!)

Note: $\Lambda_1 \subset \Lambda_2$, then $\mathcal{A}_{\Lambda_1} \subset \mathcal{A}_{\Lambda_2}$ by the identification $A \in \mathcal{A}_{\Lambda_1}$ with $A \otimes \mathbb{1}_{\mathcal{H}_{\Lambda_2 \setminus \Lambda_1}} \in \mathcal{A}_{\Lambda_2}$.

$\leadsto$ Local algebra: $\mathcal{A}_{loc} := \bigcup_{\substack{\Lambda \in \Gamma \\ |\Lambda| < \infty}} \mathcal{A}_\Lambda$; $[\mathcal{A}_\Lambda, \mathcal{A}_{\Lambda'}] = 0$ if $\Lambda \cap \Lambda' = \emptyset$

<u>Quasi-local algebra</u> : $\mathcal{A} := \overline{\mathcal{A}_{loc}}^{\|\cdot\|}$ is a C^*-algebra.

local observables and norm-limits thereof.

- <u>States</u> : these are "expectation values", i.e. positive, linear, normalized forms over $\mathcal{A}$,

$\omega : \mathcal{A} \longrightarrow \mathbb{C}$:

i) $\omega(A^*A) \geq 0$

ii) $\omega(\mathbb{1}) = 1$.

Note. For any ω and any $\Lambda \in \underset{\sim}{\mathcal{F}}(\Gamma)$, the restriction $\omega \restriction_{\mathcal{L}(\mathcal{H}_\Lambda)}$ is a positive, linear, normalized form over a matrix algebra, i.e. a density matrix ρ_Λ^ω :

the finite subsets of Γ

$$\cancel{A \in \mathcal{A}_\Lambda} \qquad \omega(A) = \mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho_\Lambda^\omega A) \quad \forall A \in \mathcal{A}_\Lambda$$

In general, there is no a priori Hilbert space in the infinite volume limit (i.e. on Γ) and hence no density matrix. Again, ω can be thought of as the family (with the mathematical structure of a net) of density matrices $\{\rho_\Lambda^\omega\}_{\Lambda\in\mathcal{F}(\Gamma)}$, together with its limit.

- Dynamics: Interaction: $\Phi: \mathcal{F}(\Gamma) \to$ self-adjoint operators

$$X \mapsto \Phi(X) \in \mathcal{A}_X$$

and $\Phi(X)$ is the interaction between spins located in X.

$\rightsquigarrow$ for any $\Lambda \in \mathcal{F}(\Gamma)$:

$$H_\Lambda := \sum_{X\subset\Lambda} \Phi(X)$$

Note: There is no limit of H_Λ as $\Lambda\to\Gamma$ in the operator norm topology

For any $\Lambda\in\mathcal{F}(\Gamma)$, the dynamics is defined as

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A\, e^{-itH_\Lambda} \qquad \forall A\in\mathcal{A}_\Lambda .$$

[add: τ_t is an automorphism: $\|\tau_t^\Lambda(A)\| = \|A\|$]

If Φ is too much long-range (insufficient decay of the interaction between far away spins) then τ_t^Λ cannot be extended as $\Lambda\to\Gamma$. Otherwise a locality estimate for the dynamics will allow for this: the Lieb-Robinson bound.

- A Banach space of interactions:

Let $F: [0,\infty) \to (0,\infty)$:

i) $\|F\| := \sup_{x\in\Gamma} \sum_{y\in\Gamma} F(d(x,y)) < \infty$ (integrability)

ii) $C := \sup_{x,y\in\Gamma} \sum_z \frac{F(d(x,z))F(d(z,y))}{F(d(x,y))} < \infty$ (convolution) and let for any $\mu > 0$:

$$F_\mu(x) := e^{-\mu x} F(x)$$

Check: if F satisfies (i, ii), then so does F_μ, and $\|F_\mu\| \le \|F\|$; $C_\mu \le C$.

Concretely: on $\Gamma = \mathbb{Z}^d$,

$$F(x) := (1+x)^{-(d+\varepsilon)} \quad , \; \varepsilon > 0.$$

With this we define ("exponential decay of $\|\Phi(X)\|$, with rate μ").

$$\|\Phi\|_\mu := \sup_{x,y \in \Gamma} \sum_{X \ni x,y} \frac{\|\Phi(X)\|}{F_\mu(d(x,y))} \quad \text{(check: it is a norm)}$$

and $\Phi \in \mathcal{B}_\mu$ if $\|\Phi\|_\mu < \infty$. $\mathcal{B}_\mu$ is a Banach space.

- A propagation bound:

<u>Theorem</u>: Let $\mu \ge 0$ and $\Phi \in \mathcal{B}_\mu$. Let $\Lambda \in \mathcal{F}(\Gamma)$. Then: for any $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$, $X, Y \subset \Lambda$:

$$\| [\tau_t^\Lambda(A), B] \| \le \frac{2\|A\|\|B\|}{C_\mu} g_\mu(t) \sum_{\substack{x \in X \\ y \in Y}} F_\mu(d(x,y))$$

where

$$g_\mu(t) = \begin{cases} e^{2\|\Phi\|_\mu C_\mu |t|} - 1 & \text{if } d(X,Y) > 0 \\ e^{2\|\Phi\|_\mu C_\mu |t|} & \text{otherwise} \end{cases}$$

In particular, if $d(X,Y) > 0$:

$$\sum_{\substack{x \in X \\ y \in Y}} F_\mu(d(x,y)) \le \min(|X|, |Y|)\, e^{-\mu d(X,Y)} \|F\| \quad \text{, i.e.}$$

$$\| [\tau_t^\Lambda(A), B] \| \leq K_\mu(A,B)\, e^{-\mu\left(d(X,Y) - v_\mu(\Phi)|t|\right)}$$

with $K_\mu(A,B) = \frac{2\|A\|\|B\|}{C_\mu} \|F\| \min\left(|X|, |Y|\right)$

and $v_\mu(\Phi) := \frac{2\|\Phi\|_\mu C_\mu}{\mu}$

In other words: outside the "sound cone" $d(X,Y) \leq v_\mu(\Phi)|t|$ the commutator is exponentially small so the support of A grows at most linearly with time.

In fact, one can show that for any $\delta > 0$ and any t, there exists an observable $A_t^\delta \in \mathcal{A}_{B_t(X,\delta)}$ s.t.

$$\| \tau_t^\Lambda(A) - A_t^\delta \| \leq C e^{-\mu\delta}$$

where $B_t(X,\delta) := \{ x \in \Lambda : d(X,x) \leq v_\mu(\Phi)|t| + \delta \}$.

so for any $t \neq 0$, the support of $\tau_t(A)$ is – in general – all of Λ, but it is still mostly localized in a $v_\mu(\Phi)|t|$ fattening of the initial support of A.
This is an "almost"-version of Einstein's relativistic causality.

- Proof of LR-bound.

Let $f : \mathbb{R} \to \mathcal{A}$: $\quad f(t) := [\tau_t^{\hat{}}(A), B]$

Then $f'(t) = i\,[\tau_t^{\hat{}}(\underbrace{[H, A]}_{=[H^X, A]}), B] = i\,[[\tau_t^{\hat{}}(H_X), \tau_t^{\hat{}}(A)], B]$, where

$$H^X := \sum_{\substack{Z \subset \Lambda: \\ Z \cap X \neq \phi}} \Phi(Z)$$

$$\overset{\text{Jacobi}}{=} i\,[f(t), \tau_t^{\hat{}}(H_X)] + i\,[\tau_t^{\hat{}}(A), [\tau_t^{\hat{}}(H_X), B]]$$

This is of the form $\quad \partial_t Y(t) = L(t) Y(t) + I(t) \qquad (*)$

where $L(t)$ is a linear operator $\mathcal{A} \to \mathcal{A}$.

If $X(t)$ is the solution of $\partial_t X(t) = L(t) X(t)$, we denote by γ_t the map $X(t) = \gamma_t(X(0))$.

Then, the solution of $(*)$ reads

$$Y(t) = \gamma_t\left(Y(0) + \int_0^t \gamma_s^{-1}(I(s))\, ds \right)$$

Hence $\quad \| Y(t) - \gamma_t(Y(0)) \| \leq \int_0^t \| \gamma_s^{-1}(I(s)) \|\, ds$

in the case $L(t) Y(t) = i\,[f(t), \tau_t^{\hat{}}(H_X)]$ preserves the operator norm (since it is generated by a self-adjoint operator) hence: $(\|\gamma_s(\ldots)\| = \|(\ldots)\|)$

$$\|f(t)\| \leq \|f(0)\| + \int_0^{|t|} \| [\tau_s^{\hat{}}(A), [\tau_s^{\hat{}}(H_X), B]] \|\, ds$$

$$\leq \|f(0)\| + 2\|A\| \int_0^{|t|} \| [\tau_s^{\hat{}}(H_X), B] \|\, ds$$

since $\tau_t^{\hat{}}$ is an automorphism.

Let now $\displaystyle C_B(X,t) := \sup_{A \in \mathcal{A}_X} \frac{\| [\tau_t^\wedge(A), B] \|}{\|A\|}$

$$C_B(X,t) \le C_B(X,0) + 2 \sum_{Z \cap X \neq \emptyset} \|\Phi(Z)\| \int_0^{|t|} C_B(Z,s)\, ds$$

which can be readily iterated !

Note. $C_B(\cancel{Z},0) \le 2\|B\| \underbrace{\delta_Y(Z)}$

$= 0$ if $Z \cap Y = \emptyset$
$= 1$ otherwise.

Hence:

$$C_B(X,t) \le 2\|B\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} a_n$$

where $\displaystyle a_n = \sum_{Z_1 \cap X \neq \emptyset} \sum_{Z_2 \cap Z_1 \neq \emptyset} \cdots \sum_{Z_n \cap Z_{n-1} \neq \emptyset} \prod_{i=1}^{n} \|\Phi(Z_i)\| \, \delta_Y(Z_n)$

now: $\displaystyle a_1 = \sum_{\substack{Z \cap X \neq \emptyset \\ Z \cap Y \neq \emptyset}} \|\Phi(Z)\| \le \sum_{\substack{x \in X \\ y \in Y}} \sum_{Z \ni x,y} \|\Phi(Z)\| \le \|\Phi\|_\mu \sum_{\substack{x \in X \\ y \in Y}} F_\mu(d(x,y))$

similarly $\displaystyle a_2 \le \sum_{\substack{x \in X \\ y \in Y}} \sum_{z \in \Lambda} \sum_{Z_1 \ni x,z} \|\Phi(Z_1)\| \sum_{Z_2 \ni z,y} \|\Phi(Z_2)\|$

$$\le \|\Phi\|_\mu^2 \sum_{\substack{x \in X \\ y \in Y}} \underbrace{\sum_{z \in \Lambda} F_\mu(d(x,z)) F_\mu(d(z,y))}_{\le C_\mu F_\mu(d(x,y))}$$

and so: $\displaystyle a_n \le \|\Phi\|_\mu^n C_\mu^{n-1} \sum_{x \in X} \sum_{y \in Y} F_\mu(d(x,y))$.

which yields the claim.

If $d(X,Y) > 0$, then $a_0 = 0$, hence the "-1" $\square$

- Note that all bounds are independent of the volume Λ.
- With this, we prove the existence of dynamics in the infinite volume limit.

<u>Theorem</u>: $\mu \geq 0$, $\Phi \in \mathcal{B}_\mu$. $\exists\ \{\tau_t\}_{t\in\mathbb{R}}$ a strongly continuous, one-parameter group of automorphisms s.t.

$$\lim_{n\to\infty} \| \tau_t^{\Lambda_n}(A) - \tau_t(A) \| = 0 \qquad \forall A \in \mathcal{A}$$

where $\Lambda_n \subset \Lambda_m$ if $n \leq m$ and for any $X \in \mathcal{P}(\Gamma)$, $\exists\ n_0 : X \subset \Lambda_n\ \forall n \geq n_0$.

<u>Proof</u>: $n \leq m$. $A \in \mathcal{A}_Y$.

$$\tau_t^{\Lambda_m}(A) - \tau_t^{\Lambda_n}(A) = \int_0^t \underbrace{\frac{d}{ds}\left(\tau_s^{\Lambda_m} \circ \tau_{t-s}^{\Lambda_n}(A)\right)}_{= \tau_s^{\Lambda_m}\left(i[\underbrace{H_{\Lambda_m} - H_{\Lambda_n}}_{\text{all interaction terms in } \Lambda_m \text{ that are not completely in } \Lambda_n},\ \tau_{t-s}^{\Lambda_n}(A)]\right)} ds$$

$$\Rightarrow \| \tau_t^{\Lambda_m}(A) - \tau_t^{\Lambda_n}(A) \| \leq \sum_{x\in\Lambda_m\setminus\Lambda_n} \sum_{X \ni x} \int_0^{|t|} \underbrace{\| [\Phi(X), \tau_s^{\Lambda_n}(A)] \|}_{\text{by LR: } \leq \frac{2\|A\|\|\Phi(X)\|}{C_\mu} g_\mu(s) \sum_{z\in X,\, y\in Y} F_\mu(d(z,y))} ds$$

we replace $\sum_{X\ni x} \sum_{z\in X}$ by $\sum_{z\in\Gamma} \sum_{X \ni x,z}$, we obtain,

$$\|\tau_t^{\Lambda_m}(A) - \tau_t^{\Lambda_n}(A)\| \leq \frac{2\|A\|}{C_\mu} \int_0^{|t|} g_\mu(s) \sum_{\substack{x \in \Lambda_m \setminus \Lambda_n \\ z \in \Gamma}} \sum_{X \ni x, z} \|\Phi(X)\| \sum_{y \in Y} F_\mu(d(z,y))$$

$$\leq 2\|A\| \, \|\Phi\|_\mu \, |Y| \underbrace{\sup_{y \in Y} \sum_{x \in \Lambda_m \setminus \Lambda_n} F_\mu(d(x,y))}_{\leq \|F\| e^{-\mu d(Y, \Lambda_m \setminus \Lambda_n)}}$$

$$\longrightarrow 0$$

hence $\tau_t^{\Lambda_n}(A)$ is a Cauchy sequence and converges. $\square$

$\leadsto$ $(\mathcal{A}, \tau_t)$ is a quantum dynamical system (in the infinite volume limit)

- As discussed, $\{H_\Lambda\}_{\Lambda \in \mathcal{F}(\Gamma)}$ does not have a limit as an operator. However, the dynamics it generates does. Another way to express this would be that $\forall A \in \mathcal{A}_{loc}$:

$$i[H_\Lambda, A] \xrightarrow{\Lambda \uparrow \Gamma} \delta(A)$$

where $\delta(A) = \frac{d}{dt} \tau_t(A) \Big|_{t=0}$ is the generator of τ_t.

For a finite range interaction, this is pretty clear: $A \in \mathcal{A}_X$

$$[H_\Lambda, A] = \sum_{\substack{Y \subset \Lambda \\ Y \cap X \neq \emptyset}} [\Phi(Y), A]$$

is independent of Λ for Λ large enough. Otherwise, one needs sufficient decay of $\|\Phi(Y)\|$ as $|Y|$ or $\mathrm{diam}(Y)$ grows, as expressed, for example, by the condition $\Phi \in \mathcal{B}_\mu$.

3) Examples

a) The Heisenberg model:

Here: $\mathcal{H}_x = \mathbb{C}^{2j+1}$ carries the $(2j+1)$-dimensional irreducible representation of $su(2)$, i.e. the three spin matrices of spin j:

$$S_x^1, \; S_x^2, \; S_x^3$$

$$[S_x^i, S_x^j] = i\varepsilon^{ijk} S_x^k$$

e.g. $j=\frac{1}{2}$: the three Pauli matrices

$$H_\Lambda := -\sum_{x,y\in\Lambda} \left(J_{xy}^1 S_x^1 S_y^1 + J_{xy}^2 S_x^2 S_y^2 + J_{xy}^3 S_x^3 S_y^3 \right)$$

i.e. a two-body interaction, with some decay of $|J_{xy}^i|$ as $d(x,y) \to \infty$.

typically: $J_{xy}^i = 0$ if $d(x,y) > 1$: nearest-neighbour interaction.

Simplest case:

$$H_\Lambda^{F/AF} = -J \sum_{\substack{x,y\in\Lambda \\ d(x,y)=1}} \overbrace{\sum_{i=1}^{3} S_x^i S_y^i}^{S_x \cdot S_y} \qquad \begin{cases} J > 0 & \text{ferromagnet} \\ J < 0 & \text{Antiferromagnet} \end{cases}$$

Note: $S_x \cdot S_y = \frac{1}{2} T_{xy} - \frac{1}{4} \qquad \left(j = \frac{1}{2}\right)$

where $T_{xy}(\varphi_x \otimes \varphi_y) = \varphi_y \otimes \varphi_x$ (transposition)

hence, $S_x \cdot S_y$ is maximized ~~by~~ on symmetric vectors and the following is not surprising:

Theorem. ($j = \frac{1}{2}$) Let $J > 0$, Λ connected. The ground state space $\mathcal{G}_\Lambda^F$ has dimension $|\Lambda|+1$ and consists of all vectors invariant under permutations of the vertices. (maximal total spin)

If e_1, e_2 is an ONB of $\mathbb{C}^2$, a basis of $\mathcal{G}_\Lambda^F$ is given by the vectors:

$$\Psi_k := \frac{1}{(|\Lambda|!)^{1/2}} \sum_{\pi \in S(\Lambda)} U_\pi \Big(\big(\underbrace{e_1 \otimes \cdots \otimes e_1}_{k}\big) \otimes \big(\underbrace{e_2 \otimes \cdots \otimes e_2}_{|\Lambda|-k}\big) \Big)$$

for $k = 0, 1, \dots, |\Lambda|$

unitary implementation of the permutation π.

Now pick a site "0" $\in \Lambda$, consider the Pauli matrix

σ_0^3

and e_1, e_2 its eigenbasis: $\sigma_0^3 e_1 = +1$, $\sigma_0^3 e_2 = -1$.

as $\sigma_0^3 \, U_\pi \big((e_1 \otimes \cdots \otimes e_1) \otimes (e_2 \otimes \cdots \otimes e_2) \big)$

$= \underset{\sim}{\pm} \, U_\pi \big(\qquad \big)$

depending on ~~whether~~ $\pi^{-1}(\{0\})$

Hence. $\langle \Psi_k, \sigma_0^3 \Psi_k \rangle = + \frac{k}{N} - \frac{(N-k)}{N} = \frac{2k-N}{N} \in [-1, 1]$.

and the various ground states are distinguishable by looking at the local magnetisation $\langle \Psi_k, \sigma_0^3 \Psi_k \rangle$. i.e. only at site $\{0\}$.

σ_0^3 is called a local order parameter.

- Note: If $J < 0$, there is only one ground state, the totally antisymmetric vector.

b) The AKLT model [Affleck–Kennedy–Lieb–Tasaki]

- 1-d model with spin 1, i.e. $\mathcal{H}_x = \mathbb{C}^3$, and

$$H_{[1,N]} = \sum_{i=1}^{N-1} \left(S_i \cdot S_{i+1} - \frac{1}{3} (S_i \cdot S_{i+1})^2 \right)$$

Note: the choice $-\frac{1}{3}$ makes the interaction equal to (up to an additive constant) the projection onto the spin-2 subspace of $D_{(1)} \otimes D_{(1)}$ (see Clebsch–Gordan decomposition:

$$D_{(1)} \otimes D_{(1)} = D_{(0)} \oplus D_{(1)} \oplus D_{(2)}$$

dimensions: $3 \cdot 3 \qquad 1 + 3 + 5$)

Result: The ground state space is 4-dimensional for all $N \geq 2$, with a basis $\{\psi_{\alpha\beta}\}_{\alpha,\beta = +,-}$ with an explicit description.

basis of $\bigotimes_{i=1}^{N} \mathcal{H}_x$: $\{\psi_A\}_{A \in W_N}$

where W_N is the set of words of length N with characters $-, 0, +$

$$\psi_A = e_{A_1} \otimes e_{A_2} \otimes \cdots \otimes e_{A_N}$$

where $\{e_-, e_0, e_+\}$ is the ONB of $\mathbb{C}^3$ that is the eigenbasis of S^3.

$$\psi_{\alpha\beta} = \sum_{A \in W_N} \psi_{\alpha\beta}(A)\, \psi_A \quad \text{with the following rules.}$$

$\psi_{\alpha\beta}(A)$ is zero unless the non-zero characters alternate, separated by an arbitrary number of "0", while the first non-zero character is α and the last one is β. "dilute Néel order"

e.g. for $\alpha = +$, $\beta = -$: $|0+-00+000-+0-\rangle$

The coefficients can be explicitly computed: e.g.

$$\Psi_{+-}(A) = (-1)^m \, 2^k$$

with $m = \#$ of odd sites having "+" or "−"
$k = \#$ of pairs of "+, −".

- Difference with the Heisenberg model: there is no local order parameter.

Proposition: Let $A \in \mathcal{A}_{[\ell,1]}$. For any choice of α, β:

$$\lim_{N\to\infty} \frac{\langle \Psi^{(N)}_{\alpha\beta}, A\, \Psi^{(N)}_{\alpha\beta}\rangle}{\|\Psi^{(N)}_{\alpha\beta}\|^2} = \frac{\sum_{\gamma,\delta} \langle \Psi^{\ell}_{\gamma\delta}, A\, \Psi^{\ell}_{\gamma\delta}\rangle}{\|\Psi^{\ell}_{\gamma\delta}\|^2}$$

Furthermore: For any choice of $\alpha, \beta, \gamma, \delta$:

$$\lim_{N\to\infty} \frac{\langle \Psi^{(N)}_{\alpha\beta}, A\, \Psi^{(N)}_{\gamma\delta}\rangle}{\|\Psi^{(N)}_{\alpha\beta}\| \, \|\Psi^{(N)}_{\gamma\delta}\|} = \delta_{\alpha\gamma}\, \delta_{\beta\delta} \cdot C_A$$

The convergence is exponential, with error bounds $\leq C\, 3^{-N}$.

In other words: in the limit of large chains, local observables have the same expectation values in all 4 ground states, and moreover, no local observable maps one ground state into another.

$\Rightarrow$ local disorder

As a consequence: let $P^{(N)}$ be the projection onto the ground state space of H_{EKNT}, namely

$$P^{(N)} = \sum_{\alpha\beta} \frac{|\Psi_{\alpha\beta}^{(N)}\rangle\langle\Psi_{\alpha\beta}^{(N)}|}{\|\Psi_{\alpha\beta}^{(N)}\|^2}.$$

Then:

$$\| P^{(N)} A P^{(N)} - \mathcal{P}(A) P^{(N)} \| \leq C_A 3^{-N} \qquad (*)$$

where $\mathcal{P}(A) = \sum_{\alpha\beta} \frac{\langle \Psi_{\alpha\beta}^{(N)}, A\Psi_{\alpha\beta}^{(N)}\rangle}{\|\Psi_{\alpha\beta}^{(N)}\|^2}$

i.e.: Restricted to the ground state space, any local observable is a multiple of the identity, as $N \to \infty$.

Notes: * A model for which the ground state space $P^{(N)}$ is such that
i) $\dim P^{(N)} > 1$
ii) $(*)$ holds for all $A \in \mathcal{A}_{loc}$ for some (exponentially) decaying r.h.s.

is called locally topologically ordered (i.e. locally disordered...!)
From that point of view, topological order may already exist in $d=1$.

* There is no long range order, but the number of $+,-$ can be read from the following sum: $\sum_{j=x+1}^{y-1} S_j^3$, which yield the following string order parameter (non local!)

$$(-1)^{d(y,x)}\, \omega\left(S_x^3\, e^{i\pi \sum_{j=x+1}^{y-1} S_j^3}\, S_y^3\right) > 0 \quad \forall x,y$$

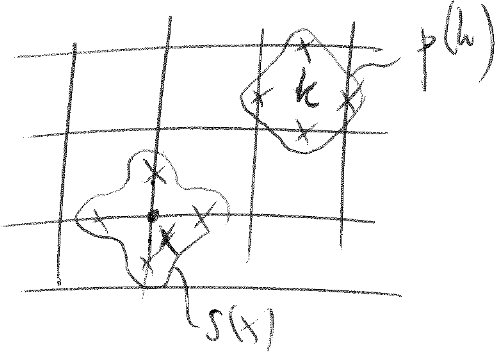
(if S_x^3 is $+$, then the value of S_y^3 is known)

c) The toric code model [Kitaev]

- Here $\Gamma = \mathbb{Z}^d$ and the spins lie on the edges of the lattice, the set of which we denote $\mathcal{E}(\Gamma)$.

$\forall e \in \mathcal{E}(\Gamma)$, $\mathcal{H}_e = \mathbb{C}^2$

For each $x \in \Gamma$, $s(x)$ is the set of edges attached to x.
— — $k \in \Gamma^*$, $p(k)$ ——— — ——— surrounds the plaquette



Let $A_x := \prod_{j \in s(x)} \sigma_j^1$; $B_k := \prod_{j \in p(k)} \sigma_j^3$ where

$\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices.

Note. $\forall x \in \Gamma$, $k \in \Gamma^*$: $[A_x, B_k] = 0$.

indeed: either $s(x) \cap p(k) = \emptyset$ and the statement is trivial,

or $s(x) \cap p(k)$ has exactly two elements, $\{i, j\}$.

$$A_x B_k = \sigma^1 \sigma^1 \underbrace{\sigma_i^1 \sigma_i^3}_{= -\sigma_i^3 \sigma_i^1} \underbrace{\sigma_j^1 \sigma_j^3}_{= -\sigma_j^3 \sigma_j^1} \sigma^3 \sigma^3$$

$$= B_k A_x$$

- The toric code Hamiltonian:

$$H_\Lambda := -\sum_{x:\, s(x)\subset\Lambda} A_x - \sum_{h:\, p(h)\subset\Lambda} B_h$$

is therefore a sum of commuting operators. Moreover:

$$Sp(A_x) = Sp(B_h) = \{-1, 1\}$$

since $A_x^2 = B_h^2 = \mathbb{1}$ $\forall x, h$. Hence, the ground state space of H_Λ is given by

$$\mathcal{G}_\Lambda := \{\Psi \in \mathcal{H}_\Lambda : A_x\Psi = \Psi,\ B_h\Psi = \Psi,\ \forall x\in\Gamma, h\in\Gamma^d : s(x)\subset\Lambda;\ p(h)\subset\Lambda\}.$$

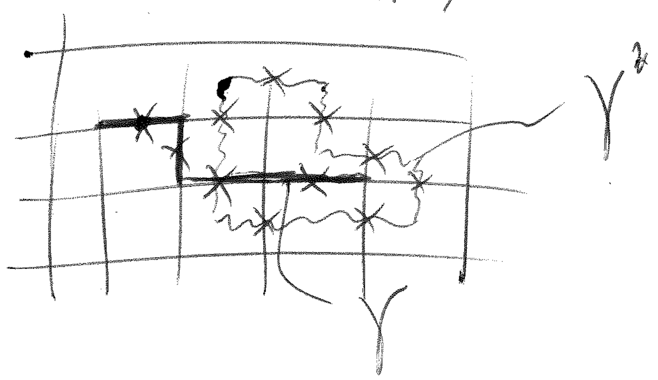
- We start with Λ being a finite square, with periodic boundary conditions, i.e. a torus.

Then, $p(h)$ are faces, or cells, of Λ and $s(x)$ are similarly faces or cells of the dual graph.

The algebra $\mathcal{A}_\Lambda$ is generated (as an algebra) by $\{\sigma_x^1, \sigma_x^3, x\in\Lambda\}$ but equivalently as a vector space by the monomials in $\{\sigma_x^1, \sigma_x^3, x\in\Lambda\}$ with coefficients in $\mathbb{C}$. Since they anticommute, a vector space basis of $\mathcal{A}_\Lambda$ is given by

$$X_{\gamma^d} Z_\gamma \quad , \quad X_{\gamma^d} := \prod_{x\in\gamma^d}\sigma_x^1;\quad Z_\gamma := \prod_{x\in\gamma}\sigma_x^3$$

γ, γ^d are paths on the lattice/dual lattice



The set of all γ^* naturally carry the mathematical structure of a 1-chain with $\mathbb{Z}_2$ coefficients, and γ would be a 1-chain of the dual graph.

Insert: CW-complex

So far: we think of a graph embedded on a manifold
Naturally: hierarchical structure: vertices of the graph, edges of the graph, faces, ...

A CW-complex K is a topological space X with a decomposition

$$X = \bigcup_{n=0}^{N} X^{(n)} \qquad N \in \mathbb{N}$$

(N may be ∞, but we shall consider only $N < \infty$, in fact $N = 2$ here), such that

i) $X^{(0)}$ is discrete (i.e. a set of vertices)

ii) $X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \dots$

and $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching a (possibly infinite) number of n-cells, where an n-cell is a space homeomorphic to the n ball $\sum_{i=1}^{n-1} x_i^2 \leq 1$ and "attaching" means that (part of) $X^{(n-1)}$ is identified with the boundary of the cell

$$\begin{array}{ccc} \partial S^n & \xrightarrow{\chi} & X^{(n-1)} \\ {\scriptstyle id}\downarrow & & \downarrow \\ S^n & \longrightarrow & X^{(n-1)} \cup_\chi S^n \end{array}$$

Typically: attaching an edge to two points, attaching a face to a certain number of edges, For us: $K_\Lambda = K_\Lambda^{(0)} \cup K_\Lambda^{(1)} \cup K_\Lambda^{(2)}$

* $K_\Lambda^{(0)} \simeq \Lambda$ the set of vertices
* $K_\Lambda^{(1)} \simeq \mathcal{E}(\Lambda)$ — " — edges
* $K_\Lambda^{(2)}$ is the set of faces i.e. plaquettes $p(\Lambda)$.

We want to define homology, which could be done by understanding the attaching in details. Heuristically (ignoring orientations, for example)

The r-chain group $C_r(K)$ is the free Abelian group generated by the r-cells, namely the set of "words"

$$c_r := a_1^{n_1} \cdots a_e^{n_e} \quad (\text{a chain})$$

with coefficient n_i in a field. Here: $\mathbb{Z}_2$

The boundary of an r-cell is an $(r-1)$ cell, and this allows for the def of a boundary operator

$$\partial_r : C_r(K) \to C_{r-1}(K)$$

mapping an r-chain to its boundary.

An r-chain is called an r-cycle if $c \in C_r(K)$ and

$$\partial_r c = 0$$

Of particular importance here: the 1-cycles which are closed loops of edges.

The set of r-cycles: $Z_r(K) := \ker(\partial_r)$

Reciprocally an r-boundary is a $c \in C_r(K)$ s.t. $\exists$ $d \in C_{r+1}(K)$ s.t.

$$c = \partial_{r+1} d.$$

Set of r-boundaries $B_r(K) := \mathrm{Ran}(\partial_{r+1})$

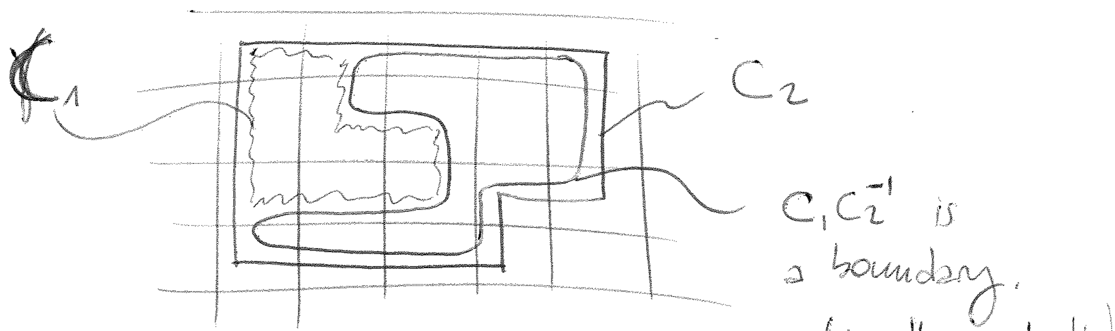
Note: $\partial_r \partial_{r+1} = 0$ so that $B_r(K) \subseteq Z_r(K)$.

The r^{th} homology group of K is:

$$H_r(K) := Z_r(K) / B_r(K)$$

namely the set of equivalence classes of r-cycles, where two cycles are equivalent if they differ by a boundary.

Case r=1: set of closed loops that cannot be "deformed" into each other.



Betti numbers: $b_r(K) := \text{rank}\, H_r(K)$ (i.e. the cardinality of a smallest generating set)

End of insert

- Let now $E_{\gamma^*\gamma} := X_{\gamma^*} Z_\gamma$ be an arbitrary element in the basis. Clearly:

$$[Z_\gamma, B_k] = 0$$
$$[X_{\gamma^*}, A_x] = 0$$

and $[X_{\gamma^*}, B_k] = 0$ if $|\gamma^* \cap p(k)|$ is even
$\{X_{\gamma^*}, B_k\} =$ if $|\gamma^* \cap p(k)|$ is odd.

Similarly $E_{\gamma^*\gamma}$ either commutes or anticommutes with any A_x / B_k.

Now if $\exists\, x$: $\{E_{\gamma^*\gamma}, A_x\} = 0$. Then, $\forall \psi, \varphi \in \mathcal{G}_1$

$$\langle \psi, E_{\gamma^*\gamma}\varphi \rangle = \langle \psi, A_x E_{\gamma^*\gamma} A_x \varphi \rangle = -\langle \psi, \underbrace{A_x^2}_{=1} E_{\gamma^*\gamma} \varphi \rangle$$

so that $\langle \psi, E_{\gamma^*\gamma}\varphi \rangle = 0$ and $E\mathcal{G}_1 \in \mathcal{G}_1^{\perp}$.

It follows that

$$E_{\gamma^*\gamma}\,\mathcal{G}_\Lambda \subseteq \mathcal{G}_\Lambda \iff [E, A_x] = 0 = [E, B_h] \quad \forall x, h$$

which yields a description of the algebra acting on the ground state space

$$\mathcal{A}(\mathcal{G}_\Lambda) := \{ O \in \mathcal{A}_\Lambda : O\mathcal{G}_\Lambda \subset \mathcal{G}_\Lambda \},$$

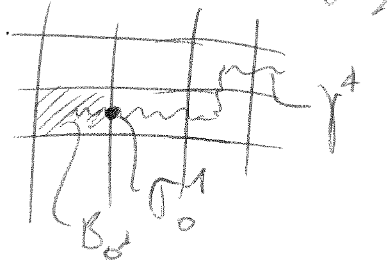
namely

$$\mathcal{A}(\mathcal{G}_\Lambda) = \underbrace{\mathcal{A}_{A,B}'}_{} \supseteq \mathcal{A}_{A,B}.$$

commutant of the algebra generated by the A_x, B_h's.

- It remains to characterise this algebra.

Suppose γ^* is not a 1-cycle. Then it has a boundary (by definition, and also because there must be at least one end of a string), and



$$[\underbrace{B_{\sigma^*}, \sigma_0^1}_{\in \partial_1 \gamma^*}] = \sigma^3\sigma^3\sigma^3[\sigma_0^3, \sigma_0^1] \neq 0.$$

Hence $\cancel{\gamma^* \in}$ $X_{\gamma^*}Z_\gamma \in \mathcal{A}(\mathcal{G}_\Lambda)$

$\Rightarrow \gamma^*, \gamma$ are 1 cycles, i.e. closed loops.

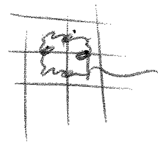
Reciprocally: if γ^* is a cycle, then it has an even number of intersections with the boundary of any face (γ^* must "enter" and "leave" the face) so that

$$[X_{\gamma^*}, B_h] = 0 \quad \forall h.$$

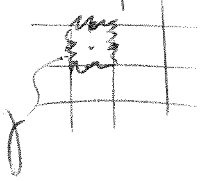
Altogether: $X_{\gamma^*}Z_\gamma \in \mathcal{A}(\mathcal{G}_\Lambda) \iff \gamma$ and γ^* are 1-cycles

- Now: how is this algebra represented on $\mathcal{G}_\Lambda$?

As we have seen: $A_x\Psi = \Psi$, $B_h\Psi = \Psi$ $\forall \Psi \in \mathcal{G}_\Lambda$,
i.e. A_x, B_h all act as the identity on $\mathcal{G}_\Lambda$.
Note that A_x is X_{γ^*} for a simple closed closed loop:



γ^* and



γ Z_γ for a B_h.

More generally: if γ^* is a "contractible" loop — precisely: $\gamma^* \in B_1(K)$ is a 1-boundary — then

$$X_{\gamma^*} = \prod_i \cancel{B_{h_i}}\, A_{x_i}$$

where $\gamma^* = \partial_2 \left(\cancel{h_1 h_2 \cdots h_n}\right) \underbrace{(s_1 \cdots s_n)}_{\text{stars}}$

the ~~faces~~ "inside" γ^*.

since any edge inside the loop will appear twice in the product, and $(\sigma^1)^2 = \mathbb{1}$

i.e. "trivial loops correspond to the trivial operator $\mathbb{1}$".

Furthermore: if γ_1^*, γ_2^* belong to the same homology group, then $\gamma_2^*(\gamma_1^*)^{-1} \in B_1(K)$ so that

$$X_{\gamma_2^*}\Psi = X_{\gamma_1^*\gamma_2^*(\gamma_1^*)^{-1}}\Psi = X_{\gamma_1^*}X_{\gamma_2^*(\gamma_1^*)^{-1}}\Psi$$

$$= X_{\gamma_1^*}\Psi$$

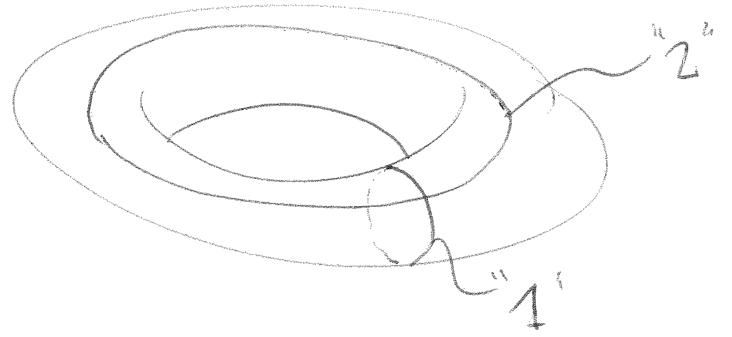
recall $C_1(K)$ is an Abelian group.

Hence $X_{\gamma^*}\mathcal{G}_\Lambda = X_{[\gamma^*]}\mathcal{G}_\Lambda$

$\Rightarrow$ The only non-trivial operators on $\mathcal{G}_\Lambda$ are associated with the equivalence classes of non-trivial loops, and

$\mathcal{A}(G_1)$ is generated on the torus, as an algebra, by

$$X_1,\ X_2,\ Z_1,\ Z_2\ .$$



Note: $[X_i, X_j] = 0$, $[Z_i, Z_j] = 0$

$[X_1, Z_2] = 0$, $[X_2, Z_1] = 0$

$\{X_1, Z_2\} = 0$ as they share one spin (or an odd number thereof).

hence: $\mathcal{A}(G_1)$ is a representation of the algebra generated by

$$\underset{X^1}{\mathbb{1} \otimes \sigma^1}\ ;\ \underset{X^2}{\sigma^1 \otimes \mathbb{1}}\ ,\ \underset{Z^1}{\sigma^3 \otimes \mathbb{1}}\ ,\ \underset{Z^2}{\mathbb{1} \otimes \sigma^3}$$

the algebra of two spin-$\frac{1}{2}$!

$\rightarrow$ $\dim G_1 = 2^2 = 4$

and more generally:

$$\dim G_K = 2^{b_1(K)}$$

for a nice embedding of a graph in a Riemannian manifold M with genus g: $g = \frac{1}{2} b_1(K)$, so that

$$\dim G = 2^{2g}$$

- Summarizing: Let K be a 2-dimensional connected CW-complex, with

$$b_1(K) := \dim(H_1(K;\mathbb{Z}_2))$$

Then:

$$\dim \mathcal{G}_K = 2^{b_1(K)}$$

$\mathcal{A}(\mathcal{G}_K)$ is generated by $\{X_i, Z_i \quad i \in H_1(K;\mathbb{Z}_2)\}$ and it is $*$-isomorphic to the $b_1(K)$-fold tensor product of the algebra of Pauli matrices.

If K corresponds to a 2-d Riemannian manifold without boundary of genus g, then

$$\dim \mathcal{G}_K = 2^{2g}$$

Hence: The ground state degeneracy depends on the topology of the lattice (CW-complex)

This is our second characterisation of topological order

- We label X_i, Z_i s.t. $\{X_i, Z_i\} = 0$.

Let $\{\Psi_{\alpha\beta}, \ \alpha, \beta \in \{-,+\}\}$ be s.t.

$$Z_1 \Psi_{\alpha\beta} = \alpha \Psi_{\alpha\beta}$$
$$Z_2 \Psi_{\alpha\beta} = \beta \Psi_{\alpha\beta}.$$

Then:

$$Z_1 X_1 \Psi_{\alpha\beta} = -X_1 Z_1 \Psi_{\alpha\beta} = -\alpha X_1 \Psi_{\alpha\beta}.$$

Hence: all four ground states can be obtained from one of them by the action of X_1, X_2:

$$\Psi_{--}, \ X_1 \Psi_{--} = \Psi_{+-}; \ X_2 \Psi_{--} = \Psi_{-+}; \ X_1 X_2 \Psi_{--} = \Psi_{++}.$$

- Let $A \in \mathcal{A}_\Lambda$ be decomposed as

$$A = \sum_{\gamma, \gamma^*} c_{\gamma\gamma^*} E_{\gamma\gamma^*} \qquad , \gamma, \gamma^* \text{ are chains and co-chains.}$$

Here: A is called local if $[\gamma] \neq 0$ or $[\gamma^*] \neq 0 \Rightarrow c_{\gamma\gamma^*} = 0$

$\forall \varphi, \psi \in \mathcal{G}_\Lambda, \ \langle \varphi, \psi \rangle = 0$

$$\langle \varphi, A\psi \rangle = \sum_{\substack{\gamma, \gamma^* \\ \text{(co)-}\underline{\text{cycles}}}} c_{\gamma\gamma^*} \langle \varphi, E_{\gamma\gamma^*} \psi \rangle$$

Now: Assume that $c_{\gamma\gamma^*} = 0$ whenever $[\gamma]$ or $[\gamma^*]$ is not the neutral element (only "contractible" loops).
Then $E_{\gamma\gamma^*}$ is a product of A's and B's, i.e.

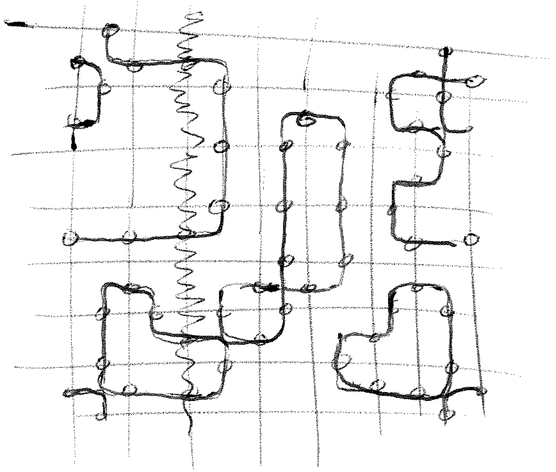
$$A \in \mathcal{A}_{A,B}$$

so that $\langle \varphi, A\psi \rangle = c(A) \cdot \langle \varphi, \psi \rangle = 0$.

$\rightsquigarrow$ This corresponds to (but it is not exactly the same!) the first characterisation of topological order: local observables are <u>diagonal</u> on the ground state space (here in fact, they are exactly so for any Λ large enough).

- Explicit description of the ground states in the product basis of σ^3: $\psi = \sum_z c_z |z\rangle$, where $z = z_1 z_2 \cdots z_{|\Lambda|}$ and $z_i \in \{-1, 1\}$.

The constraint $B_h \psi = \psi$ implies $c_z = 0$ if $\exists h$ s.t. $\prod_{j \in p(h)} z_j = -1$.
Each such conf. determines a loop configuration:



"o" indicate "−", the others are "+"

All such loop configuration are obtained from the $z_i = +1 \ \forall i$ state by acting with a product of A_x's, iff there is no non-trivial loop.

Among those, all A_x must act trivially, hence all c_z must be equal.

Furthermore, along any non-trivial loop, there is an even number of "$\neq$" spins: $Z_1\Psi = \Psi$, $Z_2\Psi = \Psi$ so that this represents the Ψ_{++} state:

$$\Psi_{++} = \sum_{\substack{\text{loop configurations } L \\ \text{all contractible loops}}} |z_L\rangle$$

and the others are obtained from this by the action of $X_{1,2}$.

⇝ ground states as a loop gas

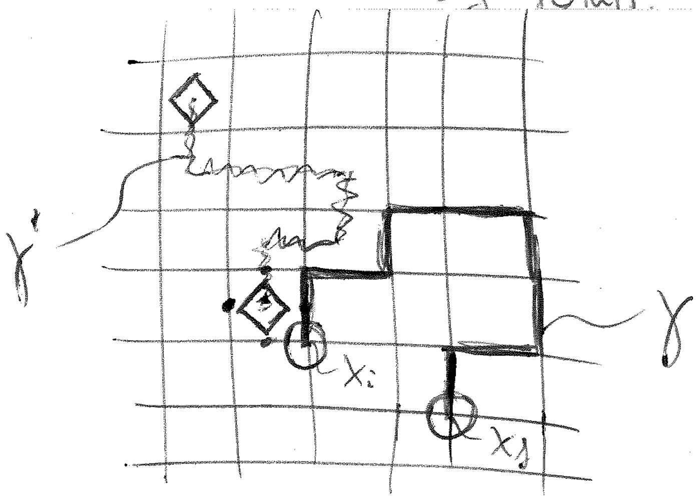
- Excitations: Since A_x, B_h's all commute, we can look for excitations in the joint eigenspaces of all A_x, B_h!
  An elementary excitation (particle) is created when $\exists x : A_x\varphi = -\varphi$ or $B_h\varphi = -\varphi$

On the torus: $\prod_x A_x = \mathbb{1}$ since every edge is shared by exactly two stars, and similarly $\prod_h B_h = \mathbb{1}$.

Hence: if $\exists x : A_x\varphi = -\varphi$, then $\exists x' : A_{x'}\varphi = -\varphi$ (on the torus), and

i.e. particles arise in pairs

- Formulated differently: if γ is a 1-chain, which is not a cycle, then it has at least two boundary points.



For any ground state Ψ: $Z_\gamma \Psi$ is such that

$$\langle Z_\gamma \Psi, H_\Lambda Z_\gamma \Psi \rangle = 4$$

Indeed: $[Z_\gamma, B_h] = 0 \quad \forall h$, while

$$H_\Lambda Z_\gamma = + \left[-(A_{x_i} + A_{x_j}) Z_\gamma + Z_\gamma \left(H_\Lambda + (A_{x_i} + A_{x_j}) \right) \right]$$
$$= Z_\gamma H_\Lambda + 2 Z_\gamma (A_{x_i} + A_{x_j})$$

so that, since $Z_\gamma^2 = \mathbb{1}$, and $A_x \Psi = \Psi$

$$\langle Z_\gamma \Psi, H_\Lambda Z_\gamma \Psi \rangle = \langle \Psi, H_\Lambda \Psi \rangle + 4 \langle \Psi, \Psi \rangle$$

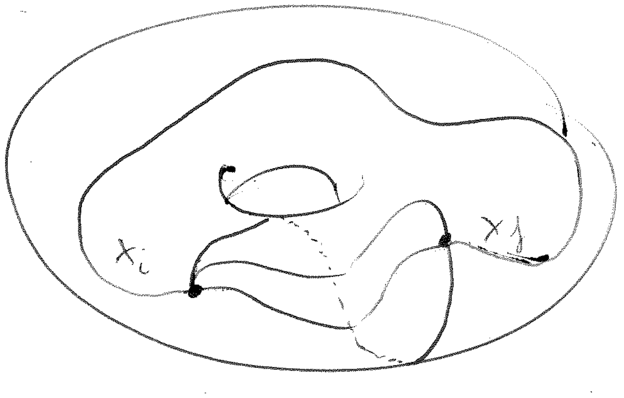
and similarly for $X_{\gamma^*} \Psi$ and $A_x \leftrightarrow B_h$.

- Proposition: The spectral gap above the ground state energy is 4. The corresponding eigenspace has dimension

$$4 \left(|\Lambda| (|\Lambda| - 1) \right).$$

Rem: This is for the genus $g=1$ surface, otherwise, one would obtain 4^g again.

Proof: It remains to compute the dimension. Any excited state is obtained from "creating a pair" from a ground state. If $\gamma, \tilde\gamma$ are two chains from x_i to x_j (the orientation is irrelevant), then $\gamma \tilde\gamma^{-1}$ is a cycle. Hence



$$Z_\gamma \Psi = Z_{[\gamma]} \Psi$$

where $[\gamma] := \{ c : c$ is a 1-chain s.t. $\partial c = \partial \gamma \}$

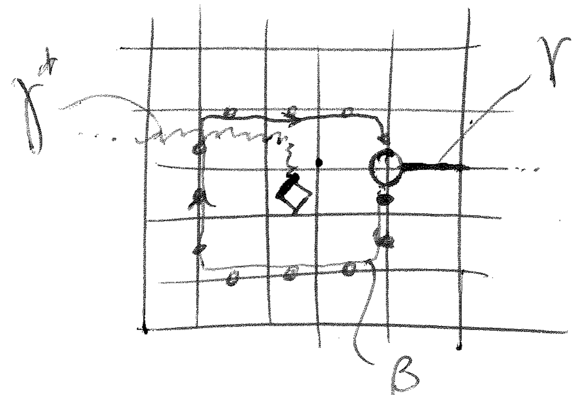
If $\tilde{\Psi}_{\alpha\beta}$ are the ground states of $X_1 \tilde{\Psi}_{\alpha\beta} = \alpha \tilde{\Psi}_{\alpha\beta}$, $X_2 \tilde{\Psi}_{\alpha\beta} = \beta \tilde{\Psi}_{\alpha\beta}$, then: if $\gamma, \tilde{\gamma}$ are s.t. $[\gamma] \neq [\tilde{\gamma}]$:

$$\langle Z_\gamma \tilde{\Psi}_{\alpha\beta}, Z_{\tilde{\gamma}} \tilde{\Psi}_{\alpha\beta} \rangle = \langle \tilde{\Psi}_{\alpha\beta}, Z_{\gamma \cdot \tilde{\gamma}} \tilde{\Psi}_{\alpha\beta} \rangle = 0$$

It follows that for each x_i, x_j, there are exactly four orthogonal state in the eigenspace, for a total of $4 \cdot \frac{1}{2} (|\Lambda| (|\Lambda| - 1))$ "Z-type" excitations.

A similar reasoning follows for "X-type" excitations and pairs of points in the dual lattice, and the result follows. $\square$

- <u>Note</u>: Given $x_i, x_j \in \Lambda$, the four possible excitations can be equivalently thought of as arising from the same ground state but with the four possible types of paths γ, or all from the same path, but from the four possible ground states.
- Strings play an important role when particles are "moved":



the initial configuration and the "braiding" of a Z-type particle around an X-type particle. Let Ψ be a ground state.

The initial state is $\Phi_i := X_{\gamma^*} Z_\gamma \Psi$ and the final state

$$\Phi_f = Z_\beta X_{\gamma^*} Z_\gamma \Psi$$

Now: β and γ^* will share an odd number of spins, so that.

$$Z_\beta X_{\gamma'} = - X_{\gamma'} Z_\beta$$

and since β is a trivial 1-cycle $Z_\beta Z_\gamma \psi = Z_\gamma \psi$.

Hence: $\quad \varphi_f = - \varphi_i$

despite describing the <u>same configuration of particles</u>

We shall call X, Z (distinguishable) anyons, they form a representation of the unoriented braid group.

- <u>Note:</u> We have just encountered another characterization of topological order, namely that elementary excitations are anyons (and in fact, such anyons are necessarily "attached to strings" in the general framework of algebraic quantum field theory)

More on braid groups

- A *braided vector space* V is a vector space V with an invertible linear map $c : V\otimes V \to V\otimes V$ s.t.

$$(c\otimes \mathbb{1})(\mathbb{1}\otimes c)(c\otimes \mathbb{1}) = (\mathbb{1}\otimes c)(c\otimes \mathbb{1})(\mathbb{1}\otimes c) \qquad (*)$$

on $V\otimes V\otimes V$.

for example: $T(v_1\otimes v_2) = v_2\otimes v_1$

(V, T) is a braided vector space since

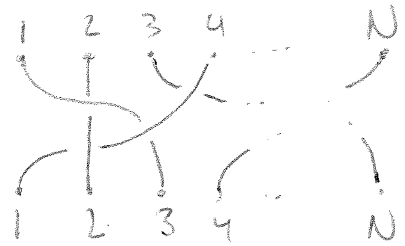
$T_{12}T_{23}T_{12} = T_{13} = T_{23}T_{12}T_{23}$

Note: $(*)$ is also called the Yang-Baxter equation.

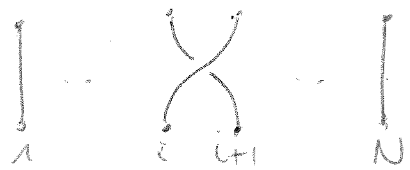
- Let $N\geq 3$. The *algebraic braid group* on N strands B_N is the group with $N-1$ generators $\sigma_1, \dots, \sigma_{N-1}$ subject to the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{if } |i-j|\geq 1$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \text{for } 1\leq i\leq N-2 \qquad (**)$$

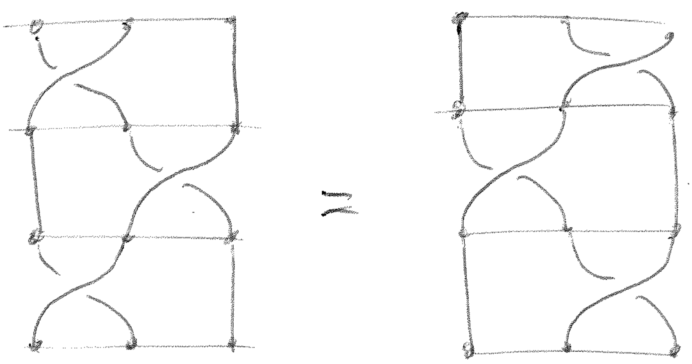
in picture: A braid $B = B_N$



with σ_i



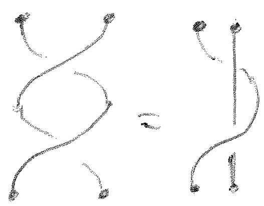
$(**)$:



<u>Note</u>: Canonical surjection: $B_N \to S_N$ to the symmetric group (of permutations):

$$\sigma_i \longmapsto \tau_{i,i+1}$$

τ transposition.



But: $(\tau_{i,i+1})^2 = 1$ while $\sigma_i^2 \neq 1$

(in fact all elements of B_N are of infinite order).

The two concepts above are closely related: a braided vector space can be used to define a representation of B_N.

$$\pi : B_N \longrightarrow \mathrm{Aut}\,(V^{\otimes N})$$

ith factors and (i+1)th

$$\sigma_i \longmapsto c_i = 1 \otimes 1 \dots \otimes c \otimes 1 \dots 1$$

- Other representation: one dimensional:

$$\chi : B_N \longrightarrow U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$$

such that $\chi(bb') = \chi(b)\chi(b')$.

<u>Lemma</u>: $\exists\, \alpha \in [0, 2\pi)$ s.t.

$$\chi_\alpha(\sigma_i) = e^{i\alpha} \qquad \forall\, 1 \leq i \leq N-1$$

Indeed. Given χ, $(**)$ implies that $\chi(\sigma_i) = \chi(\sigma_{i+1})$

Reciprocally χ_α satisfies $(**)$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ $(|i-j| \geq 2)$ $\square$

$\Rightarrow$ All unitary 1-dimensional representations of B_N are parametrized by $\alpha \in [0, 2\pi)$, with $\chi_\alpha(\sigma_i) = \exp(i\alpha)$

- Identical particles: the configuration space of N particles in $\mathbb{R}^d$

$$M^d := \{ \{x_1, \dots, x_N\} \subset \mathbb{R}^d \} \quad \text{(subset of } \mathbb{R}^d)$$

and we consider

$$M_0^d := M^d \setminus \Delta \,, \quad \Delta = \{ x \in M : x_i = x_j \text{ for some } i \neq j \}.$$

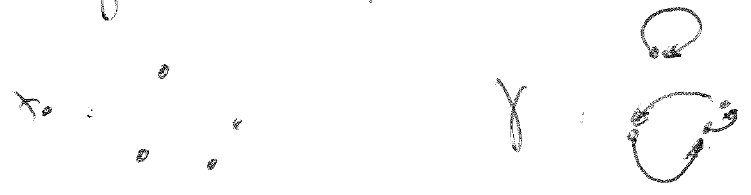
in particular: $\forall\, x_0 \in M_0^d$, $\exists$ local chart K s.t. $\exists\, U \ni x_0$:

$$U \ni x \xrightarrow{\varphi} (x_1, \dots, x_N) \in K \subset \mathbb{R}^{dN}$$

and φ is bijective

with this formulation: the arbitrariness of the numbering (identical particles) reduces to the independence on the choice of chart.

Let now γ be a loop in Π_0^d:



(∞)

The quantum mechanical state must be "parallel transported" along γ, which results in

$$\psi \longmapsto \chi(\gamma)\psi$$

at the end of the loop, where $\chi(\gamma) \in U(1)$ and $\chi(\gamma)$ depends only on the homotopy class of γ (because translations commute locally, not globally)

$\Rightarrow$ Identical particles form a <u>1-dimensional unitary representation of the homotopy group</u> (of the conf. space Π_0)

$$\chi : \pi_1(\Pi_0^d) \longrightarrow U(1)$$
$$\gamma \longmapsto \chi(\gamma)$$

- Now <u>Theorem</u>:
$$\pi_1(\Pi_0^d) \simeq S_N \qquad (d \geq 3)$$
$$\pi_1(\Pi_0^d) \simeq B_N \qquad (d = 2) \qquad \text{(Artin)}$$

- We have already classified the 1-d reps of B_N and introduced the surjective (but not injective) map $B_N \to S_N : \sigma_i \mapsto \tau_{i,i+1}$. This induces a map $\tilde{\chi} \mapsto \chi$ of reps of S_N to those of B_N:
$$\chi(b) := \tilde{\chi}(\pi) \qquad (b \in B_N, \ \pi \in S_N)$$

Thm:

$$1 = \tilde{\chi}\left((\tau_{i,i+1})^2\right) = \chi\left((\sigma_i)^2\right) = \chi(\sigma_i)\chi(\sigma_i) = e^{2i\alpha}$$

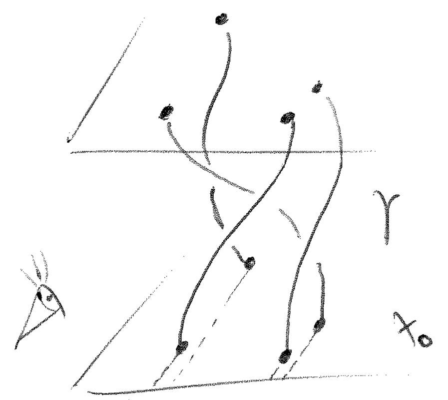
→ From the theorem:

- in 3-dimensional Euclidean space, identical particles are characterized by $\alpha = 0$ (bosons) or $\alpha = \pi$ (fermions)
- in 2-dimensional Euclidean space, they are characterized by any phase: they are <u>anyons</u>.

Proof (sketch): A loop γ induces a permutation by a continuous numbering, so that there is a homomorphism, which is surjective,

$$\pi_1(\Pi_0) \to S_N \quad ; \quad \gamma \mapsto \sigma. \qquad (P)$$

Parametrizing the loop by $t \in [0,1]$ corresponds to N "worldlines" in $[0,1] \times \mathbb{R}^d$ with fixed points $(0, x_i)$ and $(1, x_{\pi(i)})$ $(i = 1, \dots, N)$



see (ω)

Fact (transversality theorem): If $d+1 \geq 4$, any two curves with fixed endpoints can be arbitrarily deformed ~~across~~ around each other, namely

$$\pi(\gamma_1) = \sigma(\gamma_2) \Rightarrow \gamma_1 \simeq \gamma_2$$

homotopy

Hence, the map (P) is injective and therefore an isomorphism $(d \geq 3)$

Case $d=2$. Consider the projection of γ onto the first axis (e.g. s.t. initially $x^{(1)}_1 < x^{(2)}_1 < \dots < x^{(N)}_1$). This defines a braid b and the map $\gamma \mapsto b$ is injective: $\Pi_1(\Pi_0^2) \simeq B_N$. □

- Note: Anyons (those above!) appear importantly in the fractional quantum Hall effect, where the conductivity

$$\sigma_H := \frac{e^2}{2\pi\hbar}\nu \quad , \quad \nu \in \mathbb{Q}$$

can be understood as arising from anyons with $\alpha = \pi\nu$ whenever $\nu = \frac{1}{m}$, $m \in \mathbb{N}$.

- From now on, we shall concentrate on 2-dimensional compact manifolds. We have seen $B_N \simeq \Pi_1(\Pi_0^2)$. We shall continue and study the geometric braid groups.

algebraic (Artin) geometric (Fox);

Let M be a connected manifold and

$$UM_0 = \{(x_1, \dots, x_N) : x_i \in M, \; x_i \neq x_j, \; i \neq j\}$$

namely $UM_0 \subset M \times \dots \times M$ is the set of ordered N-tuples.

The geometric N-string unpermuted braid group over M is

$$P_N(M) := \Pi_1(\bar{U}M_0)$$

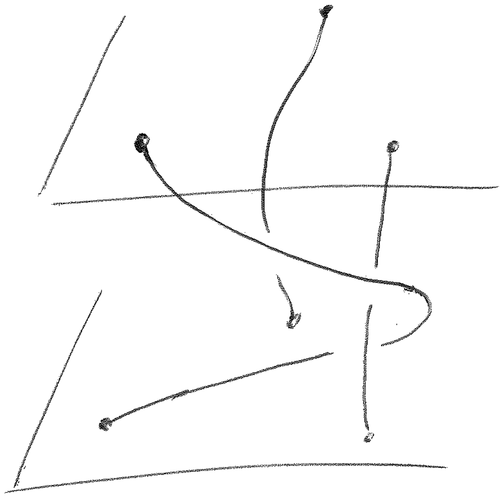
(also called the pure braid group)

The geometric N-string braid group over M is

$$B_N(M) := \Pi_1(M_0)$$

and we've seen: $B_N(\mathbb{R}^2) = B_N$, the algebraic braid group.

Pictorially:

 $\in P_N(\mathbb{R}^2)$

In other words the permutation corresponding to the loop γ is trivial in $P_N(\mathbb{R}^2)$, while it may not be in $B_N(\mathbb{R}^2)$:

$\gamma \in P_N(M)$ is represented by $(x_1(t), \dots, x_N(t))$:

$$x_i(0) = x_i(1) \quad \forall i \quad \text{and} \quad x_i(t) \neq x_j(t) \quad (i \neq j)$$

so that $x_i(t)$ is a closed loop on M.

$\rightsquigarrow$ $\gamma \in P_N(M)$ therefore involves both the topological structure of M and the entanglement between the strings.

Clearly: Distinguishable particles form 1-dimensional unitary reps of the pure braid group.

- Theorem: Let T be the torus. Then P_N is generated by $\gamma_1, \dots, \gamma_N, \ell_1, \dots, \ell_N$ with relations

$$[[\gamma_i, \gamma_j]] = [[\ell_i, \ell_j]] = 1$$

and:

$$A_{ij} := \gamma_i^{-1} \ell_j \gamma_i \ell_i^{-1} \qquad (*)$$

$$C_{ij} := \ell_i^{-1} \gamma_j \ell_i \gamma_j^{-1}$$

with

$$[[A_{jk}, \ell_i]] = [[A_{jk}, \gamma_i]] = 1$$

$$C_{ij} = (\ell_i^{-1} \ell_j^{-1}) A_{ij}^{-1} (\ell_j \ell_i),$$

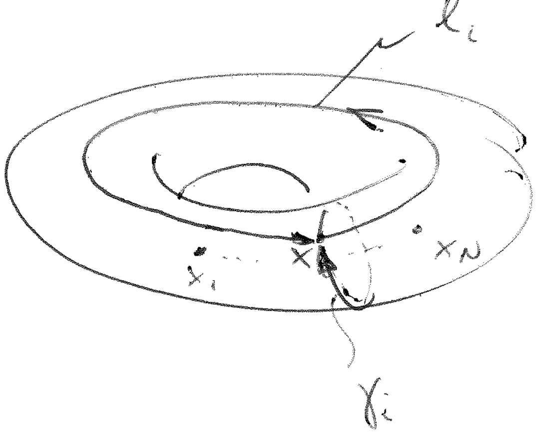
$$A_{ij} = (\gamma_i^{-1} \gamma_j^{-1}) C_{ij}^{-1} (\gamma_j \gamma_i)$$

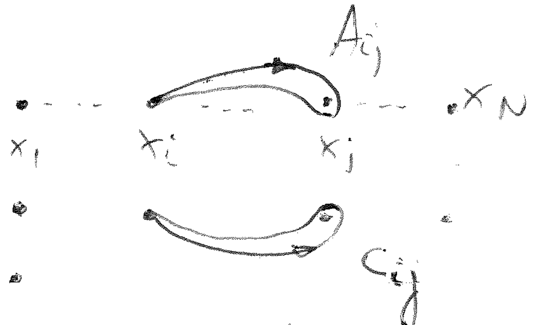
$$C_{ij} = (A_{j-1,j} A_{j-2,j} \cdots A_{i+1,j}) A_{ij}^{-1} (A_{i+1,j}^{-1} \cdots A_{j-1,j}^{-1})$$

$$\gamma_1^{-1} \ell_1^{-1} \gamma_1 \ell_1 = A_{1N} A_{1N-1} \cdots A_{13} A_{12}$$

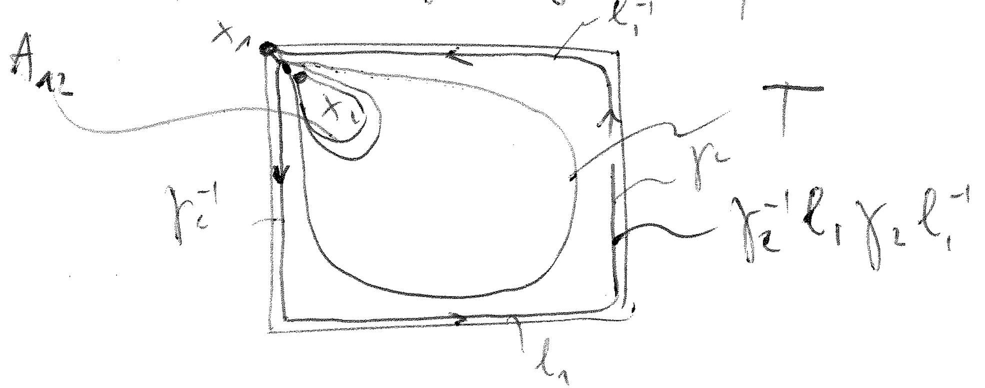
where $1 \le i < j < k \le n$, $[[a,b]] := a b a^{-1} b^{-1}$, i.e. $ab = [[a,b]] ba$.

In pictures





Note relation (d). it expresses the "local" fact of moving i around j by a combination of the topologically non-trivial loop. It is easily seen in the case of only two particles:



Note: For higher genus: not surprisingly, the generators are

$$\{ \underbrace{\gamma_{i\alpha}, \ell_{j\beta}}_{\text{all non-trivial loops.}} , 1 \le i, j \le N, 1 \le \alpha, \beta \le g \}$$

- With this: The ground state space $\mathcal{G}_T$ of the toric code model on the torus carries a representation of $P_2(T)$, namely:

$$\pi(\gamma_1) = X_1 \quad ; \quad \pi(\gamma_2) = Z_1$$
$$\pi(\ell_1) = X_2 \quad ; \quad \pi(\ell_2) = Z_2$$

$$\text{and} \quad A_{12} = -\mathbb{1} = C_{12}$$

and similarly for $\mathcal{G}_{T_g}$ and $P_2(T_g)$

$\rightsquigarrow$ better characterization of topological order: The ground state space carries a non-trivial representation of the pure braid group.

Note that "2" here arises from the group $\mathbb{Z}_2$, or equivalently from the fact that the model is a spin-$\frac{1}{2}$ model.

Also: the same observation remains true for the excited states, where now the loops are associated with the end of open strands and interpreted as particles

Hence (a little sloppily): the ground state <u>degeneracy</u> and the <u>anyonic nature</u> of the particles have the <u>same origin</u>

c) The quantum double model

- Let G be a finite group and $\mathcal{H} := \mathbb{C}[G]$ be the space of formal linear combinations of group elements with complex coefficients (the group algebra of G)
ONB: $\{g : g \in G\}$. with $\dim \mathcal{H} = |G|$.
Here we need oriented edges, with each G-spin attached to a edge of the graph (1-cell).
Define:
$$L_+^g z = gz \quad ; \quad L_-^g z = z g^{-1} \qquad \forall z \in G.$$
$$T_+^h z = \delta_{h,z} z \quad ; \quad T_-^h z = \delta_{h^{-1},z} z$$
Check: * $[L_+^g, L_-^h] = 0 = [T_+^g, T_-^h]$
* $L_+^g T_+^h = T_+^{gh} L_+^g$
$L_+^g T_-^h = T_-^{hg^{-1}} L_+^g$
$L_-^g T_+^h = T_+^{hg^{-1}} L_-^g$
$L_-^g T_-^h = T_-^{gh} L_-^g$

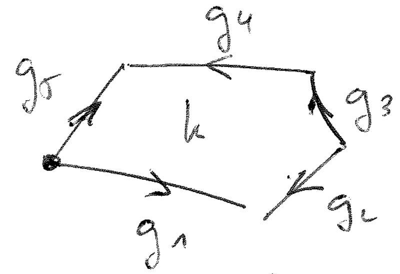
Now: 1) Star operators $A_x(g)$, x a 0-cell (site), $g \in G$.
$$A_x(g) = \prod_{j \in S(x)} L^g_{\sigma(j)}$$
where $L^g_{\sigma(j)}$ acts on $\mathcal{H}_j$ and
$$\sigma(j) = \begin{cases} + & \text{is the edge is outgoing} \\ - & \text{otherwise} \end{cases}$$
2) Plaquette operators $B_k(g)$, k a face, $g \in G$.
$$B_k(g) = \sum_{g_1 \cdots g_n = g} \prod_{m=1}^{n} T^{g_m}_{\sigma(j_m)}$$
where $T^{g_m}_{\sigma(j_m)}$ acts on $\mathcal{H}_{j_m}$ where the edges

are visited in the positive orientation, and again

$$\sigma(j) = \begin{cases} + & \text{if the edge is crossed along its orientation} \\ - & \text{otherwise} \end{cases}$$

Note : * the def of B actually depends on the "starting point" of the loop. We shall use them for $g=e$, in which case it does not matter.

* In the following situation:



$$B_k(g)\,|g_1 \cdots g_5\rangle = \delta_{g,\, g_1 g_2^{-1} g_3 g_4 g_5^{-1}}\,|g_1 \cdots g_5\rangle$$

i.e. $B_k(g)$ is the projection onto states on the plaquette with "total flux" equal to g.

• Algebraic relations:

* $A_x^g A_x^h = A_x^{gh}$ (the A's are a representation of G)
* $B_k^g B_k^h = \delta_{g,h} B_k^g$
* If x is the initial vertex of k: $A_x^g B_k^h = B_k^{ghg^{-1}} A_x$ (*)
* $(A_x^g)^\dagger = A_x^{g^{-1}}$

⇝ these are the relations of Drinfield's quantum double $D(G)$ of the algebra $\mathbb{C}[G]$.

• Now: Let $B_k := B_k^e$ (e : the neutral element of G)

$$A_x := \frac{1}{|G|} \sum_{g\in G} A_x^g$$

as noted, B_k is a projection; so is A_x since : if

g runs over G, so does gg' for any $g' \in G$.

The Hamiltonian:

$$H_\Lambda := -\sum_{x : s(x) \subset \Lambda} A_x - \sum_{h : p(h) \subset \Lambda} B_h.$$

is a sum of commuting projections by (*) with $h = e$.
Note also that A_x is the projection onto the "trivial sector" of the representation of G on $\mathcal{H}_{s(x)}$:

$$A_x(g) A_x |g_1 \cdots g_n\rangle = A_x |g_1, \dots, g_n\rangle$$

Ground state space.

$$\mathcal{G}_\Lambda := \{\psi \in \mathcal{H}_\Lambda : A_x \psi = \psi \ ; \ B_h \psi = \psi \quad \forall x, h\}.$$

- The case $G = \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ commutative group and any element is its own inverse, in particular $L_+^\# = L_-^\#$; $T_+^\# = T_-^\#$.
Writing $G = \{0, 1\}$ and $g \cdot g' = g + g' \pmod 2$

$$L^1 = \mathbb{1} \quad ; \quad L^{-1} = \sigma^1$$
$$T^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ ; \ T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the A_x, B_h defined here agree with those of the toric code model (up to addition of a constant and scaling)
- The case $G = \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ $(d \in \mathbb{N})$, still abelian.
Here it is practical to think of the representation

$$g \mapsto \omega^j \qquad \omega = \frac{2\pi i}{d} \ ; \ j \in \{0, 1, \dots d-1\}.$$
$$g \cdot g' \mapsto \omega^{j+j'}$$

Here: $L_+^j |k\rangle = |j+k\rangle$, so that $L_+^j = (L_+^1)^j$
and $(L_+^j)^* = L_-^j = (L_+^j)^{-1}$ unitary representation of G.

Similarly: $T_+^j = |j\rangle\langle j|$, $T_-^j = T_+^{-j} = |-j\rangle\langle -j|$

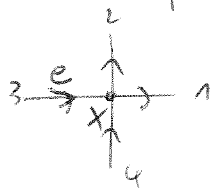
We define:

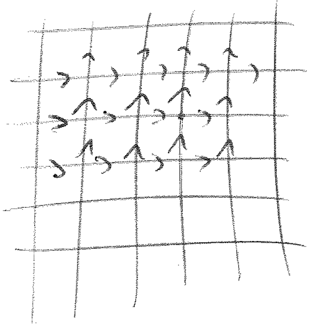
$$X := L_+^1 \quad \text{and is represented as} \quad \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}$$

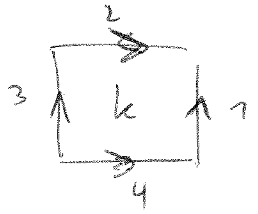
$$Z := \sum_{j=0}^{d-1} \omega^j T_+^j$$

for which: $$ZX = \omega X Z$$

For simplicity, we consider the square lattice:







$$A_x(1) = X_1 X_2 X_3^\dagger X_4^\dagger$$

$$A_x = \frac{1}{d}\sum_{j=0}^{d-1} \left(A_x(1)\right)^j$$

$$B_k(j) = \sum_{j_1 - j_2 - j_3 + j_4 = j} T_1^{j_1} T_2^{-j_2} T_3^{-j_3} T_4^{j_4}$$

and $$B_k = B_k(0) = \frac{1}{d}\sum_{j=0}^{d-1}\left(Z_1 Z_2^\dagger Z_3^\dagger Z_4\right)^j$$

after a bit of calculation. (+) see p. 43.

Now: $$Z_3 A_x |j\rangle = \frac{1}{d}\sum_{k=0}^{d-1} Z_3 |j_1+k, j_2+k, j_3-k, j_4-k\rangle$$

$$= \frac{\omega^{j_3}}{d}\sum_{k=0}^{d-1} |j_1+k, j_2+k, j_3-k, j_4-k\rangle \cdot \omega^{-k}$$

$$A_x Z_3 |j\rangle = A_x \omega^{j_3}|j\rangle = \frac{\omega^{j_3}}{d}\sum_{k=0}^{d-1} |j_1+k, j_2+k, j_3-k, j_4-k\rangle$$

Hence: If γ is an <u>oriented</u> path (1-chain), and

$$Z_\gamma = \prod_{j\in\gamma} Z_j^{\sigma(j)}, \quad \sigma(j) = \begin{cases} +1 \\ -1 \end{cases}$$

then $[A_x, Z_\gamma] = 0 \iff$ s(x) has an even number of intersections with γ

hence: $Z_\gamma \mathcal{G}_1 \subseteq \mathcal{G}_1$ iff γ, γ' are 1-cycles.
$X_{\gamma'} \mathcal{G}_1 \subseteq \mathcal{G}_1$

The difference with the $G = \mathbb{Z}_2$ case: for each element of the homology group there is one Z and one X:

$$\mathcal{A}(\mathcal{G}_1) = \mathrm{span}\left\{ X^j_{\gamma_1^*}, X^j_{\gamma_2^*}, Z^j_{\gamma_1}, Z^j_{\gamma_2} ; j \in \{0, \dots, d-1\} \right\}.$$

with non-trivial commutation relations:

$$X^j_{\gamma_1^*} Z^k_{\gamma_2} = \omega^{j+k} Z^k_{\gamma_2} X^j_{\gamma_1^*}$$

Hence $\mathcal{A}(\mathcal{G}_1)$ is the representation of $\mathcal{P}_{2d}(T)$, with $A_{jk} = \omega^{jk}$. ∎

□

(*) $$\frac{1}{d}\sum_{j=0}^{d-1} \left(\sum_{k_1,k_2,k_3,k_4=0}^{d-1} \omega^{k_1-k_2-k_3+k_4} T_1^{k_1} T_2^{-k_2} T_3^{-k_3} T_4^{k_4} \right)^j$$

note: $T_1^k T_1^j = \delta_{jk} T_1^j$, so that

$$(\cdot)^j = \sum_{k_1,k_2,k_3,k_4=0}^{d-1} \omega^{j(k_1-k_2-k_3+k_4)} T_1^{k_1} T_2^{-k_2} T_3^{-k_3} T_4^{k_4}$$

and $$\frac{1}{d}\sum_j \sum_{k\dots} (\cdot) = \sum_{k_1k_2k_3k_4} T_1^{k_1} T_2^{-k_2} T_3^{-k_3} T_4^{k_4} \underbrace{\frac{1}{d}\sum_{j=0}^{d-1} \omega^{j(k_1-k_2-k_3+k_4)}}_{=\delta_{0,k_1-k_2-k_3+k_4}}$$

$= B(0)$ indeed.

The toric code on $\mathbb{Z}^2$

- Recall: there exists $(\mathcal{A}, \tau_t)$, the algebra and dynamics in infinite volume. Let $\delta : \mathcal{A} \to \mathcal{A}$ be defined:
$$\delta(A) := \frac{d}{dt} \tau_t(A) \Big|_{t=0}$$
with $D(\delta) := \{ A \in \mathcal{A} : \lim_{t \to 0} \frac{1}{t}(\tau_t(A) - A) \text{ exists} \}$.
If $A \in \mathcal{A}_\Lambda$ for a finite set Λ, its action is given by
$$\delta(A) = i [H^\Lambda, A]$$
where $H^\Lambda := \sum_{X \cap \Lambda \neq \emptyset} \Phi(X)$

finite sum, since the interaction is of finite range.

Def: A ground state of $(\mathcal{A}, \tau_t)$ is a state such that
$$-i\, \omega(A^* \delta(A)) \geq 0 \qquad (*)$$
for all $A \in \mathcal{A}_{loc}$

Why is this a good definition? Recall the GNS construction: Given $(\mathcal{A}, \tau_t)$ and a state ω, $\exists$ a Hilbert space $\mathcal{H}_\omega$, a representation map $\pi_\omega : \mathcal{A} \to \mathcal{L}(\mathcal{H}_\omega)$ and a unit vector Ω_ω:
$$\omega(A) = \langle \Omega_\omega, \pi_\omega(A) \Omega_\omega \rangle$$
If ω is invariant, $\omega \circ \tau_t = \omega$, there exists a unitary group U_t on $\mathcal{H}_\omega$ s.t.
$$\pi_\omega(\tau_t(A)) = U_t^* \pi_\omega(A) U_t, \quad U_t \Omega_\omega = \Omega_\omega$$
(one says that τ_t is unitarily implementable in the representation)

by Stone's theorem: $\exists\, H_\omega = H_\omega^*$ on $\mathcal{H}_\omega$:

$$U_t = \exp(-itH_\omega)$$

and $H_\omega \Omega_\omega = 0$.

Back to $(*)$: in the GNS representation of ω:

$$-i\omega(A^*\delta(A)) = \langle \Omega_\omega, \pi_\omega(A^*)\,(H_\omega\pi(A) - \underbrace{\pi(A)H_\omega}_{\text{vanishes on } \Omega_\omega})\,\Omega_\omega\rangle$$

$$= \langle \Psi_A, H_\omega \Psi_A\rangle \quad \text{with } \Psi_A := \pi_\omega(A)\Omega_\omega$$

But $\{\pi_\omega(A)\Omega_\omega : A \in \mathcal{A}_{loc}\}$ is dense in $\mathcal{H}_\omega$ by the GNS construction (and the density of $\mathcal{A}_{loc}$ in $\mathcal{A}$) and its is a core for $\mathcal{H}_\omega$. Hence: $-i\omega(A^*\delta(A)) \geq 0$ implies that H_ω is a positive operator, $H_\omega \geq 0$ s.t. $H_\omega\Omega_\omega = 0$, i.e. Ω_ω is a ground state in the usual sense.

- <u>Proposition</u>: There is a unique translation-invariant ground state ω_0. It is the unique state satisfying $\omega_0(A_x) = \omega_0(B_h) = 1$ $\forall x, h$.

<u>Elements of proof</u>: Let $\tilde{\mathcal{A}}$ be the algebra generated by $\{A_x, B_h\}$. It is an abelian algebra, and in fact a <u>maximal</u> abelian algebra, namely

$$\tilde{\mathcal{A}}' \cap \mathcal{A} = \tilde{\mathcal{A}}$$

(true on $\mathbb{Z}^2$, not on the torus: the X and Z operators there belong to $\tilde{\mathcal{A}}'$ but are not in $\tilde{\mathcal{A}}$)

Let $\tilde{\omega} \in \tilde{\mathcal{A}}^*$ be s.t. $\tilde{\omega}(A_x) = 1 = \tilde{\omega}(B_h)$, and let ω_0 be an extension of $\tilde{\omega}$ to all of $\mathcal{A}$, which exists by Hahn-Banach.

Then:
$$-i\omega_0(S^*\delta(T)) = \sum_x \left(\omega_0(S^*TA_x) - \omega_0(S^*A_xT)\right) + \sum_h \left(\omega_0(S^*TB_h) - \omega_0(S^*B_hT)\right)$$

But $A_x \le \mathbb{1}$, so that $\mathbb{1} - A_x \geq 0$ and

$$|\omega_0((S^*T)(\mathbb{1}-A_x)^{1/2}(\mathbb{1}-A_x)^{1/2})|^2$$
$$\le \omega_0((S^*T)(\mathbb{1}-A_x)(S^*T)^*)\underbrace{\omega_0(\mathbb{1}-A_x)}_{=0} = 0$$

hence $\omega_0((S^*T)(\mathbb{1}-A_x)) = 0$

($\Diamond$) so that $\omega_0((S^*T)A_x) = \omega_0((S^*T))$. It follows that

$$\omega_0(S^*TA_x) - \omega_0(S^*A_xT) = \omega_0(S^*(\mathbb{1}-A_x)T)$$

which is ≥ 0 in the case $S = T$. Since the same conclusion holds for the B_p terms, this shows that

$$-i\omega_0(S^*\delta(S)) \geq 0$$

for all $S \in \mathcal{A}_{loc}$ and hence all $S \in \mathcal{A}$, so that any extension ω_0 is a ground state. It remains to prove uniqueness. Expanding any observable in the basis $E_{\gamma\gamma'} = X_{\gamma'} Z_{\gamma}$, we again see that ~~this~~

$$\omega_0(Z_\gamma) \overset{(\Diamond)}{=} \omega_0(A_x Z_\gamma A_x) = -\omega_0(Z_\gamma)$$

for any Z_γ : $\exists x : \{Z_\gamma, A_x\} = 0$

Hence $\omega_0(E_{\gamma\gamma'}) \neq 0 \iff E_{\gamma\gamma'} \in \widetilde{\mathcal{A}}' \cap \mathcal{A}$

Since $\widetilde{\mathcal{A}}' \cap \mathcal{A} = \widetilde{\mathcal{A}}$, ω is completely determined by its ~~value~~ or restriction $\widetilde{\omega}$ on $\widetilde{\mathcal{A}}$. □

- Remark: If H_0 denotes the GNS Hamiltonian, then
$$Sp(H_0) \subset \{0\} \cup [4, \infty)$$
In the algebraic framework, this is implied by
$$-i\omega_0(A^*\delta(A)) \geq 4\left(\omega_0(A^*A) - |\omega_0(A)|^2\right)$$
for all $A \in \mathcal{A}_{loc}$. Indeed, going to the GNS representation again this reads

$$\text{l.h.s.} = \langle \pi(A)\Omega, H_0 \pi(A)\Omega \rangle$$

and since $H_0 \Omega = 0$, $\pi(A)\Omega$ can be replaced by $(\mathbb{1} - |\Omega\rangle\langle\Omega|)\pi(A)\Omega$

$$\text{r.h.s.} = \langle \pi(A)\Omega, \pi(A)\Omega \rangle - \langle \pi(A)\Omega, \Omega\rangle\langle\Omega, \pi(A)\Omega\rangle$$
$$= \langle \pi(A)\Omega, (\mathbb{1} - |\Omega\rangle\langle\Omega|)\pi(A)\Omega\rangle$$

which shows that

$$\inf_{\Psi \perp \Omega} \frac{\langle \Psi, H_0 \Psi\rangle}{\langle \Psi, \Psi\rangle} \geq 4 \quad \text{, namely } Sp(H_0)\setminus\{0\} \subset [4, \infty)$$

indeed by the variational principle.

- In the infinite volume limit one can imagine creating strings γ or γ^* extending to infinity. If γ is such a string, let $\gamma^{(n)}$ be the string truncated after n edges. Then:

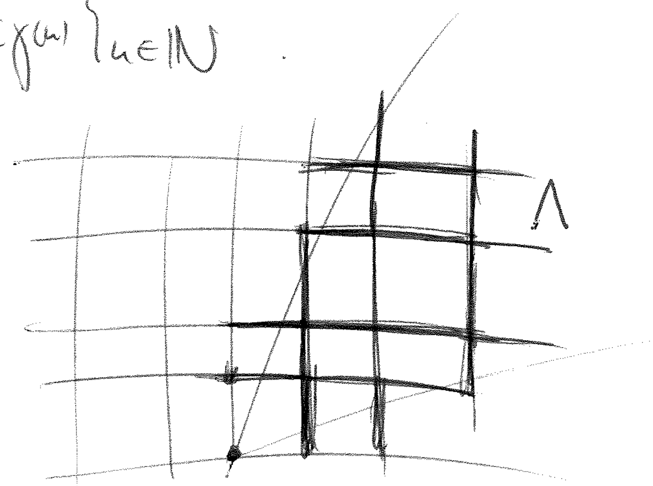
$$\| Z_{\gamma^{(m)}} - Z_{\gamma^{(n)}} \| = \| \prod_{i=m+1}^{n} \sigma^3_{e_i} \| = 1$$

so that $\{Z_{\gamma^{(n)}}\}_{n\in\mathbb{N}}$ cannot converge.

Just as with the dynamics, the infinite string can still be studied using the action induced by Z_γ on the algebra:

$$\{ Z^{+}_{\gamma^{(n)}} A Z_{\gamma^{(n)}} \}_{n\in\mathbb{N}}.$$

We need a cone Λ



A path γ is in Λ if $e \in \gamma \Rightarrow e \in \Lambda$.

Proposition: Let Λ be a cone and γ be a string in Λ.
Then the limit

$$\lim_{n\to\infty} Z_{\gamma(n)}^{*} A Z_{\gamma(n)} \qquad A \in \mathcal{A}$$

exists in norm and

$$\rho_\gamma(A) := \lim_{n\to\infty} Z_{\gamma(n)}^{*} A Z_{\gamma(n)}$$

defines an automorphism of $\mathcal{A}$.
Moreover, ρ_γ is localised in Λ

$$\rho_\gamma(A) = A \qquad \text{for all } A \in \mathcal{A}_{\Lambda^c}$$

Proof: If $A \in \mathcal{A}_{loc}$, namely $A \in \mathcal{A}_{\Lambda_0}$ for a Λ_0 finite, then $\exists\, n_0 : (\gamma_n \setminus \gamma_{n_0}) \cap \Lambda_0 = \emptyset \quad \forall n \geq n_0$. But then $Z_{\gamma(n)}^{*} A Z_{\gamma(n)} = Z_{\gamma(n_0)}^{*} A Z_{\gamma(n_0)}$ since $(\sigma^3)^2 = \mathbb{1}$ and the support properties. Hence $\rho_{\gamma(n)}(A)$ converges. Moreover, $\|\rho_\gamma(A)\| = \|A\|$ on $\mathcal{A}_{loc}$. ρ_Λ is therefore a bounded linear map that is densely defined on $\mathcal{A}$, and it has a unique, continuous extension to $\mathcal{A}$. It is invertible since $\rho_\gamma \circ \rho_\gamma = id$. $\square$

• Def: The state ω_z^e, for an edge e is defined by

$$\omega_z^e(A) := \omega_0(\rho_\gamma(A))$$

where γ is a string starting at e and extending to infinity. $\rightsquigarrow$ one particle states

Remark: As the notation indicates ω_z^e depends on e, but not on the choice of γ. This follows again by noting that a path can be deformed by acting with B_h's and that $\omega_0(B_h \cdot B_h) = \omega_0(\cdot)$.

- Further remarks on the GNS representation:
  * Given ω, $(\mathcal{H}, \pi, \Omega)$ is <u>unique</u> up to unitary equivalence: if $(\mathcal{H}', \pi', \Omega')$ is another GNS triplet, $\exists\, U : \mathcal{H}' \to \mathcal{H}$, unitary, s.t.
$$U^* \pi(A) U = \pi'(A)$$
$$\Omega = U \Omega'$$
  * Given two states ω_1, ω_2 on $\mathcal{A}$, they may generate <u>inequivalent</u> representations of $\mathcal{A}$. In that case, the state ω_2 cannot be represented as a vector in the GNS Hilbert space of ω_1. These correspond to physically globally different states of the system (e.g. magnetized in direction $\uparrow$ vs $\downarrow$).

For quantum spin systems ω_1 & ω_2, ~~two~~ pure states, are equivalent iff $\exists\, \varepsilon > 0$, $\exists\, \Lambda$ finite s.t. for all $B \in \mathcal{A}_{\Lambda^c}$:
$$|\omega_1(B) - \omega_2(B)| < \varepsilon \|B\|$$
i.e. ω_1 & ω_2 are "equal at infinity".

There is a remarkable theorem:

<u>Theorem</u>: Let Λ be a cone, e an edge both arbitrary. The representations π_{ω_0} and $\pi_{\omega_z^e}$ are inequivalent. However,
$$\pi_{\omega_0} \upharpoonright \mathcal{A}_{\Lambda^c} \underset{\text{unitary equivalence}}{\simeq} \pi_{\omega_z^e} \upharpoonright \mathcal{A}_{\Lambda^c}$$

In other words: the particle can be moved anywhere and the string deformed, but it cannot be removed.

In fact: the states ω_z^e are ground states in the algebraic sense. Indeed, the excess energy needed to create the particle cannot be removed locally.

Proof. Let γ be a path in Λ, starting at $\tilde{e}$ and extending to ∞.
Claim: $(\mathcal{H}_{\omega_0}, \Pi_{\omega_0} \circ \rho_\gamma, \Omega_{\omega_0})$ is a GNS triplet for $\omega_z^{\tilde{e}}$.
Indeed:

$$\langle \Omega_{\omega_0}, \Pi_{\omega_0} \circ \rho_\gamma(A) \Omega_{\omega_0} \rangle = \omega_0(\rho_\gamma(A)) = \omega_z^{\tilde{e}}(A) \qquad (i)$$

by definition

Moreover, if $A \in \Lambda^c$:

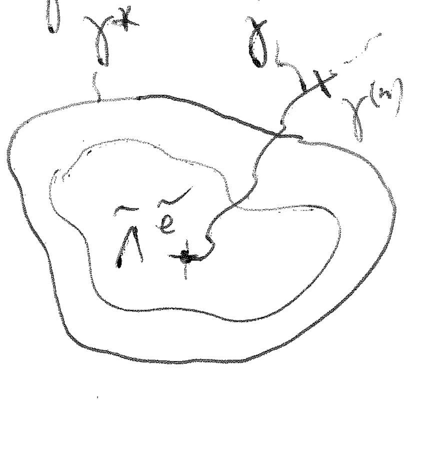
$$\Pi_{\omega_0} \circ \rho_\gamma(A) = \Pi_{\omega_0}(A) \qquad (ii)$$

Hence $\Pi_{\omega_z^{\tilde{e}}} \restriction \mathcal{A}_{\Lambda^c} \underset{(i)}{\cong} \Pi_{\omega_0} \circ \rho_\gamma \restriction \mathcal{A}_{\Lambda^c} \overset{(ii)}{=} \Pi_{\omega_0} \restriction \mathcal{A}_{\Lambda^c}$

Now if $e \neq \tilde{e}$, let γ' be a path from e to $\tilde{e}$, and

$$\rho_{\gamma' + \gamma}(A) = \rho_\gamma(Z_{\gamma'}^* A Z_{\gamma'})$$

so that $\omega_z^e(A) = \omega_z^{\tilde{e}}(Z_{\gamma'}^* A Z_{\gamma'})$. This shows that ω_z^e and $\omega_z^{\tilde{e}}$ are unitary conjugates of each other, thus $Z_{\gamma'}$ is unitary (γ' is a fixed path)
It remains to prove the inequivalence of ω_0 & ω_z.
Let $\varepsilon = 1$, and $\tilde{\Lambda}$ be a finite set with $\tilde{e} \in \tilde{\Lambda}$.
$\exists\, \gamma^*$ a closed 1-boundary whose interior contains $\tilde{\Lambda}$.



Recall: $X_{\gamma^*} = \prod_{x \in \mathrm{int}\, \gamma^*} A_x$

Since γ^* is closed: $\omega_0(X_{\gamma^*}) = \mathbb{1}$
Let $A_{\tilde{e}}$ be the star operator which intersects γ only at $\tilde{e}$; since all others have an even number of shared edges with γ:

$$\omega_z^{\tilde{e}}(X_{\gamma^*}) = \omega_0\Big(Z_{\gamma^{(n)}}^* \prod_x A_x \underbrace{Z_{\gamma^{(n)}}}\Big) = -\omega_0\Big(\underbrace{Z_{\gamma^{(n)}}^* Z_{\gamma^{(n)}}}_{=\mathbb{1}} \prod_x A_x\Big)$$

all commute, but for $A_{\tilde{e}}$;

$$= -\omega_0(X_{\gamma^+})$$

Hence, for $X_{\gamma^+} \in \mathcal{A}_{\tilde{\Lambda}^c}$, we have that

$$|\omega_0(X_{\gamma^+}) - \omega_Z^{\tilde{e}}(X_{\gamma^+})| = 2|\omega_0(X_{\gamma^+})| = 2 > \|X_{\gamma^+}\| \quad \square$$

- Summarizing: The toric code on the plane is characterized by four superselection sectors, namely four inequivalent ground state representations:

$$\pi_0, \ \pi_Z, \ \pi_X, \ \pi_{ZX}$$

In fact: $\rho_\gamma(A_x) = -A_x$ if x is the origin of the path γ, and $\rho_\gamma(A_{\tilde{x}}) = A_{\tilde{x}}$, $\rho_\gamma(B_u) = B_u$ for all other stars and plaquettes. Hence

$$\rho_\gamma(H_\Lambda) = H_\Lambda + 2A_x \quad \text{and further}$$

$$\rho_\gamma(e^{itH_\Lambda} A e^{-itH_\Lambda}) = e^{it(H_\Lambda + 2A_s)} \rho_\gamma(A) e^{-it(H_\Lambda + 2A_s)}$$

for all $A \in \mathcal{A}$. We conclude that

$$(\pi_0 \circ \rho_\gamma)(\tau_t(A)) = e^{itH_0} \pi_0\left(e^{it2A_s} \rho_\gamma(A) e^{-it2A_s}\right) e^{-itH_0}$$

$\uparrow$ GNS Hamiltonian in π_0.

$$= e^{it(H_0 + 2\pi_0(A_s))} (\pi_0 \circ \rho_\gamma)(A) e^{-it(H_0 + 2\pi_0(A_s))}$$

That is: the dynamics $\tau_t(\cdot)$ is implemented in ~~$(\mathcal{H}_0, \pi_0, \Omega)$~~ $(\mathcal{H}_0, \pi_0 \circ \rho_\gamma, \Omega_0)$ by $(H_0 + 2\pi_0(A_s))$ and since this is a GNS triple for ω_Z, we have found the GNS Hamiltonian. We have

$$Sp(H_0 + 2\pi_0(A_s)) \subset [2, \infty) \quad \text{and } \Omega_0$$

is a ground state vector. This proves that

ω_2 is a ground state.

Rem: the eigenvalue 0 has "disappeared" in the charged superselection sector π_2.

- With braiding becomes doable. Below, we shall consistently write A instead of $\pi_0(A)$

We first observe that ρ_γ is transportable: If γ_1, γ_2 are two paths extending to infinity, then $\exists V_{12}$:

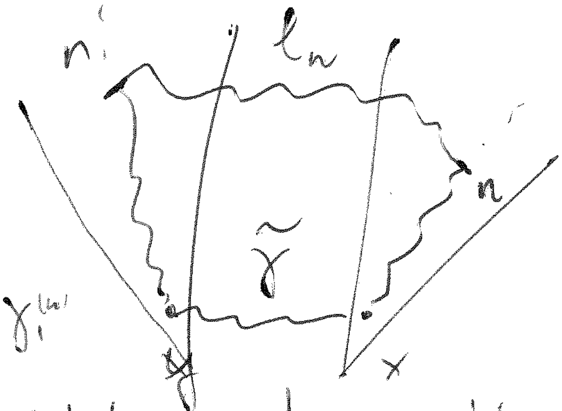
i) $V_{12}\rho_{\gamma_2}(A) = \rho_{\gamma_1}(A)V_{12}$

ii) If γ_1 starts at x, γ_2 at y, then for any path $\tilde\gamma$ from x to y:

$$Z_{\tilde\gamma} V_{12}\Omega_0 = \Omega_0$$

Moreover:

$$V_{12} = \text{s-}\lim_{n\to\infty} V_n$$



$(*)$ where $V_n := Z_{\gamma_2^{(n)}} Z_{l_n} Z_{\gamma_1^{(n)}}$

which gives an explicit bound on the intertwiner of representations π_{γ_1} and π_{γ_2}, since $\pi_{\gamma_1} = \pi_0 \circ \rho_{\gamma_1}$

Proof. Let $V: \{A\Omega_0 : A \in \mathcal{A}_{loc}\} \to \mathcal{H}_0$

$$A\Omega_0 \mapsto (\rho_2\circ\rho_1)(A) Z_{\tilde\gamma}\Omega_0$$

We first have

$$V\rho_1(A)B\Omega_0 = \rho_2(A)(\rho_2\circ\rho_1)(B)Z_{\tilde\gamma}\Omega_0 = \rho_2(A)VB\Omega_0$$

so that V intertwines ρ_1 and ρ_2 on $\mathcal{A}_{loc}\Omega_0$.

Let V_n be as in $(*)$. For any $A \in \mathcal{A}_{loc}$, say

$A \in \mathcal{A}_\Lambda$: $\exists\, n_0 : \forall n \geq n_0 : \ell_n \cap \Lambda = \emptyset$, so that

$$V_n \rho_1(A) = \rho_2(A) V_n \qquad \forall n \geq n_0$$

Furthermore, $\gamma_2^{(n)} + \ell_n + \gamma_1^{(n)} + \tilde{\gamma}$ is a loop, so that

$$Z_{\tilde{\gamma}} V_n \Omega_0 = \Omega_0 \quad , \quad \text{and} \qquad (*)$$

$$\|(V_n - V) A \Omega_0\| = \| V_n \rho_1(\rho_1(A)) \Omega_0 - (\rho_2 \circ \rho_1)(A) Z_{\tilde{\gamma}} \Omega_0 \| = 0$$

since $V_n \rho_1(\rho_1(A)) \Omega_0 = \rho_2(\rho_1(A)) V_n \Omega_0 = (\rho_2 \circ \rho_1) Z_{\tilde{\gamma}} \Omega_0$ by $Z_{\tilde{\gamma}}^2 = \mathbb{1}$ and $(*)$

Hence $V_n \xrightarrow{s} V$ $(n \to \infty)$ on the dense space

$$\{ A\Omega_0 : A \in \mathcal{A}_{loc} \}.$$

Since $\|V_n\| = 1$, $\|V\| = 1$ so that V extends to a bounded operator on $\mathcal{H}_0$. It is unitary since V_n are so.

That $Z_{\tilde{\gamma}} V \Omega_0 = \Omega_0$ follows from the definition with $A = \mathbb{1}$ $\square$

- Note: as weak limits of elements in $\mathcal{A}$, $V_{12} \in \mathcal{A}''$ (it is a deep theorem that the weak closure is exactly $\mathcal{A}''$) but $V_{12} \notin \mathcal{A}$, as discussed. This makes the following formal, although this can be remedied by considering suitable cone algebras.
- Let ρ be a fixed, arbitrary string automorphism, and ρ_1, ρ_2 obtained from ρ by the action of V_1, V_2.

Then: if ρ_1 & ρ_2 are localised in disjoint cones:

$$\rho_1 \circ \rho_2(A) = \rho_2 \circ \rho_1(A)$$

but $(\rho_1 \circ \rho_2)(A) = V_1\, \rho(V_2)\, \rho^2(A)\, \rho(V_2^*)\, V_1^*$

and similarly for $(\rho_2 \circ \rho_1)(A)$. Hence,

$$\rho^2(A) = \rho(V_1^\dagger)V_2^\dagger\,(\rho_2\circ\rho_1)(A)\,V_2\,\rho(V_1)$$
$$= \rho(V_1^\dagger)V_2^\dagger\,(\rho_1\circ\rho_2)(A)\,V_2\,\rho(V_1)$$
$$= \left(\rho(V_1^\dagger)V_2^\dagger V_1\,\rho(V_2)\right)\rho^2(A)\left(\cdots\right)^\dagger$$

hence

$$\varepsilon := \rho(V_1^\dagger)V_2^\dagger V_1\,\rho(V_2)$$

is called the <u>statistics operator</u>. Why?

For simplicity: $V_1 = \mathbb{1}$, $\rho = \rho_1$ localized in Λ_0,
ρ_2 localized in Λ,
$$\varepsilon = V^\dagger \rho(V) \in \mathcal{A}_{\Lambda_0}$$

Note: $\varepsilon\,\rho(\varepsilon) = V^\dagger\rho(V)\rho(V^\dagger)\rho^2(V) = V^\dagger\rho^2(V)$
so that:

i) $\varepsilon\,\rho(\varepsilon)\,\varepsilon = V^\dagger\rho^2(V)\,\varepsilon \underset{\uparrow}{=} V^\dagger\varepsilon\,\rho^2(V)$
ε is the intertwiner

ii) $\rho(\varepsilon)\,\varepsilon\,\rho(\varepsilon) = \rho(\varepsilon)V^\dagger\rho^2(V)$
$$= V^*\left(\underbrace{V\rho(\varepsilon)V^\dagger}\right)\rho^2(V)$$
$$= \rho_2(\varepsilon) = \varepsilon$$
since $\Lambda\cap\Lambda_0 = \emptyset$.

Hence $\varepsilon\,\rho(\varepsilon)\,\varepsilon = \rho(\varepsilon)\,\varepsilon\,\rho(\varepsilon)$

Letting $\pi^{(\rho)}(\sigma_i) := \rho^{i-1}(\varepsilon)$ we have obtained

* $\pi^{(\rho)}(\sigma_i)\pi^{(\rho)}(\sigma_j) = \pi^{(\rho)}(\sigma_j)\pi^{(\rho)}(\sigma_i)$ $|i-j|\geq 2$
from $\varepsilon\rho^2(\varepsilon) = \rho^2(\varepsilon)\varepsilon$

$$\Pi^{(\rho)}(\sigma_i)\Pi^{(\rho)}(\sigma_{i+1})\Pi^{(\rho)}(\sigma_i) = \Pi^{(\rho)}(\sigma_{i+1})\Pi^{(\rho)}(\sigma_i)\Pi^{(\rho)}(\sigma_{i+1})$$

is Yang-Baxter's equation.

$\to$ representation of the braid group!

In our case: $\rho(V) = V$ so that $\varepsilon = \mathbb{1}$
$\Rightarrow$ bosons.

2 particles are bosons.

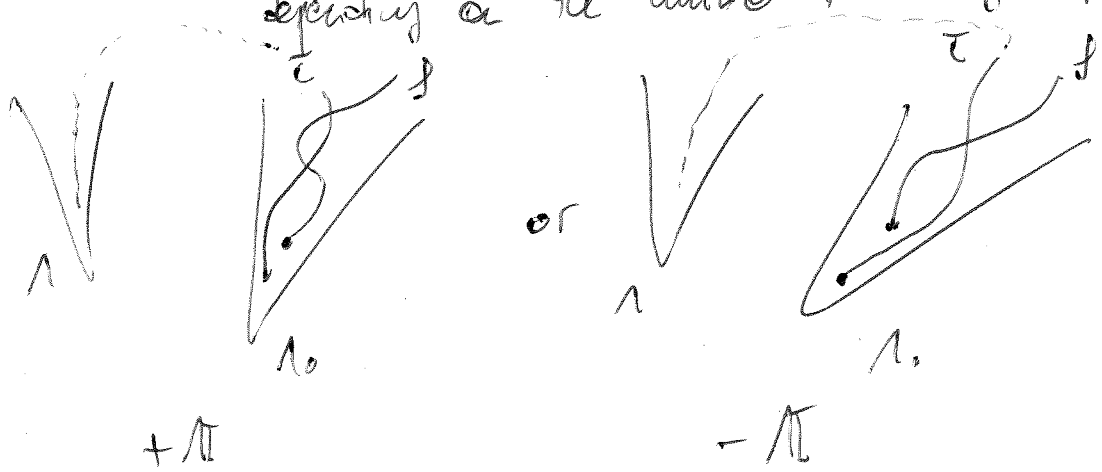
Similarly for particles of different types.
$\rho, \tilde{\tau}$ localized in the same cone Λ_0, with
$\tau = V\tilde{\tau}(\cdot)V^*$ localized in $\Lambda \cap \Lambda_0 = \emptyset$:

$$\begin{aligned}
\tilde{\tau}\circ\rho(A) &= V^*((\tau\circ\rho)(A))V \\
&= V^*((\rho\circ\tau)(A))V \\
&= V^*\rho(V\tilde{\tau}(A)V^*)V \\
&= (V^*\rho(V))(\rho\circ\tilde{\tau})(A)(V^*\rho(V))^*
\end{aligned}$$

with intertwiner

$$\varepsilon_{12} := V^*\rho(V) = \pm\mathbb{1} = \pm\varepsilon_{21}$$

depending on the number of crossings of $\rho, \tilde{\tau}$



6) The entanglement entropy

- What is entanglement? For a bipartite system with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and a pure state $\Psi \in \mathcal{H}$, <u>entanglement</u> is a (vague) measure of how far Ψ is from a product state.

One means of measuring it is through the <u>entropy</u>.

Let $\Psi_\Lambda \in \mathcal{H}_\Lambda$, a vector state of a quantum spin system in a finite volume Λ. For any $X \subset \Lambda$, its reduced density matrix is the unique matrix ρ^X s.t.

$$\mathrm{Tr}_{\mathcal{H}_X}(\rho^X A) = \langle \Psi_\Lambda, A \Psi_\Lambda \rangle \quad \forall A \in \mathcal{A}_X$$

Indeed: the map

$$\begin{aligned} \mathcal{A}_X &\to \mathbb{C} \\ A &\mapsto \langle \Psi_\Lambda, A\Psi_\Lambda \rangle \end{aligned}$$

defines a state over the finite dimensional algebra $\mathcal{A}_X$, which is given by a density matrix. Explicitly:

$$(\rho^X)_{ij} = \langle \Psi_\Lambda, e^{ji} \Psi_\Lambda \rangle \quad 1 \le i,j \le N^{|X|}$$

wobei $(e^{ji})_{kl} = \delta_{jk}\delta_{il}$

The map $|\Psi_\Lambda\rangle\langle\Psi_\Lambda| \to \rho^X$ is called the <u>partial trace</u>.

Note that ρ^X also depends on Λ.

We define the <u>entanglement entropy</u>:

$$S(X) := -\mathrm{Tr}_{\mathcal{H}_X}(\rho^X \log \rho^X)$$

so that

$$S(X) = -\sum_{i=1}^{N^{|X|}} \lambda_i \log \lambda_i$$

where $\{\lambda_i\}_i$ are the eigenvalues of ρ^X and $0\log(0) = 0$.

Since $\rho^X \geq 0$, and $\mathrm{Tr}\,\rho^X = \sum_i \lambda_i = 1$, $S(X) \geq 0$.

Its maximal value is obtained for $\lambda_i = N^{-|X|}$ for all i:

$$S_{max}(X) = -N^{|X|} \cdot \frac{1}{N^{|X|}} \log \frac{1}{N^{|X|}} = |X| \log N$$

<u>Note</u>: * $S_{max}(X)$ is proportional to the <u>volume</u> $|X|$.

* The minimal value is reached for a pure state $\lambda_1 = 1$, $\lambda_i = 0$ $(i \neq 1)$:

$$S_{min}(X) = 0.$$

It turns out (see later) that ρ^X is pure iff Ψ_Λ is a product state $\Psi_\Lambda = \Psi_X \otimes \Psi_{\Lambda\setminus X}$, hence $S(X) > 0$ is a measure of the entanglement of the sites in X with those outside of X.

- <u>Conjecture (vague)</u> Let Φ be an interaction s.t. $\Phi \in \mathcal{B}_\mu$ (i.e. τ satisfies a Lieb-Robinson bound) and assume that the system is <u>gapped</u>. Let ω_0 be a ground state of the spin system (as a limit of $\langle \Psi_\Lambda, \cdot\, \Psi_\Lambda \rangle$). Then, for any reasonable finite set X:

$$S(X) \leq C\,|\partial X|, \quad C > 0$$

namely: the entanglement is proportional to the surface of X, not its volume.

Proved in 1d only, namely $\exists\, S_0 > 0$ s.t.

$$S([-n,m]) \leq S_0 \qquad \forall\, n,m \in \mathbb{N}$$

(Hastings, 2007).

- $S(X)$ can usually best be computed with the following <u>Schmidt decomposition</u>:

<u>Lemma</u>: Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim \mathcal{H}_i < \infty$, $i=1,2$. Let $\Psi \in \mathcal{H}$. Then there exist orthonormal sets $\{e_i\}_{i=1}^r$; $\{f_j\}_{j=1}^r$, $r \leq \dim \mathcal{H}_i$ s.t.

$$\Psi = \sum_{k=1}^r s_k\, e_k \otimes f_k$$

where $s_k > 0$ and $\sum_{k=1}^r s_k^2 = 1$.

<u>Proof</u>: Since $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, and given two ONB's of $\mathcal{H}_1$ & $\mathcal{H}_2$ $\{\xi_i\}_i$, $\{\zeta_j\}_j$:

$$\Psi = \sum_{i=1}^{\dim \mathcal{H}_1} \sum_{j=1}^{\dim \mathcal{H}_2} \Psi_{ij}\, \xi_i \otimes \zeta_j$$

By the singular value decomposition, there exist

* a <u>unitary</u> $\dim \mathcal{H}_1 \times \dim \mathcal{H}_1$ matrix U
* — $\dim \mathcal{H}_2 \times \dim \mathcal{H}_2$ — V
* a diagonal $\dim \mathcal{H}_1 \times \dim \mathcal{H}_2$ — Σ.

s.t. $\Psi = U \Sigma V$, $\Sigma = \begin{pmatrix} s_1 & & & \\ & \ddots & & 0 \\ & & s_r & \\ & 0 & & \end{pmatrix}$

Hence

$$\Psi = \sum_{k=1}^r s_k \underbrace{\Big(\sum_i U_{ik}\, \xi_i\Big)}_{=:\, e_k} \otimes \underbrace{\Big(\sum_j V_{kj}\, \zeta_j\Big)}_{=:\, f_k}$$

$\square$

With this: if $\rho = |\Psi\rangle\langle\Psi|$

$$\rho_1 := \mathrm{Tr}_{\mathcal{H}_2}\rho = \sum_{i,j,h} \langle f_i, e_j \otimes f_j\rangle\langle e_h \otimes f_h, f_i\rangle s_j \bar{s}_h$$
$$= \sum_i |s_i|^2 |e_i\rangle\langle e_i|$$

so that s_i^2 are precisely the eigenvalues of ρ_1

and $S_1 = -\mathrm{Tr}(\rho_1 \log \rho_1) = -\sum_{h=1}^{\dim \mathcal{H}_1} s_i^2 \log s_i^2$

where we allowed for $s_i = 0$ in the formula above.

Notes: * S_1 is also the Shannon entropy of the probability distribution over $\{1, \dots, \dim \mathcal{H}_1\}$: $\{s_i^2\}_{i=1}^{\dim \mathcal{H}_1}$.

* An area law will depend on the decay of the Schmidt coefficients.

* ρ_1 is pure $\Leftrightarrow$ $s_1 = 1$ & $s_i = 0$ $(i \neq 1)$, namely $\Psi = \xi_1 \otimes \eta_1$ is a product state.

- Back to the toric code model, on the torus Λ

The $\{A_x, x \in \Lambda\}$ ~~form~~ generate a group under multiplication:

$$A_x^2 = \mathbb{1} \quad \text{for all } x$$

There are $|\Lambda|$ star operators, but only $|\Lambda| - 1$ suffice to generate $\mathcal{G}_\Lambda^0$ since — on the torus! — $\prod_{x \in \Lambda} A_x = \mathbb{1}_{\mathcal{H}_\Lambda}$.

<u>Lemma</u>: $|\mathcal{G}_\Lambda^0| = 2^{|\Lambda| - 1}$

<u>Proof</u>: Any $g \in \mathcal{G}_\Lambda^0$ can be represented as an element in 2^Λ, namely: $g = \prod_{x \in X} A_x \longmapsto \chi_X$

this mapping is 1-to-2 since

$$g = g \cdot \mathbb{1} = \prod_{x \in X} A_x \cdot \prod_{y \in \Lambda} A_y = \prod_{x \in \Lambda \setminus X} A_x \longmapsto 1 - \chi_X$$

Hence $|\mathcal{G}_\Lambda^0| = \frac{1}{2} 2^{|\Lambda|} = 2^{|\Lambda| - 1}$ $\square$

Two remarks on the entropy.

1) The convexity of $f: [0,1] \to \mathbb{R}:$
$$t \mapsto f(t) = t \log t$$
yields: For a density matrix ρ on $\mathcal{H}$ with $n = \dim \mathcal{H}$:
$$-S(\rho) = \sum_{i=1}^{n} \lambda_i \log \lambda_i = n \sum_{i=1}^{n} \left(\tfrac{1}{n}\right) f(\lambda_i)$$
$$\geq n \left(\sum_i \tfrac{1}{n} \lambda_i\right) \log \left(\sum_i \left(\tfrac{1}{n}\right) \lambda_i\right) = -\log n$$
since $\sum_i \lambda_i = 1$.

Hence $S(\rho) \leq \log n$ for all ρ and the maximal value is attained for $\lambda_i = \frac{1}{n}$ $\forall i$.

2) We consider the lattice $\mathbb{Z}^d$, and $X_L := [1, L]^d$, with on-site Hilbert space $\mathcal{H}_x$: $\dim \mathcal{H}_x = N$.

Then $\dim \mathcal{H}_{X_L} = N^{|X_L|} = N^{L^d}$

Let λ_i, $1 \leq i \leq N^{L^d}$ be the eigenvalues of ρ_{X_L}, arranged in decreasing order. Let
$$\mathcal{B}_\alpha = \{ i \in \mathbb{N} : N^{(\alpha-1)L^{d-1}} \leq i < N^{\alpha L^{d-1}} \}$$
for $1 \leq \alpha \leq L$ (where $\mathcal{B}_L$ is defined with $.. \leq i \leq ..$).

We assume for simplicity: $\lambda_i = \lambda_j$ whenever $i, j \in \mathcal{B}_\alpha$.

Furthermore we assume the following decay: Let $\varepsilon > 0$:
$$i \in \mathcal{B}_\alpha \Rightarrow \lambda_i = \frac{K(\varepsilon)}{|\mathcal{B}_\alpha| \, \alpha^{2+\varepsilon}}$$
where $K(\varepsilon)$ is s.t. $\sum_i \lambda_i = 1$, namely:

$$1 = \sum_{\alpha=1}^{L} |B_\alpha| \frac{K(\varepsilon)}{|B_\alpha| \alpha^{2+\varepsilon}} \quad , \quad \text{i.e.}$$

$$K(\varepsilon) = \sum_{\alpha=1}^{L} \frac{1}{\alpha^{2+\varepsilon}} < \sum_{\alpha=1}^{\infty} \alpha^{-(2+\varepsilon)}$$

The entropy is given by

$$S_\varepsilon(L) = \sum_{\alpha=1}^{L} |B_\alpha| \frac{K(\varepsilon)}{|B_\alpha| \alpha^{2+\varepsilon}} \left(-\log \frac{K(\varepsilon)}{|B_\alpha| \alpha^{2+\varepsilon}} \right)$$

Now: $|B_\alpha| = N^{\alpha L^{d-1}} (1 - N^{-L^{d-1}}) \leqq N^{\alpha L^{d-1}}$, so that

$$S_\varepsilon(L) \leq \sum_{\alpha=1}^{L} K(\varepsilon) \alpha^{-(2+\varepsilon)} \left(\alpha L^{d-1} \log N - \log \frac{K(\varepsilon)}{\alpha^{2+\varepsilon}} \right)$$

$$\leqq K_1(\varepsilon) L^{d-1} - K_2(\varepsilon) \qquad \text{for all } L \in \mathbb{N}.$$

In other words: an area law follows from a sufficient decay of the Schmidt coefficients.

Recall also: any $g \in \mathcal{G}_\Lambda^0$ corresponds to a loop configuration, and the ground state vectors: (normalized)

$$\Psi_{ij} = \sum_{g \in \mathcal{G}_\Lambda^0} g X_1^i X_2^j \Omega \cdot \frac{1}{|\mathcal{G}_\Lambda^0|^{1/2}}$$

equal weight superposition of closed loops, upon $\Omega = |+ \cdots +\rangle$.

Now: $g = t_g^X \otimes t_g^{X^c}$ where $X^c = \Lambda \setminus X$

with $t_g^Y \in \mathcal{A}_Y$ but t_g^X is not necessarily in $\mathcal{G}_\Lambda^0$

Theorem: Let $S_{ij}(X) = \mathrm{Tr}_{\Lambda\setminus X} |\Psi_{ij}\rangle\langle\Psi_{ij}|$. Then:

i) $S_{ij}(X)$ is indep. of i,j.

ii) $S_{ij}(X) = \log \dfrac{|\mathcal{G}_\Lambda|}{|\mathcal{G}_X||\mathcal{G}_Y|}$

where $\mathcal{G}_X$ is generated by $\{A_x : x \in X\}$

Proof: i) Recall that X_k are unitary so that

$$|\Psi_{ij}\rangle\langle\Psi_{ij}| = X_1^i X_2^j |\Psi_{++}\rangle\langle\Psi_{++}| X_1^{i*} X_2^{j*}$$

has the same spectrum as $|\Psi_{++}\rangle\langle\Psi_{++}|$ and hence the same entropy.

ii) We compute $S_{++}(X) \equiv S$ and $\rho_{++}^X = \rho$:

$$\rho = \frac{1}{|\mathcal{G}_\Lambda^0|} \mathrm{Tr}_{\Lambda\setminus X} \sum_{g,g'} g P_\Omega g'$$

(since $g^* = g$ since A_x are self-adjoint and they all commute)

Since Ω is a product state $\Omega = \Omega^X \otimes \Omega^{X^c}$, and

$$\rho = \frac{1}{|\mathcal{G}^\circ_\Lambda|} \mathrm{Tr}_{\Lambda \setminus X} \sum_{g,g'} t^X_g P_{\Omega^X} t^X_{g'} \otimes t^Y_g P_{\Omega^Y} t^Y_{g'} \quad , Y = X^c$$

$$= \frac{1}{|\mathcal{G}^\circ_\Lambda|} \sum_{g,g'} t^X_g P_{\Omega^X} t^X_{g'} \langle \Omega^Y, t^Y_{g'} t^Y_g \Omega^Y \rangle$$

now: t^Y_g are products of σ^x matrices, which flip the spins, so that $\langle \Omega^Y, t^Y_{g'} t^Y_g \Omega^Y \rangle \neq 0$

iff $t^Y_{g'} = t^Y_g$

Hence: the only non-zero terms in the sum are s.t.

$$g = t^X_g \otimes t^Y_g \quad \text{with} \quad t^Y_g = t^Y_{g'}$$
$$g' = t^X_{g'} \otimes t^Y_{g'}$$

namely $\quad g' = (S^X \otimes \mathbb{1})\, g$

and so $\quad \underbrace{(S^X \otimes \mathbb{1})}_{=:\, \tilde{g}} = g' g^{-1} \in \mathcal{G}_X$

$$\Rightarrow \quad \rho = \frac{1}{|\mathcal{G}^\circ_\Lambda|} \sum_{\substack{g \in \mathcal{G}^\circ_\Lambda \\ \tilde{g} \in \mathcal{G}_X}} \left(t^X_g P_{\Omega^X} t^X_g \right) \tilde{g}$$

We further claim: $\quad t^X_{g'} = t^X_g \iff g' = hg$ with $h \in \mathcal{G}_Y$

as above. Hence:

$$\rho = \frac{|\mathcal{G}_Y|}{|\mathcal{G}^\circ_\Lambda|} \sum_{\substack{g \in \mathcal{G}^\circ_\Lambda \setminus \mathcal{G}_Y \\ \tilde{g} \in \mathcal{G}_X}} \left(t^X_g P^X_\Omega t^X_g \right) \tilde{g}$$

Claim: $$\rho^2 = \frac{|\mathcal{G}_X||\mathcal{G}_Y|}{|\mathcal{G}_\Lambda^\circ|}\,\rho \,.$$

Indeed: $$\rho^2 = \frac{|\mathcal{G}_Y|^2}{|\mathcal{G}_\Lambda^\circ|^2}\sum_{\substack{g_1,g_2\\ \tilde g_1,\tilde g_2}} t^X_{g_1} P_{\Omega^X} t^X_{g_2}\,\tilde g_2 \cdot \underbrace{\langle \Omega^X, t^X_{g_1}\tilde g_1 t^X_{g_2}\Omega^X\rangle}_{\neq 0 \text{ iff } t^X_{g_2} = t^X_{g_1}\tilde g_1}$$

so that:

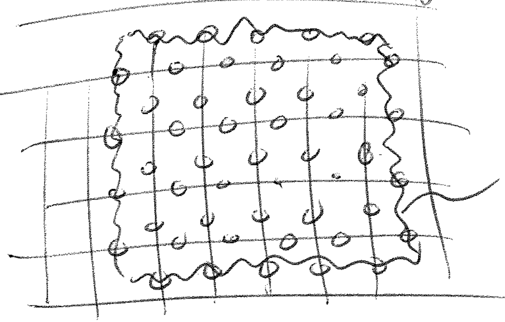
$$\rho^2 = \frac{|\mathcal{G}_Y|^2}{|\mathcal{G}_\Lambda^\circ|^2}\sum_{\substack{g_1\\ \tilde g_1,\tilde g_2}} t^X_{g_1} P_{\Omega^X} t^X_{g_1}\,\tilde g_1\tilde g_2$$

$$= \frac{|\mathcal{G}_Y|^2}{|\mathcal{G}_\Lambda^\circ|^2}\,|\mathcal{G}_X| \sum_{\substack{g_1\in \mathcal{G}_\Lambda^\circ\setminus \mathcal{G}_Y\\ \tilde g\in \mathcal{G}_X}} \left(t^X_g P_{\Omega^X} t^X_g\right)\tilde g \quad \text{as claimed.}$$

From the matrix equality $\rho\left(\rho - \frac{|\mathcal{G}_X||\mathcal{G}_Y|}{|\mathcal{G}_\Lambda^\circ|}\right) = 0$, one concludes that the non-zero eigenvalues of ρ are given by $\frac{|\mathcal{G}_X||\mathcal{G}_Y|}{|\mathcal{G}_\Lambda^\circ|}$ and there must be $1/(\cdots)$ of them by normalization. Hence:

$$S = -\sum_{i=1}^{1/(\cdots)} (\cdots)\log(\cdots) = -\log\frac{|\mathcal{G}_X||\mathcal{G}_Y|}{|\mathcal{G}_\Lambda^\circ|} \qquad \square$$

- We specifically consider X being a rectangle:



$\partial X \subset X$ and $X_0 := X\setminus\partial X$

Now: as in the lemma but without constraint
(i.e. $\mathcal{G}_X$ is freely generated by $\{A_x, x \in X_\circ\}$)

$$|\mathcal{G}_X| = 2^{|X_\circ|} \; ; \; |\mathcal{G}_{\Lambda\setminus X}| = 2^{|(\Lambda\setminus X)_\circ|}$$

Hence if $\log = \log_2$:

$$S_X = |\Lambda| - 1 - |(\Lambda\setminus X)_\circ| - |X_\circ| = \underbrace{|\partial X|}_{\substack{= \#\text{ of sqns at}\\ \text{the boundary}\\ \text{of } X.}} - 1$$

In general, on a torus of genus g:

$$\underbrace{n_v}_{\#\text{vertices}} + \underbrace{n_p}_{\#\text{plaquette}} - \underbrace{n_e}_{\#\text{edges}} = 2 - 2g \qquad \text{(Euler's formula)}$$

If $n_v = n_p$: $\quad 2(n_v - 1) = n_e - 2g$

so that $|\mathcal{G}^\circ_{\Lambda g}| = 2^{n_v - 1} = 2^{n_e/2 - g}$

and $S_X = \frac{n_e}{2} - |X| - |\Lambda\setminus X| - g$

hence: in all cases

$$S_X = \underbrace{c_1 |\partial X|}_{\text{area law}} - \underbrace{g}_{\substack{\text{topological}\\ \text{entanglement}\\ \text{entropy!}}}$$

The presence of a universal constant of topological origin is our final characterization of topological order.