TMP Master Core Module:
Mathematical Statistical Physics I

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Chapter 1

Introduction

1.1 The Ising model: Peierls argument

Chapter 2

C*-algebras, states and representations

2.1 C*-algebras

Let $\mathcal{A}$ be an associative algebra over $\mathbb{C}$. $\mathcal{A}$ is a normed algebra if there is a norm $\mathcal{A} \ni x \mapsto \|x\| \in \mathbb{R}_+$ such that $\|xy\| \leq \|x\|\|y\|$. A complete normed algebra is a Banach algebra. A mapping $x \mapsto x^*$ of $\mathcal{A}$ into itself is an involution if

$$\begin{aligned}
&(x^*)^* = x; \\
&(x+y)^* = x^* + y^*; \\
&(xy)^* = y^* x^*; \\
&(\lambda x)^* = \overline{\lambda} x^*.
\end{aligned}$$

An algebra with an involution is a *-algebra.

Definition 1. *A Banach *-algebra $\mathcal{A}$ is called a* C*-algebra *if*

$$\|x^* x\| = \|x\|^2, \qquad x \in \mathcal{A}.$$

Proposition 1. *Let $\mathcal{A}$ be a C*-algebra.*

1. $\|x^*\| = \|x\|$*;*
2. *If $\mathcal{A}$ does not have an identity, let $\widetilde{\mathcal{A}}$ be the algebra obtained from $\mathcal{A}$ by adjoining an identity* 1*. Then $\widetilde{\mathcal{A}}$ is a C*-algebra with norm $\|\cdot\|$ defined by*

$$\|\lambda 1 + x\| = \sup_{y \neq 0} \frac{\|\lambda y + xy\|}{\|y\|}, \qquad \lambda \in \mathbb{C}.$$

Proof. Exercise. □

In the following, $\mathcal{A}$ will always denote a C*-algebras with an identity if not specified otherwise.

Definition 2. *The* spectrum $\mathrm{Sp}(x)$ *of $x \in \mathcal{A}$ is the set*

$$\mathrm{Sp}(x) := \{\lambda \in \mathbb{C} : x - \lambda 1 \textit{ is not invertible in } \mathcal{A}\}.$$

If $|\lambda| > \|x\|$, then the series $\lambda^{-1}\sum_{n\in\mathbb{N}}(x/\lambda)^n$ is norm convergent and sums to $(\lambda 1 - x)^{-1}$. Hence, $\mathrm{Sp}(x) \subset B_{\|x\|}(0)$. Assume now that $x \in \mathcal{A}$ is a self-adjoint element and that $a + \mathrm{i}b \in \mathrm{Sp}(x)$, $a, b \in \mathbb{R}$. Then $a+\mathrm{i}(b+t) \in \mathrm{Sp}(x+\mathrm{i}t1)$. Since $\|x+\mathrm{i}t1\|^2 = \|x+\mathrm{i}t1\|\|x-\mathrm{i}t1\| = \|x^2+t^21\| \leq \|x\|^2 + t^2$, and by the remark above, $|a + \mathrm{i}(b+t)|^2 \leq \|x\|^2 + t^2$, and further $2bt \leq \|x\| - a^2 - b^2$ for all $t \in \mathbb{R}$, so that $b = 0$. For any polynomial P over $\mathbb{C}$, $P(\mu) - \lambda = A\prod_{i=1}^n(\mu - z_i)$, and $P(x) - \lambda 1 = A\prod_{i=1}^n(x - z_i) \in \mathcal{A}$ for any $x \in \mathcal{A}$. Hence, $\lambda \in \mathrm{Sp}(P(x))$ iff $z_j \in \mathrm{Sp}(x)$ for a $1 \leq j \leq n$. Since $P(z_j) = \lambda$, we have that $\lambda \in \mathrm{Sp}(P(x))$ iff $\lambda \in P(\mathrm{Sp}(x))$. We have proved

Proposition 2. *Let $x \in \mathcal{A}$.*

1. $\mathrm{Sp}(x) \subset B_{\|x\|}(0)$*;*
2. *if $x = x^*$, then* $\mathrm{Sp}(x) \subset [-\|x\|, \|x\|]$*;*
3. *if $xx^* = x^*x$, i.e. x is normal, then $\|x\| = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\}$;*
4. *for any polynomial P,* $\mathrm{Sp}(P(x)) = P(\mathrm{Sp}(x))$*;*

The proof of 3. is left as an exercise. Note that the condition holds in particular for $x = x^*$

An element $x \in \mathcal{A}$ is positive if it is self-adjoint and $\mathrm{Sp}(x) \subset \mathbb{R}_+$.

Proposition 3. *Let $x \in \mathcal{A}$, $x \neq 0$. The following are equivalent:*

1. *x is positive;*
2. *there is a self-adjoint $z \in \mathcal{A}$ such that $x = z^2$;*
3. *there is $y \in \mathcal{A}$ such that $x = y^*y$;*

Proof. (3) $\Rightarrow$ (2) by choosing $y = z$. (3) $\Rightarrow$ (1) since z^2 is self-adjoint and since, by Proposition 2, $\mathrm{Sp}(z^2) \subset [0, \|z\|^2]$. To show (1) $\Rightarrow$ (3), we note that[1] for any $\mu > 0$,

$$\mu = \left[\frac{1}{\pi}\int_0^\infty \sqrt{\lambda}\left(\frac{1}{\lambda} - \frac{1}{\lambda+\mu}\right)d\lambda\right]^2 \tag{2.1}$$

Since x is positive, $(x+\lambda 1)$ is invertible for all $\lambda > 0$ so that $z := \pi\int_0^\infty \sqrt{\lambda}\left(\lambda^{-1} - (x+\lambda 1)^{-1}\right)d\lambda$ is well defined as a norm convergent integral, and $x = z^2$. Using again (2.1) with $\mu = 1$, we have that

$$\|x\|^{1/2}1 - z = \frac{\|x\|^{1/2}}{\pi}\int_0^\infty \frac{\sqrt{\lambda}}{\lambda+1}(\hat{x}+\lambda 1)^{-1}(\hat{x}-1)d\lambda, \qquad \hat{x} = x\|x\|^{-1}$$

But $\hat{x}$ positive implies $\mathrm{Sp}(\hat{x}) \subset [0,1]$, hence $\mathrm{Sp}(1-\hat{x}) \subset [0,1]$ and $\|1-\hat{x}\| \leq 1$. Moreover, since $\mathrm{Sp}((\hat{x}+\lambda 1)^{-1}) = (\mathrm{Sp}(\hat{x}+\lambda 1))^{-1} \subset [(1+\lambda)^{-1}, \lambda^{-1}]$, we have $\|(\hat{x}+\lambda 1)^{-1}\| \leq \lambda^{-1}$ for $\lambda > 0$. Hence, $\|1 - z\|x\|^{-1/2}\| \leq 1$ so that $\mathrm{Sp}(z) \subset [0, 2\|x\|^{1/2}]$ and finally z is positive.

It remains to prove (2) $\Rightarrow$ (1). Since $x = y^*y$ is self-adjoint, x^2 is positive and we denote by $|x|$ its positive square root defined by the integral above. Then $x_\pm := (|x| \pm x)/2$ is positive and $x_-x_+ = x_+x_- = 0$. Decomposing $yx_- = s + \mathrm{i}t$, with self-adjoint s, t, we have $(yx_-)^*(yx_-) + (yx_-)(yx_-)^* = 2(s^2+t^2) \geq 0$. But $-(yx_-)^*(yx_-) = -x_-(-x_-+x_+)x_- = x_-^3$ is positive, so that $(yx_-)(yx_-)^*$ is positive. On the other hand, $\mathrm{Sp}((yx_-)(yx_-)^*)\cup\{0\} = \mathrm{Sp}((yx_-)^*(yx_-))\cup\{0\} \subset \mathbb{R}_-$, hence $(yx_-)^*(yx_-) = 0$ so that $x_- = 0$, and finally $x = x_+$ is positive. □

[1] Convergence follows from the asymptotics $O(s^{-1/2})$ as $s \to 0$ and $O(s^{-3/2})$ as $s \to \infty$, while the change of variables $\lambda = \mu\xi$ yields immediately that the integral is $\sqrt{\mu}$, up to a constant.

Definition 3. *A* *-morphism *between two *-algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *is a linear map* $\pi : \mathcal{A} \to \mathcal{B}$ *such that* $\pi(A_1A_2) = \pi(A_1)\pi(A_2)$ *and* $\pi(A^*) = \pi(A)^*$*, for all* $A, A_1, A_2 \in \mathcal{A}$*. It is called a* *-isomorphism *if it is bijective. A *-isomorphism* $\mathcal{A} \to \mathcal{A}$ *is an* automorphism*.*

Proposition 4. *Let* $\mathcal{A}, \mathcal{B}$ *be two C*-algebras and* $\pi : \mathcal{A} \to \mathcal{B}$ *a *-morphism. Then* $\|\pi(x)\|_{\mathcal{B}} \leq \|x\|_{\mathcal{A}}$*, and the range* $\{\pi(A) : A \in \mathcal{A}\}$ *is a *-subalgebra of* $\mathcal{B}$*.*

Proof. If x is self-adjoint, so is $\pi(x)$ and $\|\pi(x)\| = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(\pi(x))\}$. If $x - \lambda 1$ is invertible, then $1 = \pi((x-\lambda 1)^{-1}(x-\lambda 1)) = \pi((x-\lambda 1)^{-1})\pi(x-\lambda 1)$ so that $\pi(x) - \lambda 1$ is invertible, whence $\mathrm{Sp}(\pi(x)) \subset \mathrm{Sp}(x)$, we have that $\|\pi(x)\| \leq \sup\{|\lambda| : \lambda \in \mathrm{Sp}(x)\} = \|x\|$. The general case follows from $\|\pi(x)\|^2 = \|\pi(x^*x)\| \leq \|x^*x\| = \|x\|^2$. □

Let Γ be a locally compact Hausdorff space, and let $C_0(\Gamma)$ be the algebra, under pointwise multiplication, of all complex valued continuous functions that vanish at infinity.

Theorem 5. *If* $\mathcal{A}$ *is a commutative C*-algebra, then there is a locally compact Hausdorff space* Γ *such that* $\mathcal{A}$ *is *-isomorphic to* $C_0(\Gamma)$*.*

Proof. See Robert's lectures. □

In classical mechanics, the space Γ is usually referred to as the phase space.

If $\mathcal{A}$ is a commutative C*-algebra with an identity, then $\mathcal{A}$ is isomorphic to $C(K)$, the algebra of continuous functions on a compact Hausdorff space K.

Let $\mathcal{U}$ be a *-subalgebra of $\mathcal{L}(\mathcal{H})$. The commutant $\mathcal{U}'$ is the subset of $\mathcal{L}(\mathcal{H})$ of operators that commute with every element of $\mathcal{U}$, and so forth with $\mathcal{U}'' := (\mathcal{U}')'$. In particular, $\mathcal{U} \subset \mathcal{U}''$, and further $\mathcal{U}' = \mathcal{U}'''$.

Definition 4. *A* von Neumann algebra *or W*-algebra on* $\mathcal{H}$ *is a *-subalgebra* $\mathcal{U}$ *of* $\mathcal{L}(\mathcal{H})$ *such that* $\mathcal{U}'' = \mathcal{U}$*. Its* center *is* $\mathcal{Z}(\mathcal{U}) := \mathcal{U} \cap \mathcal{U}'$*, and* $\mathcal{U}$ *is a* factor *if* $\mathcal{Z}(\mathcal{U}) = \mathbb{C} \cdot 1$*.*

Theorem 6. *Let* $\mathcal{U}$ *be a *-subalgebra of* $\mathcal{L}(\mathcal{H})$ *such that* $\mathcal{U}\mathcal{H} = \mathcal{H}$*. Then* $\mathcal{U}$ *is a von Neumann algebra iff* $\mathcal{U}$ *is weakly closed.*

Note that $\mathcal{U}\mathcal{H} = \mathcal{H}$ is automatically satisfied if $1 \in \mathcal{U}$. Furthermore, for any *-subalgebra $\mathcal{U}$ of $\mathcal{L}(\mathcal{H})$, let $\overline{\mathcal{U}}$ be its weak closure for which $\overline{\mathcal{U}}'' = \overline{\mathcal{U}}$ by the theorem. Since $\mathcal{U} \subset \overline{\mathcal{U}}$, we have $\overline{\mathcal{U}}' \subset \mathcal{U}'$. Furthermore, if $x \in \mathcal{U}'$, and $y \in \overline{\mathcal{U}}$, with $\mathcal{U} \ni y_n \rightharpoonup y$, then x commutes with y_n for all n and hence with y, so that $\mathcal{U}' \subset \overline{\mathcal{U}}'$. It follows that $\overline{\mathcal{U}}' = \mathcal{U}'$, whence $\overline{\mathcal{U}}'' = \mathcal{U}''$ and so:

Corollary 7. $\mathcal{U}$ *is weakly dense in* $\mathcal{U}''$*, namely* $\overline{\mathcal{U}} = \mathcal{U}''$*.*

2.2 Representations and states

Definition 5. *Let* $\mathcal{A}$ *be a C*-algebra and* $\mathcal{H}$ *a Hilbert space. A* representation *of* $\mathcal{A}$ *in* $\mathcal{H}$ *is a *-morphism* $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$*. Moreover,*

1. *Two representations* π, π' *in* $\mathcal{H}, \mathcal{H}'$ *are* equivalent *if there is a unitary map* $U : \mathcal{H} \to \mathcal{H}'$ *such that* $U\pi(x) = \pi'(x)U$*;*

2. *A representation* π *is* topologically irreducible *if the only closed subspaces that are invariant under* $\pi(\mathcal{A})$ *are* $\{0\}$ *and* $\mathcal{H}$*;*

3. *A representation* π *is* faithful *if it is an isomorphism, namely* $\mathrm{Ker}\pi = \{0\}$*.*

Note that in general $\|\pi(x)\| \leq \|x\|$, with equality if and only if π is faithful. One can further show that for any C*-algebra, there exists a faithful representation.

If $(\mathcal{H}, \pi)$ is a representation of $\mathcal{A}$ and $n \in \mathbb{N}$, then $n\pi(A)(\oplus_{i=1}^n \psi_i) := \oplus_{i=1}^n \pi(A)\psi_i$ defines a representation $n\pi$ on $\oplus_{i=1}^n \mathcal{H}$.

Proposition 8. *Let π be a representation of $\mathcal{A}$ in $\mathcal{H}$. T.f.a.e*

1. *π is topologically irreducible;*
2. *$\pi(\mathcal{A})' := \{B \in \mathcal{L}(\mathcal{H}) : [B, \pi(x)] = 0,$ for all $x \in \mathcal{A}\} = \mathbb{C} \cdot 1$;*
3. *Any $\xi \in \mathcal{H}$, $\xi \neq 0$ is cyclic: $\overline{\pi(x)\xi} = \mathcal{H}$, or $\pi = 0$.*

Proof. (1) $\Rightarrow$ (3) : If $\pi(\mathcal{A})\xi$ is not dense, then $\pi(\mathcal{A})\xi = \{0\}$. It follows that $\mathbb{C}\xi$ is an invariant subspace, and hence $\mathcal{H} = \mathbb{C}\xi$ and $\pi = 0$
(3) $\Rightarrow$ (1) : Let $\mathcal{K} \neq \{0\}$ be a closed invariant subspace. For any $\xi \in \mathcal{K}$, $\pi(\mathcal{A})\xi \subset \mathcal{K}$ and since ξ is cyclic, $\pi(\mathcal{A})\xi$ is dense in $\mathcal{H}$
(2) $\Rightarrow$ (1) : Let $\mathcal{K} \neq \{0\}$ be a closed invariant subspace, and let $P_\mathcal{K}$ be the orthogonal projection on $\mathcal{K}$. Then $P_\mathcal{K} \in \pi(\mathcal{A})'$, since for $\xi \in \mathcal{K}, \eta \in \mathcal{K}^\perp$, $\langle \xi, \pi(x)\eta \rangle = \langle \pi(x^*)\xi, \eta \rangle = 0$ so that $\pi(x)\eta \in \mathcal{K}^\perp$ for any $x \in \mathcal{A}$. Hence $P_\mathcal{K} = 0$ or $P_\mathcal{K} = 1$, i.e. $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$.
(1) $\Rightarrow$ (2) : Let $c \in \pi(\mathcal{A})'$ be self-adjoint. Then all spectral projectors of c belong to $\pi(\mathcal{A})'$, so that they are all either 0 or 1 by (1), and c is a scalar. If c is not self-adjoint, apply the above to $c \pm c^*$. □

A triple $(\mathcal{H}, \pi, \xi)$ where ξ is a cyclic vector is called a cyclic representation.

Recall that $\mathcal{A}^* := \{\omega : \mathcal{A} \to \mathbb{C} : \omega \text{ is linear and bounded}\}$. For any $\xi \in \mathcal{H}$, the map $\mathcal{A} \ni x \mapsto \langle \xi, \pi(x)\xi \rangle$ is an element of $\mathcal{A}^*$ since $|\langle \xi, \pi(x)\xi \rangle| \leq \|\xi\|_\mathcal{H}^2 \|x\|_\mathcal{A}$ and it is positive: If x is positive, then $x = y^*y$ and $\langle \xi, \pi(x)\xi \rangle = \|\pi(y)\xi\|_\mathcal{H}^2 \geq 0$. We shall denote it $\omega_{\pi,\xi}$. If $0 \leq T \leq 1$ is a self-adjoint operator in $\mathcal{H}$ and $T \in \pi(\mathcal{A})'$, then the form $x \mapsto \omega_{\pi,T\xi}$ is positive, and $\omega_{\pi,T\xi}(y^*y) = \|\pi(y)T\xi\|^2 = \|T\pi(y)\xi\|^2 \leq \|\pi(y)\xi\|^2 = \omega_{\pi,\xi}(y^*y)$, so that $\omega_{\pi,T\xi} \leq \omega_{\pi,\xi}$.

Lemma 9. *Let ω be a positive linear functional on $\mathcal{A}$. Then*

$$\omega(x^*y) = \overline{\omega(y^*x)}, \qquad |\omega(x^*y)|^2 \leq \omega(x^*x)\omega(y^*y).$$

Proof. This follows from the positivity of the quadratic form $\lambda \mapsto \omega((\lambda x + y)^*(\lambda x + y)) \geq 0$. □

In fact, any positive linear form ν bounded above by $\omega_{\pi,\xi}$ is of the form above. Indeed,

$$|\nu(x^*y)|^2 \leq \nu(x^*x)\nu(y^*y) \leq \omega_{\pi,\xi}(x^*x)\omega_{\pi,\xi}(y^*y) \leq \|\pi(x)\xi\|^2 \|\pi(y)\xi\|^2$$

so that $\pi(x)\xi \times \pi(y)\xi \mapsto \nu(x^*y)$ is a densely defined, bounded, symmetric linear form on $\mathcal{H} \times \mathcal{H}$. By Riesz representation theorem, there exists a unique bounded operator T such that $\nu(x^*y) = \langle \pi(x)\xi, T\pi(y)\xi \rangle$, and $0 \leq T \leq 1$. Moreover,

$$\langle \pi(x)\xi, T\pi(z)\pi(y)\xi \rangle = \nu(x^*zy) = \nu((z^*x)^*y) = \langle \pi(x)\xi, \pi(z)T\pi(y)\xi \rangle,$$

so that $T \in (\pi(\mathcal{A}))'$.

Definition 6. *A* state *ω on a C*-algebra $\mathcal{A}$ is a positive element of $\mathcal{A}^*$ such that*

$$\|\omega\| = \sup_{x \in \mathcal{A}} \frac{\omega(x)}{\|x\|} = 1.$$

A state ω is called

- pure *if the only positive linear functionals majorised by* ω *are* $\lambda\omega$, $0 \le \lambda \le 1$,
- faithful *if* $\omega(x^*x) = 0$ *implies* $x = 0$.

If ω is normalised and $\mathcal{A}$ has an identity, then $\omega(1) = 1$. Reciprocally, $|\omega(x)|^2 \le \omega(1)\omega(x^*x)$. Since $\|x^*x\|1 - x^*x \ge 0$, we further have $|\omega(x)|^2 \le \|x^*x\|\omega(1)^2$, i.e. $\|\omega\| \le \omega(1)$, which proves:

Proposition 10. *If* $\mathcal{A}$ *has an identity, and* ω *is a positive linear form on* $\mathcal{A}$*, then* $\|\omega\| = 1$ *if and only if* $\omega(1) = 1$.

By Corollary 7, $\pi(\mathcal{A})''$ is a von Neumann algebra for any state ω. ω is called a factor state if $\pi(\mathcal{A})''$ is a factor, i.e. if $\pi(\mathcal{A})' \cap \pi(\mathcal{A})'' = \mathbb{C} \cdot 1$.

We shall denote $\mathcal{E}(\mathcal{A})$ the set of states over $\mathcal{A}$ and $\mathcal{P}(\mathcal{A})$ the set of pure states.

Proposition 11. $\mathcal{E}(\mathcal{A})$ *is a convex set, and it is weakly-* compact iff* $\mathcal{A}$ *has an identity. In that case,* $\omega \in \mathcal{P}(\mathcal{A})$ *iff it is an extremal point of* $\mathcal{E}(\mathcal{A})$.

Proof. We only prove the second part, the first part being is a version of the Banach-Alaoglu theorem. Let $\omega \in \mathcal{P}(\mathcal{A})$. Assume that $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$. Then $\omega \ge \lambda\omega_1$, hence $\lambda\omega_1 = \mu_1\omega$, $0 \le \mu_2 \le 1$ and similarly for ω_2. Hence $\omega = (\mu_1 + \mu_2)\omega$ and ω is extremal. Reciprocally, assume that ω is not pure, in which case there is a linear functional $\tilde{\nu}_1 \ne \tilde{\lambda}\omega$ such that $\omega \ge \tilde{\nu}_1$. In particular, $\lambda := \tilde{\nu}_1(1) \le \omega(1) = 1$. Since $\nu_1 := \lambda^{-1}\tilde{\nu}_1$ is a state, $\nu_2 := (\omega - \lambda\nu_1)/(1-\lambda)$ defines a state, and $\omega = \lambda\nu_1 + (1-\lambda)\nu_2$. Hence ω is not extremal. □

In particular, if $\{\omega_i\}_{i\in I}$ is an arbitrary infinite family of states, then there exists at least one weak-* accumulation point. Note that $\omega_n \rightharpoonup \omega$ in the weak-* topology if $\omega_n(x) \to \omega(x)$ for all $x \in \mathcal{A}$. In fact, it is defined as the weakest topology in which this holds, namely in which the map $x : \omega \mapsto \omega(x)$ are continuous.

Theorem 12. *Let* $\mathcal{A}$ *be a C*-algebra and* $\omega \in \mathcal{E}(\mathcal{A})$*. Then there exists a cyclic representation* $(\mathcal{H}, \pi, \Omega)$ *such that*

$$\omega(x) = \langle \Omega, \pi(x)\Omega \rangle$$

for all $x \in \mathcal{A}$*. Such a representation is unique up to unitary isomorphism.*

Proof. We consider only the case where $\mathcal{A}$ has an identity. Let $\mathcal{N} := \{a \in \mathcal{A} : \omega(a^*a) = 0\}$. Since, by Lemma 9, $0 \le \omega(a^*x^*xa) \le \omega(a^*a)\|x\|^2 = 0$, we have $a \in \mathcal{N}, x \in \mathcal{A}$ implies $xa \in \mathcal{N}$ is a left ideal. On $h := \mathcal{A} \setminus \mathcal{N}$, we denote ψ_x the equivalence class of $x \in \mathcal{A}$, and the bilinear form $(\psi_x, \psi_y) \mapsto \omega(x^*y)$ is positive and well-defined, since $\omega((x+a)^*, y+b) = \omega(x^*y) + \omega(a^*y) + \omega(x^*b) + \omega(a^*b) = \omega(x^*y)$ for any $x, y \in \mathcal{A}$; $a, b \in \mathcal{N}$. Let $\mathcal{H}$ be the Hilbert space completion of h. For any $\psi_x \in h$, let $\pi(y)\psi_x := \psi_{yx}$. The map $\pi : \mathcal{A} \to \mathcal{L}(h)$ is linear and bounded since $\|\pi(y)\psi_x\|^2 = \langle \psi_{yx}, \psi_{yx} \rangle = \omega(x^*y^*yx) \le \|y\|^2\|x\|^2$ and thus has a bounded closure. It is a *-homomorphism since

$$\langle \psi_y, \pi(z^*)\psi_x \rangle = \langle \psi_y, \psi_{z^*x} \rangle = \omega(y^*z^*x) = \langle \psi_{zy}, \psi_x \rangle = \langle \pi(z)\psi_y, \psi_x \rangle$$

and $\pi(xy)\psi_z = \psi_{xyz} = \pi(x)\pi(y)\psi_z$ and defines a representation of $\mathcal{A}$ in $\mathcal{H}$. Moreover, $\langle \psi_1, \pi(x)\psi_1 \rangle = \langle \psi_1, \psi_x \rangle = \omega(x)$, so that $\Omega = \psi_1$. Cyclicity follows from $\{\pi(x)\Omega : x \in \mathcal{A}\} = \{\psi_x : x \in \mathcal{A}\}$, which is the dense set of equivalence classes by construction. Finally, let $(\mathcal{H}', \pi', \Omega')$ be another such representation. Then the map $U : \mathcal{H} \to \mathcal{H}'$ defined by $\pi'(x)\Omega' = U\pi(x)\Omega$ is a densely defined isometry, since

$$\langle \pi(y)\Omega, \pi(x)\Omega \rangle_{\mathcal{H}} = \omega(y^*x) = \langle \pi'(y)\Omega', \pi'(x)\Omega' \rangle_{\mathcal{H}'} = \langle U\pi(y)\Omega, U\pi(x)\Omega \rangle_{\mathcal{H}'},$$

and hence extends to a unitary operator. □

Corollary 13. *Let $\mathcal{A}$ be a C*-algebra and α a *-automorphism. If $\omega \in \mathcal{E}(\mathcal{A})$ is α-invariant, $\omega(\alpha(x)) = \omega(x)$ for all $x \in \mathcal{A}$, then there is a unique unitary operator U on the GNS Hilbert space $\mathcal{H}$ such that, for all $x \in \mathcal{A}$,*

$$U\pi(x) = \pi(\alpha(x))U, \quad \textit{and} \quad U\Omega = \Omega.$$

One says that α is unitarily implementable in the GNS representation.

Proof. The corollary follows from the uniqueness part of Theorem 12 applied to $(\mathcal{H}, \pi \circ \alpha, \Omega)$, since $\langle \Omega, \pi(x)\Omega \rangle = \omega(x) = \omega(\alpha(x)) = \langle \Omega, \pi \circ \alpha(x)\Omega \rangle$. □

Proposition 14. *Let $\mathcal{A}$ be a C*-algebra, $\omega \in \mathcal{E}(\mathcal{A})$ and $(\mathcal{H}, \pi, \Omega)$ the associated representation. Then π is irreducible and $\pi \neq 0$ iff ω is a pure state.*

Proof. Let ν be majorised by $\omega = \omega_{\pi,\Omega}$. There is a $0 \leq T \leq 1$ such that $\nu(x^*y) = \langle \pi(x)\xi, T\pi(y)\xi \rangle$ with $T \in (\pi(\mathcal{A}))'$. If π is irreducible, then $T = \sqrt{\lambda} \cdot 1$ so that $\nu = \lambda\omega$, $0 \leq \lambda \leq 1$ and ω is pure. Reciprocally, if ν is not a multiple of ω, then T is not a multiple of the identity, so that $(\mathcal{H}, \pi)$ is not irreducible. □

Definition 7. *Let $(\mathcal{H}, \pi)$ be a representation of $\mathcal{A}$. A state ω is π-*normal *if there exists a density matrix ρ_ω in $\mathcal{H}$ such that $\omega(A) = \mathrm{Tr}(\rho_\omega \pi(A))$. Two representations $(\mathcal{H}_1, \pi_1), (\mathcal{H}_2, \pi_2)$ are* quasi-equivalent *if every π_1-normal state is π_2-normal and conversely.*

Further, two states ω_1, ω_2 are said to be quasi-equivalent if their GNS representations are quasi-equivalent. These correspond to thermodynamically equivalent states.

2.3 Examples: Quantum spin systems, the CCR and CAR algebras

2.3.1 Quantum spin systems

Let Γ be a countable set. Denote $\Lambda \Subset \Gamma$ the finite sets of Γ and $\mathcal{F}(\Gamma)$ the set of finite subsets. For each $x \in \mathcal{H}$, let $\mathcal{H}_x$ be a finite dimensional Hilbert space, and assume that $\sup_{x\in\Gamma} \dim(\mathcal{H}_x) < \infty$. The Hilbert space of $\Lambda \Subset \Gamma$ is given by $\mathcal{H}_\Lambda := \otimes_{x\in\Lambda}\mathcal{H}_x$. The associated algebra of local observables is

$$\mathcal{A}_\Lambda := \mathcal{L}(\mathcal{H}_\Lambda) \simeq \otimes_{x\in\Gamma}\mathcal{L}(\mathcal{H}_x).$$

Inclusion defines a partial order on $\mathcal{F}(\Gamma)$, which induces the following imbedding:

$$\Lambda \subset \Lambda' \quad \Longrightarrow \quad \mathcal{A}_\Lambda \subset \mathcal{A}_{\Lambda'}$$

where $x \in \mathcal{A}_\Lambda$ is identified with $x \otimes 1_{\Lambda'\setminus\Lambda} \in \mathcal{A}_{\Lambda'}$. Note that $\Lambda \cap \Lambda' = \emptyset$ implies $xy = x \otimes y = yx$ for all $x \in \mathcal{A}_\Lambda, y \in \mathcal{A}_{\Lambda'}$. Finally, the algebra of quasi-local observables is given by

$$\mathcal{A} := \overline{\bigcup_{\Lambda\in\mathcal{F}(\Gamma)} \mathcal{A}_\Lambda}^{\|\cdot\|} \equiv \overline{\mathcal{A}_{\mathrm{loc}}}^{\|\cdot\|}$$

and it is a C*-algebra. Note that $\mathcal{A}$ has an identity. In other words, $\mathcal{A}$ is obtained as a limit of finite-dimensional matrix algebras, which is referred to as a uniformly hyperfine algebra (UHF). From the physical point of view, a finite dimensional Hilbert space is the state space of a physical system with a finite number of degrees of freedom, namely a few-levels atom or a spin. In the latter case, $\mathcal{H}_x = \mathbb{C}^{2S_x+1}$ is the state space of a spin-S, with $S \in 1/2\mathbb{N}$, and it carries the $(2S_x+1)$-dimensional irreducible representation of the quantum mechanics rotation group $SU(2)$. A UHF is therefore the algebra of observables of atoms in an optical lattice or of magnetic moments of nuclei in a crystal.

A state ω on $\mathcal{A}$ has the property that it is generated by a a family of density matrices defined by: if $x \in \mathcal{A}_\Lambda$, then $\omega(x) = \mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho^\omega_\Lambda x)$. Such a state is called locally normal. We have:

Proposition 15. *If ω is a state of a quantum spin system $\mathcal{A}$, then the density matrices ρ^ω_Λ obey*

1. *$\rho^\omega_\Lambda \in \mathcal{A}_\Lambda$, $\rho^\omega_\Lambda \geq 0$ and $\mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho^\omega_\Lambda) = 1$*
2. *the consistency condition $\Lambda \subset \Lambda'$ and $x \in \mathcal{A}_\Lambda$, then $\mathrm{Tr}_{\mathcal{H}_\Lambda}(\rho^\omega_\Lambda x) = \mathrm{Tr}_{\mathcal{H}_{\Lambda'}}(\rho^\omega_{\Lambda'} x)$*

Conversely, given a family $\{\rho_\Lambda\}_{\Lambda\in\mathcal{F}(\Gamma)}$ satisfying $(1,2)$, there is a unique state ω^ρ on $\mathcal{A}$.

Proof. Since $\mathcal{A}_\Lambda$ is a finite dimensional matrix algebra, the restriction of ω to $\mathcal{A}_\Lambda$ is given by a density matrix satisfying (1). (2) follows from the identification $\mathcal{A}_\Lambda \simeq \mathcal{A}_\Lambda \otimes 1_{\Lambda'\setminus\Lambda}$. Conversely, a family of ρ_Λ defines is a bounded linear functional on the dense subalgebra $\mathcal{A}_{\mathrm{loc}}$. Hence it extends uniquely to a linear functional on $\mathcal{A}$ with the same bound. □

Theorem 16. *Let ω_1, ω_2 be two pure states of a quantum spin system. Then ω_1 and ω_2 are equivalent if and only if for all $\epsilon > 0$, there is $\Lambda \Subset \Gamma$ such that*

$$|\omega_1(x) - \omega_2(x)| \leq \epsilon \|x\|,$$

for all $x \in \mathcal{A}_{\Lambda'}$ with $\Lambda \cap \Lambda' = \emptyset$.

In other words, two pure states of a quantum spin system are equivalent if and only if they are 'equal at infinity', namely thermodynamically equal. More generally, the theorem holds – with quasi-equivalence – for any two factor states. Note that if a state is pure, then it is irreducible, i.e. $\pi(A)' = \mathbb{C}\cdot 1$, so that $\pi(A)'' = \mathcal{L}(\mathcal{H})$ and $\pi(A)' \cap \pi(A)'' = \mathbb{C}\cdot 1$, hence ω is a factor state.

In practice, one is given a family of vectors Ψ^i_Λ, $i = I_\Lambda$ an index set, typically the set of thermal/ground states of a finite volume Hamiltonian H_Λ on $\mathcal{H}_\Lambda$. All states $\omega^i_\Lambda := \langle \Psi^i_\Lambda, \cdot\, \Psi^i_\Lambda\rangle$ on $\mathcal{A}_\Lambda$ can be extended to a state on $\mathcal{A}$ (by Hahn-Banach), that we still denote ω^i_Λ. The set $\mathcal{S} := \{\omega^i_\Lambda : \Lambda \Subset \Gamma, i \in I_\Lambda\}$ is a subset of $\mathcal{E}(\mathcal{A})$, which is weakly-* compact, hence there are weak-* accumulation points, denoted ω^i_Γ, $i \in I_\Gamma$. These are usually taken as the thermodynamic thermal/ground states of the quantum spin system.

Finally, let $\Gamma = \mathbb{Z}^d$. There is a natural notion of translations on $\mathcal{A}$ which defines a group of automorphisms $\mathbb{Z}^d \ni z \mapsto \tau_z$: If $\Lambda \Subset \Gamma$ and $x \in \mathcal{A}_\Lambda$, $\tau_z(x)$ is the same observable on $\Lambda + z$. This defines an automorphism on the dense subalgebra $\mathcal{A}_{\mathrm{loc}}$, which can be extended by continuity to τ_z on all of $\mathcal{A}$. If a state is translation invariant, $\omega \circ \tau_z = \omega$ for all $z \in \mathbb{Z}$, then τ_z is unitarily implementable in the GNS representation, namely there is $\mathbb{Z}^d \ni z \mapsto U(z)$, where $U(z)$ are unitary operators on $\mathcal{H}$ with $U(z)\Omega = \Omega$, such that $\pi(\tau_z(x)) = U(z)^*\pi(x)U(z)$ for all $z \in \mathbb{Z}^d$ and $x \in \mathcal{A}$. Furthermore, $\tau_{z_1+z_2} = \tau_{z_1} \circ \tau_{z_1}$ implies $U(z_1+z_2) = U(z_1)U(z_2)$.

The following proposition is usually referred to as the asymptotic abelianness of $\mathcal{A}$

Proposition 17. *Let $\mathcal{A}$ and $z \mapsto \tau_z$ be as above. Then for each $x, y \in \mathcal{A}$,*

$$\lim_{|z|\to\infty} [\tau_z(x), y] = 0.$$

Proof. Exercise. □

2.3.2 Fermions: the CAR algebra

The algebra of canonical anticommutation relations (CAR) is the algebra of creation and annihilation operators of fermions

Definition 8. *Let $\mathcal{D}$ be a prehilbert space. The* CAR algebra $\mathcal{A}_+(\mathcal{D})$ *is the C^*-algebra generated by* 1 *and elements $a(f)$, $f \in \mathcal{D}$ satisfying*

$$\begin{aligned} &f \longmapsto a(f) \text{ is antilinear} \\ &\{a(f), a(g)\} = 0, \quad \{a(f)^*, a(g)^*\} = 0 \\ &\{a(f)^*, a(g)\} = \langle g, f\rangle 1 \end{aligned}$$

for all $f, g \in \mathcal{D}$.

It follows from the CAR relations that $(a(f)^*a(f))^2 = a(f)^*\{a(f), a(f)^*\}a(f) = \|f\|^2 a(f)^*a(f)$, and the C*-property then implies $\|a(f)\| = \|f\|$ so that $f \mapsto a(f)$ is a continuous map.

Proposition 18. *Let $\mathcal{D}$ be a prehilbert space with closure $\overline{\mathcal{D}} = \mathcal{H}$. Then*

1. $\mathcal{A}_+(\mathcal{D}) = \mathcal{A}_+(\mathcal{H})$
2. *$\mathcal{A}_+(\mathcal{D})$ is unique: If $\mathcal{A}_1, \mathcal{A}_2$ both satisfy the above definition, then there exists a unique *-isomorphism $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $a_2(f) = \gamma(a_1(f))$ for all $f \in \mathcal{D}$*
3. *If L is a bounded linear operator in $\mathcal{H}$ and A a bounded antilinear operator in $\mathcal{H}$ satisfying*[2]

$$\begin{aligned} L^*L + A^*A = LL^* + AA^* = 1, \\ LA^* + AL^* = L^*A + A^*L = 0, \end{aligned}$$

*there is a unique *-automorphism $\gamma_{L,A}$ of $\mathcal{A}_+(\mathcal{H})$ such that $\gamma_{L,A}(a(f)) = a(Lf) + a(Af)^*$.*

Proof. Since $\mathcal{D}$ is a subset of $\mathcal{H}$, we have that $\mathcal{A}_+(\mathcal{D}) \subset \mathcal{A}_+(\mathcal{H})$. Moreover, if $f \in \mathcal{H}$, there is a sequence $f_n \in \mathcal{D}$ such that $f_n \to f$. By linearity and continuity, $\|a(f) - a(f_n)\| = \|a(f - f_n)\| = \|f - f_n\| \to 0$, showing that $a(f) \in \mathcal{A}_+(\mathcal{D})$, and $\mathcal{A}_+(\mathcal{H}) \subset \mathcal{A}(\mathcal{D})$, proving (1).

Assume now that $\dim\mathcal{H} < \infty$, and that $\{f_i\}_{i=1}^n$ is an orthonormal basis. Then the map $\mathcal{I} : \mathcal{A}_+(\mathcal{H}) \to \mathcal{M}_2^{\otimes n}$ defined by

$$\begin{aligned} \mathcal{I}(a(f_k)a(f_k)^*) = e_{11}^k \quad \mathcal{I}(V_{k-1}a(f_k)) = e_{12}^k \\ \mathcal{I}(V_{k-1}a(f_k)^*) = e_{21}^k \quad \mathcal{I}(a(f_k)^*a(f_k)) = e_{22}^k \end{aligned}$$

where e_{ij}^k is the canonical basis matrix in $\mathcal{M}_2^{\otimes n}$ which is non-trivial on the k-th factor, and $V_k = \prod_{i=1}^k (1 - 2a(f_i)^*a(f_i))$, is an algebra isomorphism. In particular, the CAR imply that $e_{ij}^k e_{ab}^k = \delta_{ja} e_{ib}^k$ and $[e_{ij}^k, e_{ab}^l] = 0$ if $k \neq l$ as it should. Furthermore, it is invertible with inverse

$$a(f_k) = \mathcal{I}^{-1}\left(\prod_{i=1}^{k-1}(e_{11}^i - e_{22}^i)e_{12}^k\right).$$

This proves (2) for the finite dimensional case. If $\mathcal{H}$ is infinite dimensional, there is a basis $\{f_\alpha\}_{\alpha\in A}$ of $\mathcal{H}$, not necessarily countable, and the above construction can be made with any finite subset of A. We conclude in this case by (1) since the vector space of finite linear combinations of f_α is dense in $\mathcal{H}$.

[2]By definition, $\langle f, Ag\rangle = \langle g, A^*f\rangle$ for an antilinear operator

Finally,

$$\{a(Lf) + a(Af)^*, a(Lg)^* + a(Ag)\} = \langle Lf, Lg\rangle + \langle Ag, Af\rangle = \langle f, g\rangle,$$

and similar computations for other anticommutators show that 1 and $a(Lf) + a(Af)^*$ for all $f \in \mathcal{H}$ also generate $\mathcal{A}_+(\mathcal{H})$, conluding the proof by (2). □

Note that the proof of (2) shows that the CAR algebra is a UHF algebra.

The transformation $\gamma_{L,A}$ is called a Bogoliubov transformation. Its unitary implementability in a given representation is a separate question, which can be completely answered in the case of so-called quasi-free representations. A particularly simple case is given by $A = 0$ and a unitary L, corresponding to the non-interacting evolution of single particles under L.

2.3.3 Bosons: the CCR algebra

The algebra of canonical commutation relations (CCR) is the algebra of creation and annihilation operators of bosons. Being unbounded operators, they do not form a C*-algebra, but their exponentials do so and it is usually referred to, in this form, as the Weyl algebra.

Definition 9. *Let $\mathcal{D}$ be a prehilbert space. The* Weyl algebra $\mathcal{A}_-(\mathcal{D})$ *is the C*-algebra generated by $W(f)$, $f \in \mathcal{D}$ satisfying*

$$W(-f) = W(f)^*$$
$$W(f)W(g) = \exp\left(-\frac{\mathrm{i}}{2}\mathrm{Im}\langle f, g\rangle\right) W(f+g)$$

for all $f, g \in \mathcal{D}$.

Note the commutation relation $W(f)W(g) = \exp(-\mathrm{i}\mathrm{Im}\langle f, g\rangle)W(g)W(f)$.

Proposition 19. *Let $\mathcal{D}$ be a prehilbert space with closure $\overline{\mathcal{D}} = \mathcal{H}$. Then*

1. *$\mathcal{A}_-(\mathcal{D}) = \mathcal{A}_-(\mathcal{H})$ if and only if $\mathcal{D} = \mathcal{H}$*
2. *$\mathcal{A}_-(\mathcal{D})$ is unique: If $\mathcal{A}_1, \mathcal{A}_2$ both satisfy the above definition, then there exists a unique *-isomorphism $\gamma : \mathcal{A}_1 \to \mathcal{A}_2$ such that $W_2(f) = \gamma(W_1(f))$ for all $f \in \mathcal{D}$*
3. *$W(0) = 1$, $W(f)$ is a unitary element and $\|W(f) - 1\| = 2$ for all $f \in \mathcal{D}$, $f \neq 0$*
4. *If S is a real linear, invertible operator in $\mathcal{D}$ such that $\mathrm{Im}\langle Sf, Sg\rangle = \mathrm{Im}\langle f, g\rangle$, then there is a unique *-automorphism γ_S of $\mathcal{A}_-(\mathcal{D})$ such that $\gamma_S(W(f)) = W(Sf)$.*

In fact, $\mathcal{D}$ only needs to be a real linear vector space equipped with a symplectic form, and S is a symplectic map. This is the natural structure of phase space and its Hamiltonian dynamics in classical mechanics, and the map $f \mapsto W(f)$ is called the Weyl quantisation[3]. (3) shows in particular that it is a discontinuous map. We only prove (3) and (4). The difference between the CAR and CCR algebra with respect to closure of the underlying space is due to the lack of continuity of $f \mapsto W(f)$.

Proof. The definition implies that $W(f)W(0) = W(f) = W(0)W(f)$ so that $W(0) = 1$. Moreover, $W(f)W(-f) = W(-f)W(f) = W(0) = 1$ so that $W(f)$ is unitary. In turn, this implies

$$W(g)W(f)W(g)^* = \exp(\mathrm{i}\mathrm{Im}\langle f, g\rangle)W(f).$$

[3]In fact, it is also an algebra isomorphism between $\mathcal{D} = C^\infty(X)$ equipped with a Poisson bracket and $\mathcal{A}_-(\mathcal{D})$

Hence, the spectrum of $W(f)$ is invariant under arbitrary rotations for any $f \neq 0$, so that $\mathrm{Sp}(W(f)) = S^1$. Hence, $\sup\{|\lambda| : \lambda \in \mathrm{Sp}(W(f) - 1)\} = 2$, which concludes the proof of (3) since $W(f) - 1$ is a normal operator. Finally, (4) follows again from (2) and the invariance of the Weyl relations. □

Definition 10. *A representation* $(\mathcal{H}, \pi)$ *of* $\mathcal{A}_-(\mathcal{D})$ *is* regular *if* $t \mapsto \pi(W(tf))$ *is a strongly continuous map on* $\mathcal{H}$ *for all* $f \in \mathcal{D}$.

In a regular representation, $\mathbb{R} \ni t \mapsto \pi(W(tf))$ is a strongly continuous group of unitaries by the Weyl relations, so that Stone's theorem yields the existence of a densely defined, self-adjoint generator $\Phi_\pi(f)$ such that $\pi(W(tf)) = \exp(it\Phi_\pi(f))$ for all $f \in \mathcal{D}$. In fact, for any finite dimensional subspace $\mathcal{K} \subset \mathcal{D}$ there is a common dense space of analytic vectors of $\{\Phi_\pi(f), \Phi_\pi(\mathrm{i}f), f \in \mathcal{K}\}$, namely for which $\sum_{n=0}^\infty \|\Phi_\pi^n \psi\| t^n/n! < \infty$ for t small enough. The creation and annihilation operators can be defined

$$a_\pi^*(f) := 2^{-1/2}\left(\Phi_\pi(f) - \mathrm{i}\Phi_\pi(\mathrm{i}f)\right), \qquad a_\pi(f) := 2^{-1/2}\left(\Phi_\pi(f) + \mathrm{i}\Phi_\pi(\mathrm{i}f)\right)$$

on $D(a_\pi^*(f)) = D(a_\pi(f)) = D(\Phi_\pi(f)) \cap D(\Phi_\pi(\mathrm{i}f))$, which is dense. Note that $a_\pi^*(f) \subset a_\pi(f)^*$. In fact, equality holds.

By construction $f \mapsto \Phi_\pi(f)$ is real linear, so that $f \mapsto a_\pi(f)$ is antilinear and $f \mapsto a_\pi^*(f)$ is linear. Now, taking the second derivative of the Weyl relations applied on any vector $\xi \in D(\Phi_\pi(f)) \cap D(\Phi_\pi(g))$ at $t = t' = 0$, one obtains $(\Phi_\pi(f)\Phi_\pi(g) - \Phi_\pi(g)\Phi_\pi(f))\,\xi = \mathrm{i}\mathrm{Im}\langle f, g\rangle \xi$, so that

$$(a_\pi(f) a_\pi^*(g) - a_\pi^*(g) a_\pi(f))\,\xi = \langle f, g\rangle \xi$$

the usual form of the canonical commutation relations (CCR). Finally, we prove that the creation/annihilation operators are closed. Indeed, $\|\Phi_\pi(f)\xi\|^2 + \|\Phi_\pi(\mathrm{i}f)\xi\|^2 = \|a_\pi(f)\xi\|^2 + \|a_\pi^*(f)\xi\|^2$, while the commutation relations yield $\|a^*(f)\xi\|^2 - \|a(f)\xi\|^2 = \|f\|^2\|\xi\|^2$. Together, $\|\Phi_\pi(f)\xi\|^2 + \|\Phi_\pi(\mathrm{i}f)\xi\|^2 = 2\|a_\pi(f)\xi\|^2 + \|f\|^2\|\xi\|^2$. Hence, for any sequence $\psi_n \in D(a_\pi(f))$ such that $\psi_n \to \psi$ and $a_\pi(f)\psi_n$ converges, we have that $\Phi_\pi(f)\psi_n, \Phi_\pi(\mathrm{i}f)\psi_n$ converge. Since Φ_π are self-adjoint and hence closed, we have that $\psi \in D(a_\pi(f))$, and $\Phi_\pi(f)\psi_n \to \Phi_\pi(f)\psi$ and $\Phi_\pi(\mathrm{i}f)\psi_n \to \Phi_\pi(\mathrm{i}f)\psi$. By the norm equality again, $a_\pi(f)\psi_n \to a_\pi(f)\psi$ and $a_\pi(f)$ is closed.

2.3.4 Fock spaces and the Fock representation

The set $\mathcal{D}^{\otimes n}$ carries an action Π of the permutation group S_n

$$\Pi_\pi : \psi_1 \otimes \cdots \otimes \psi_n \longmapsto \psi_{\pi^{-1}(1)} \otimes \cdots \otimes \psi_{\pi^{-1}(n)}$$

for any $\pi \in S_n$ and we denote $\mathcal{D}_\pm^{(n)} := \{\Psi^{(n)} \in \mathcal{D}^{\otimes n} : \Pi_\pi \Psi^{(n)} = (\pm 1)^{\mathrm{sgn}\pi}\Psi^{(n)}\}$, namely the symmetric, respectively antisymmetric subspace of $\mathcal{D}^{\otimes n}$. Let also $\mathcal{D}_\pm^{(0)} := \mathbb{C}$. The bosonic, respectively fermionic Fock space over $\mathcal{D}$ is denoted $\mathcal{F}_\pm(\mathcal{D}) := \oplus_{n=0}^\infty \mathcal{D}_\pm^{(n)}$. That is, a vector $\Psi \in \mathcal{F}_\pm(\mathcal{D})$ can be represented as a sequence $(\Psi^{(n)})_{n\in\mathbb{N}}$ such that $\Psi^{(n)} \in \mathcal{D}_\pm^{(n)}$, with $\sum_{n\in\mathbb{N}} \|\Psi^{(n)}\| < \infty$. The vector $\Omega := (1, 0, \ldots)$ is called the vacuum. We further denote $\mathcal{F}_\pm^{\mathrm{fin}}(\mathcal{D}) := \{\Psi \in \mathcal{F}_\pm(\mathcal{D}) : \exists N \in \mathbb{N} \text{ with } \Psi^{(n)} = 0, \forall n \geq N\}$, which is dense. Note that the probability to find more than N particles in any vector Ψ vanishes as $N \to \infty$,

$$P_{\geq N}(\Psi) := \sum_{n \geq N} \|\Psi^{(n)}\|^2 \longrightarrow 0, \qquad (N \to \infty),$$

which we interpret as follows: In Fock space, there is an arbitrarily large but finite number of particles. In particular, there is no vector representing a gas at non-zero density in the thermodynamic limit. We define $N : \mathcal{F}_\pm^{\mathrm{fin}}(\mathcal{D}) \to \mathcal{F}_\pm^{\mathrm{fin}}(\mathcal{D})$ by $N\Psi = n\Psi$ whenever $\Psi \in \mathcal{D}_\pm^{(n)}$.

For $f \in \mathcal{D}$, let $b_\pm(f) : \mathcal{D}^{\otimes n} \to \mathcal{D}^{\otimes n-1}$ be defined by

$$b_\pm(f)(\psi_1, \dots \psi_n) = \sqrt{n}\langle f, \psi_1\rangle(\psi_2, \dots \psi_n),$$

which maps $\mathcal{D}_\pm^{(n)}$ to $\mathcal{D}_\pm^{(n-1)}$, with $b_\pm(f)\mathcal{D}_\pm^{(0)} = 0$, and hence $b_\pm(f) : \mathcal{F}_\pm(\mathcal{D}) \to \mathcal{F}_\pm(\mathcal{D})$. Its adjoint $b_\pm^*(f) := b_\pm(f)^* : \mathcal{D}_\pm^{(n-1)} \to \mathcal{D}_\pm^{(n)}$ such that

$$b_\pm^*(f)\Psi^{(n-1)} = \frac{1}{\sqrt{n}}\sum_{k=1}^n (\pm 1)^{k-1}\Pi_{\pi_k} f \otimes \Psi^{(n-1)}$$

where $\pi_k^{-1} = (k, 1, 2, \dots, k-1, k+1, \dots, n)$. Indeed, the right hand side $\tilde\Psi$ is in $\mathcal{D}_\pm^{(n)}$: $(\pi_{\sigma(k)} \circ \sigma^{-1} \circ \pi_k^{-1})(1) = 1$ and the signature of the permutation is $(k-1) + \mathrm{sgn}(\sigma) + (\sigma(k) - 1)$, so that $\Pi_{\pi_{\sigma(k)}^{-1}}\Pi_\sigma\Pi_{\pi_k}(f \otimes \Psi^{(n-1)}) = (\pm 1)^{\mathrm{sgn}(\sigma)+(\sigma(k)-k)} f \otimes \Psi^{(n-1)}$, which implies that $\Pi_\sigma\tilde\Psi = (\pm 1)^{\mathrm{sgn}(\sigma)}\tilde\Psi$. Moreover, for any $\Upsilon^{(n)} \in \mathcal{D}_\pm^{(n)}$,

$$\langle b_\pm(f)\Upsilon^{(n)}, \Psi^{(n-1)}\rangle = \sqrt{n}\langle \Upsilon^{(n)}, f \otimes \Psi^{(n-1)}\rangle = \frac{1}{\sqrt{n}}\sum_{k=1}^n \langle \Pi_{\pi_k}\Upsilon^{(n)}, \Pi_{\pi_k}(f \otimes \Psi^{(n-1)})\rangle = \langle \Upsilon^{(n)}, \tilde\Psi\rangle$$

where we used that $\Pi_{(\cdot)}$ is unitary, proving that $\tilde\Psi = b_\pm^*(f)\Psi^{(n-1)}$.

Proposition 20. *1. $f \mapsto b_\pm(f)$ is antilinear, $f \mapsto b_\pm^*(f)$ is linear*

2. $Nb_\pm(f) = b_\pm(f)(N-1)$

3. $b_\pm(f), b_\pm^*(g)$ satisfy the canonical commutation, resp. anticommutation relations

Proof. We denote $[A, B]_\pm := AB \mp BA$ and prove $[b_\pm(f), b_\pm^*(g)]_\pm = \langle f, g\rangle$. Indeed, for $\Psi^{(n-1)} = \psi_1 \otimes \cdots \otimes \psi_{n-1}$,

$$\frac{1}{\sqrt{n}} b_\pm(f)\Pi_{\pi_{k+1}}(g \otimes \Psi^{(n-1)}) = \langle f, \psi_1\rangle \psi_2 \otimes \cdots \otimes \psi_k \otimes g \otimes \cdots \psi_{n-1} = \frac{1}{\sqrt{n-1}}\Pi_{\pi_k}(g \otimes b_\pm(f)\Psi^{(n-1)}),$$

so that

$$\begin{aligned} b_\pm(f)b_\pm(g)\Psi^{(n-1)} &= \frac{1}{\sqrt{n}}\sum_{k=1}^n (\pm 1)^{k-1} b_\pm(f)\Pi_{\pi_k}(g \otimes \Psi^{(n-1)}) \\ &= \langle f, g\rangle\Psi^{(n-1)} \pm \frac{1}{\sqrt{n-1}}\sum_{k=1}^{n-1}(\pm 1)^{k-1}\Pi_{\pi_k}(g \otimes b_\pm(f)\Psi^{(n-1)}) \\ &= \langle f, g\rangle\Psi^{(n-1)} \pm b_\pm(g)b_\pm(f)\Psi^{(n-1)}, \end{aligned}$$

where the second equality follows by extracting the first term in the sum and using the observation above in the remaining terms. □

In particular, $\{b_-(f) : f \in \mathcal{D}\}$ form a representation of the CAR algebra. Furthermore, The operators $\Phi_+(f) := 2^{-1/2}(b_+(f) + b_+^*(f))$ are symmetric on $\mathcal{F}_+^{\mathrm{fin}}(\mathcal{D})$ and extend to self-adjoint operators, so that $W_+(f) := \exp(\mathrm{i}\Phi_+(f))$ are well-defined unitary operators on $\mathcal{F}_+(\mathcal{D})$, yielding a representation of the Weyl algebra. They are the fermionic and bosonic Fock representations associated to the Fock state

$$\begin{cases} \omega_F^{\mathrm{CAR}}(a(f)^*a(g)) := \left\langle \Omega, b_-^*(f)b_-(g)\Omega\right\rangle = 0 \text{ and } \omega_F^{\mathrm{CAR}}(a(f)) := 0 & \text{(fermions)} \\ \omega_F^{\mathrm{CCR}}(W(f)) := \langle \Omega, W_+(f)\Omega\rangle = \mathrm{e}^{-\|f\|^2/4} & \text{(bosons)} \end{cases}$$

In other words, Fock spaces are the GNS Hilbert spaces for the Fock states.

Quantum mechanics in one dimension for one particle is usually associated with the Schrödinger representation, defined on the Hilbert space $L^2(\mathbb{R})$. It arises as the regular representation of the Weyl algebra $\mathcal{A}_-(\mathbb{C})$ given by

$$\pi_S(W(s+\mathrm{i}t)) := \mathrm{e}^{\frac{\mathrm{i}}{2}st}U(s)V(t),$$

where

$$(U(s)\psi)(x) = \mathrm{e}^{\mathrm{i}st}\psi(x), \qquad (V(t)\psi)(x) = \psi(x+t),$$

with self-adjoint generators $X := \Phi_S(1)$ and $P := \Phi_S(\mathrm{i}) = -\mathrm{i}\partial_x$.

In fact, $L^2(\mathbb{R})$ carries a Fock space structure, obtained by introducing $a_S := 2^{-1/2}(X + \mathrm{i}P)$ and $a_S^* := 2^{-1/2}(X - \mathrm{i}P)$, which satisfy the CCR (strongly on a dense set such as $C_c^\infty(\mathbb{R})$. The vacuum vector Ω_S is the L^2-normalised solution of $a_S\Omega_S = 0$, namely

$$(x + \partial_x)\Omega_S(x) = 0, \qquad \text{i.e.} \qquad \Omega_S(x) = \pi^{-1/4}\mathrm{e}^{-x^2/2}.$$

With $\mathcal{H}^n := \mathrm{span}\{(a_S^*)^n\Omega_S\}$, namely the span of the nth Hermite function, one obtains $L^2(\mathbb{R}) = \oplus_{n=0}^\infty \mathcal{H}^n$. In other words, $L^2(\mathbb{R}) \simeq \mathcal{F}_+(\mathbb{C})$ and the Schrödinger and Fock representations are equivalent, the unitary map being $(a_S^*)^n\Omega_S \mapsto (b_+^*)^n\Omega$.

Hence, the dimension of $\mathcal{D}$ has the interpretation of 'the number of degrees of freedom' of the system and N-body quantum mechanics in $\mathbb{R}^d$ corresponds to the algebra $\mathcal{A}_-(\mathbb{C}^{Nd})$, which has a Schrödinger representation, namely the (Nd)-fold tensor product representation of that given above. In fact, this is the only one:

Theorem 21. *Let $\mathcal{H}$ be a finite dimensional Hilbert space,* $\dim\mathcal{H} = n$. *Then, any irreducible representation of $\mathcal{A}_-(\mathcal{H})$ is equivalent to the Schrödinger representation.*

In other words, the algebraic machinery is useless in quantum mechanics. Whenever $\dim\mathcal{H} = \infty$, typically $\mathcal{H} = L^2(\mathbb{R}^d)$ itself, there are truly inequivalent representations: these are in particular those arising in quantum statistical mechanics.

Chapter 3

Equilibrium: KMS states

In a quantum system with Hamiltonian H such that $\mathrm{Tr}\exp(-\beta H)$ is finite for some $\beta > 0$, the Gibbsian rule is as follows: The system in thermal equilibrium is in a state given by a density matrix ρ_β on $\mathcal{H}$,

$$\rho_\beta = Z(\beta)^{-1}\mathrm{e}^{-\beta H}, \qquad Z(\beta) := \mathrm{Tr}\mathrm{e}^{-\beta H}.$$

Among its many properties, we concentrate on an a priori rather coincidental properties. Let ω_β denote the state associated to the density matrix ρ_β, and $\tau_t(A) = \exp(\mathrm{i}tH)A\exp(-\mathrm{i}tH)$. Consider the function $F_\beta(A,B;t) = Z(\beta)^{-1}\mathrm{Tr}(\exp(-\mathrm{i}(t-\mathrm{i}\beta)H)A\exp(\mathrm{i}tH)B)$. Using the cyclicity of the trace, $F_\beta(A,B;t) = \omega_\beta(A\tau_t(B))$. On the other hand, $F_\beta(A,B;t)$ can be analytically continued into the complex plane to $t+\mathrm{i}\beta$ to give $F_\beta(A,B;t+\mathrm{i}\beta) = Z(\beta)^{-1}\mathrm{Tr}(\exp(-\mathrm{i}tH)A\exp(\mathrm{i}(t+\mathrm{i}\beta)H)B) = \omega_\beta(\tau_t(B)A)$. Hence, there is an analytic function $F_\beta(A,B;z)$ defined on the strip $\{z\in\mathbb{C} : 0 \leq \mathrm{Im}z \leq \beta\}$ with boundary values

$$F_\beta(A,B;t) = \omega_\beta(A\tau_t(B)), \qquad F_\beta(A,B;t+\mathrm{i}\beta) = \omega_\beta(\tau_t(B)A). \tag{3.1}$$

This turns our to be the property that extends naturally to the algebraic setting.

3.1 Definition

It will be useful to first introduce some terminology.

Definition 11. *A pair $(\mathcal{A},\tau_t)$ is a* C*-dynamical system *if $\mathcal{A}$ is a C*-algebra with an identity and $\mathbb{R}\ni t\mapsto\tau_t$ is a strongly continuous group of *-automorphisms of $\mathcal{A}$, namely*

$$\|\tau_{t+\epsilon}(A) - \tau_t(A)\| \to 0 \qquad (\epsilon\to 0)$$

for all $A\in\mathcal{A}$.

It follows from the strong continuity that τ_t is generated by a *-derivation, $\tau_t(A) = \mathrm{e}^{t\delta}A$:

Proposition 22. *Let $\delta_t : \mathcal{A}\to\mathcal{A}, A\mapsto\delta_t(A) = t^{-1}(\tau_t(A) - A)$, let*

$$D(\delta) := \{A\in\mathcal{A} : \lim_{t\to 0^+}\delta_t(A) \text{ exists}\},$$

and define

$$\begin{aligned}\delta : D(\delta) &\to \mathcal{A}\\ A &\mapsto \delta(A) = \lim_{t\to 0^+} t^{-1}(\tau_t(A) - A).\end{aligned}$$

Then, δ is a closed, densely defined map such that

$$\begin{aligned}
&1 \in D(\delta) \text{ and } \delta(1) = 0,\\
&\delta(AB) = \delta(A)B + A\delta(B),\\
&\delta(A^*) = \delta(A)^*.
\end{aligned}$$

In fact, just as there is a one-to-one correspondence between self-adjoint generators and strongly continuous unitary groups on a Hilbert space, there a correspondence between *-derivations and strongly continuous groups of *-automorphisms on a C*-algebra. This is Hille-Yosida's theorem. Recall that A is an analytic element for a derivation δ if $A \in D(\delta^n)$ for all $n \in \mathbb{N}$ and $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\delta^n A\| < \infty$, for $0 \leq t < t_A$.

Theorem 23. *Let $\mathcal{A}$ be a C*-algebra with a unit. A densely defined, closed operator δ on $\mathcal{A}$ generates a strongly continuous groups of *-automorphisms if and only if*

$$\begin{aligned}
&\delta \text{ is a *-derivation}\\
&\delta \text{ has a dense set of analytic elements}\\
&\|A + \lambda\delta(A)\| \geq \|A\|,\ \forall \lambda \in \mathbb{R}, A \in D(\delta).
\end{aligned}$$

In the case of quantum mechanics with a finite number of degrees of freedom, $\tau_t(A) = \exp(\mathrm{i}tH)A\exp(-\mathrm{i}tH)$ is strongly continuous if and only if H is bounded, in which case it is also norm continuous (see exercises). The associated derivation $\delta := \mathrm{i}[H, \cdot]$ is bounded and everywhere defined. In fact, as a consequence of the closed graph theorem, an everywhere defined derivation necessarily generates a norm-continuous *-automorphism.

Definition 12. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system. A state ω on $\mathcal{A}$ is a (τ, β)-*KMS state *for $\beta > 0$ if, for any $A, B \in \mathcal{A}$, there exists a function $F_\beta(A, B, z)$, analytic in $S_\beta := \{z \in \mathbb{C} : 0 < \mathrm{Im} z < \beta\}$, continuous on $\overline{S_\beta}$, and satisfying the KMS boundary condition (3.1).*

We shall say that A is an analytic element for τ_t if the map $A \mapsto \tau_t(A)$ extends to an analytic function on $\mathbb{C}$.

Theorem 24. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system. A state ω on $\mathcal{A}$ is a (τ, β)-*KMS state *if and only if there exists a dense, τ-invariant *-subalgebra $\mathcal{D}$ of analytic elements for τ_t such that*

$$\omega(BA) = \omega(A\tau_{\mathrm{i}\beta}(B)). \tag{3.2}$$

The following proposition shows that the condition of analyticity is never a true restriction.

Proposition 25. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system, and let δ be its generator. For any $A \in \mathcal{A}$ and $m \in \mathbb{N}$, let*

$$A_m := \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \tau_t(A) \mathrm{e}^{-mt^2} dt.$$

Then

1. *A_m is analytic for τ_t*
2. *A_m is analytic for δ*
3. *The *-subalgebra $\mathcal{A}_\tau := \{A_m : A \in \mathcal{A}, m \in \mathbb{N}\}$ is dense*

Proof. First of all, $\|\tau_t(A)\| \exp(-mt^2) = \|A\| \exp(-mt^2) \in L^1(\mathbb{R})$, so that A_m is well-defined, $A_m \in \mathcal{A}$ and $\|A_m\| \leq \|A\|$. Moreover,

$$\tau_s(A_m) = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \tau_t(A) \mathrm{e}^{-m(t-s)^2} dt.$$

The right-hand-side extends to an analytic function, with $\|\text{r.h.s.}\| \leq \|A\| \exp(-m(\text{Im}z)^2)$, which can be used to extend $\tau_s(A_m)$ to $\tau_z(A_m)$ for all $z \in \mathbb{C}$. Moreover, by dominated convergence,

$$\frac{d^n}{ds^n}\tau_s(A_m)\Big|_{s=0} = \sqrt{\frac{m}{\pi}} \int_{\mathbb{R}} \tau_t(A) \frac{d^n}{ds^n} \mathrm{e}^{-m(t-s)^2}\Big|_{s=0} dt = \sqrt{\frac{m^{1+n}}{\pi}} \int_{\mathbb{R}} \tau_t(A) H_n(t) \mathrm{e}^{-mt^2} dt$$

were H_n are the Hermite polynomials, so that $A_m \in D(\delta^n)$ for all $n \in \mathbb{N}$, and A_m is analytic for δ. Finally,

$$A_n - A = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} \left(\tau_{t/\sqrt{n}}(A) - A\right) \mathrm{e}^{-t^2} dt \to 0 \qquad (n \to \infty)$$

by the strong continuity of τ_t and dominated convergence. □

Proof of Theorem 24. Necessity. Let $A \in \mathcal{A}, B \in \mathcal{A}_\tau$ and ω be a (τ, β)-KMS state. Then $z \mapsto G(z) = \omega(A\tau_z(B))$ is analytic and $G(t) = F_\beta(A, B; t)$ for $t \in \mathbb{R}$. Hence $z \mapsto G(z) - F_\beta(A, B; z)$ is analytic on S_β, continuous on $S_\beta \cup \mathbb{R}$ and vanishes on $\mathbb{R}$. By the Schwarz reflection principle, it extends to an analytic function on the double strip $S_\beta \cup S_{-\beta}$ that vanishes on $\mathbb{R}$. Hence it equals zero everywhere, and by continuity also on $\overline{S_\beta}$, that is $F_\beta(A, B; z) = \omega(A\tau_z(B))$ for all $z \in \overline{S_\beta}$. In particular, setting $z = \mathrm{i}\beta$ yields $\omega_\beta(BA) = \omega_\beta(A\tau_{\mathrm{i}\beta}(B))$.
Sufficiency. First, for $A, B \in \mathcal{D}$, $z \mapsto F(A, B; z) := \omega(A\tau_z(B))$ is analytic on $\mathbb{C}$. Since $\tau_t(B) \in \mathcal{D}$,

$$F(A, B; t) = \omega(A\tau_t(B)), \qquad F(A, B; t + \mathrm{i}\beta) = \omega(A\tau_{\mathrm{i}\beta}(\tau_t(B))) = \omega(\tau_t(B)A),$$

by (3.2). Now, $|\omega(A\tau_z(B))| \leq \|A\|\|\tau^{\mathrm{iIm}z}(B)\|$ so that $F(A, B; z)$ is bounded on S_β and Hadamard's three lines theorem yields $\sup_{z\in\overline{S_\beta}} F(A, B; z) \leq \|A\|\|B\|$. For arbitrary $A, B \in \mathcal{A}$, let $A_n \to A, B_n \to B$, with $A_n, B_n \in \mathcal{D}$. Since

$$F(A_n, B_n; z) - F(A_m, B_m; z) = F(A_n - A_m, B_n; z) + F(A_m, B_n - B_m; z),$$

so that $F(A_n, B_n; z)$ is uniformly Cauchy in $\overline{S_\beta}$. Its limit is therefore analytic on S_β and continuous on its closure, and it still satisfies the KMS boundary condition. □

Clearly, the Gibbs state on a finite dimensional Hilbert space satisfies the KMS condition. As we shall see later, it is also the unique KMS state in this case[1].

The following theorem shows that a KMS state passes the simplest test for an equilibrium state: it is invariant under time evolution. Mathematically, this is also useful as it implies the unitary implementability of the dynamics in the GNS representation.

Proposition 26. *Let $(\mathcal{A}, \tau_t)$ be a C^*-dynamical system and let ω be a (τ, β)-KMS state. Then $\omega \circ \tau_t = \omega$ for all $t \in \mathbb{R}$.*

Proof. Let $A \in \mathcal{A}_\tau$. The function $z \mapsto g(z) = \omega(\tau_z(A))$ is analytic. By Theorem 24,

$$g(z + \mathrm{i}\beta) = \omega(1\tau_{\mathrm{i}\beta}(\tau_z(A))) = \omega(\tau_z(A)1) = g(z).$$

Hence, g is a periodic function along the imaginary axis, and moreover, $|g(t+\mathrm{i}\alpha)| \leq \|\tau_{t+\mathrm{i}\alpha}(A)\| = \|\tau_{\mathrm{i}\alpha}(A)\| \leq \sup_{0\leq\gamma\leq\beta} \|\tau_{\mathrm{i}\gamma}(A)\|$, which is finite. Hence, g is analytic and bounded on $\mathbb{C}$, so that it is constant by Liouville's theorem. This extends to all observables by continuity. □

[1] This shows once again that there cannot be a phase transition for quantum spin systems in finite volume.

3.2 The energy-entropy balance inequality

Just as the Gibbs state is characterised by the variational principle as being a minimiser of the free energy, general KMS states are equivalently defined by satisfying the *energy-entropy balance inequality* (EEB). In this section, $(\mathcal{A}, \tau_t)$ is a C*-dynamical system, and δ is the generator of τ_t. We start with a simple observation.

Lemma 27. *If a state ω over $\mathcal{A}$ is such that $-\mathrm{i}\omega(A^*\delta(A)) \in \mathbb{R}$ for all $A \in D(\delta)$, then $\omega \circ \tau_t = \omega$.*

Proof. Since $\omega(B^*\delta(B))$ is purely imaginary, and $\delta(B^*) = \delta(B)^*$, $\omega(\delta(B^*B)) = \omega(B^*\delta(B)) + \overline{\omega(B^*\delta(B))} = 0$. Hence, with the continuity of ω,

$$\omega(\tau_t(A^*A)) - \omega(A^*A) = \int_0^t \omega(\delta(\tau_s(A^*A)))ds = 0.$$

Hence, the statement holds for all positive elements of $\mathcal{A}$, and further extends to all of $\mathcal{A}$ by noting that any observable is a linear combination of four positive elements. □

Let f be the Fourier transform of $\check{f} \in C_c^\infty(\mathbb{R})$. By Paley-Wiener's theorem, f is analytic in $\mathbb{C}$ and $|f(z)| \leq C_n(1 + |z|^n)\exp(R|\mathrm{Im}(z)|)$ for all $n \in \mathbb{N}$. Let

$$\tau_f(A) := \int_{\mathbb{R}} f(t)\tau_t(A)dt \in D(\delta)$$

since it is analytic for δ. Let ω be a τ_t-invariant state and $H = \int_{\mathbb{R}} \lambda dP(\lambda)$ is the GNS Hamiltonian satisfying $H\Omega = 0$. We have

$$\omega(A^*\tau_f(A)) = \int_{\mathbb{R}} f(t)\langle\pi(A)\Omega, \mathrm{e}^{\mathrm{i}tH}\pi(A)\Omega\rangle dt = \int_{\mathbb{R}} \check{f}(\lambda)d\mu_A(\lambda) \tag{3.3}$$

where $d\mu_A(\lambda) = \langle\pi(A)\Omega, dP(\lambda)\pi(A)\Omega\rangle$ is the spectral measure associated with $\pi(A)\Omega$. Similary, $\omega(\tau_f(A)A^*) = \int_{\mathbb{R}} \check{f}(\lambda)d\nu_A(\lambda)$ where $d\nu_A(\lambda) = \langle\pi(A^*)\Omega, dP(-\lambda)\pi(A^*)\Omega\rangle$. Moreover, the analyticity of $z \mapsto f(z)\omega(A^*\tau_z(A))$ and the KMS condition yield

$$\omega(A^*\tau_f(A)) = \int_{\mathbb{R}} f(t + \mathrm{i}\beta)\omega(\tau_t(A)A^*)dt.$$

The right hand side is also equal to $\int_{\mathbb{R}} \check{f}(\lambda)\exp(\beta\lambda)d\nu_A(\lambda)$. Since this and (3.3) hold for any test function $\check{f}$, we obtain

$$\frac{d\mu_A}{d\nu_A}(\lambda) = \mathrm{e}^{\beta\lambda}. \tag{3.4}$$

Theorem 28. *A state ω over $\mathcal{A}$ is a (τ, β)-KMS state if and only if*

$$-\mathrm{i}\beta\omega(A^*\delta(A)) \geq \omega(A^*A)\ln\frac{\omega(A^*A)}{\omega(AA^*)}$$

for all $A \in D(\delta)$.

Proof. We only prove ⇒. First observe that $\omega(A^*\delta(A)) = \mathrm{i}\langle\pi(A)\Omega, H\pi(A)\Omega\rangle$ and

$$\beta\frac{\langle\pi(A)\Omega, H\pi(A)\Omega\rangle}{\langle\pi(A)\Omega, \pi(A)\Omega\rangle} = \frac{\int_{\mathbb{R}} \beta\lambda d\mu_A(\lambda)}{\int_{\mathbb{R}} d\mu_A(\lambda)}.$$

By Jensen's inequality,

$$\exp\left(-\frac{\int_{\mathbb{R}} \beta\lambda d\mu_A(\lambda)}{\int_{\mathbb{R}} d\mu_A(\lambda)}\right) \leq \frac{\int_{\mathbb{R}} \exp(-\beta\lambda)d\mu_A(\lambda)}{\int_{\mathbb{R}} d\mu_A(\lambda)} = \frac{\int_{\mathbb{R}} d\nu_A(\lambda)}{\int_{\mathbb{R}} d\mu_A(\lambda)} = \frac{\omega(AA^*)}{\omega(A^*A)}$$

if ω is a (τ, β)-KMS state by (3.4). Hence, $\exp(\mathrm{i}\beta\omega(A^*\delta(A))/\omega(A^*A)) \leq \omega(AA^*)/\omega(A^*A)$. □

Corollary 29. *Let $\mathcal{A}$ be a C^*-algebra with a unit and $\{\tau^n\}_{n\in\mathbb{N}}$ be a sequence of strongly continuous one-parameter groups of automorphisms of $\mathcal{A}$ such that*

$$\tau_t^n(A) \to \tau_t(A) \qquad (n \to \infty)$$

for all $A \in \mathcal{A}, t \in \mathbb{R}$, where τ_t is a strongly continuous one-parameter group of automorphisms of $\mathcal{A}$. If $\{\omega_n\}_{n\in\mathbb{N}}$ is a sequence of (τ^n, β)-KMS states, then any weak- limit point of $\{\omega_n\}$ is a (τ, β)-KMS state.*

Proof. See exercises. □

Simple example. $\tau^n = \tau^{\Lambda_n}$ the dynamics of a quantum spin system in a finite volume Λ_n, such that $\Lambda_n \to \Gamma$ as $n \to \infty$[2], generated by a Hamiltonian H_{Λ_n}. The unique KMS state is the Gibbs state with density matrix $Z_n(\beta)^{-1}\exp(-\beta H_{\Lambda_n})$. If the infinite volume dynamics exists, $\tau^{\Lambda_n}(A) \to \tau^\Gamma(A)$, then the limiting thermodynamic states are (τ^Γ, β)-KMS states.

3.3 Passivity and stability

Definition 13. *Let $(\mathcal{A}, \tau_t)$ be a C^*-dynamical system. A state ω on $\mathcal{A}$ is a* passive state *if $-\mathrm{i}\omega(U^*\delta(U)) \geq 0$ for any $U \in \mathcal{U}_0(\mathcal{A}) \cap D(\delta)$. Here, $\mathcal{U}_0(\mathcal{A})$ is the connected component of the identity in the set of all unitary elements of $\mathcal{A}$.*

Proposition 30. *If ω is a (τ, β)-KMS state, then ω is passive.*

Proof. Choose $A = U \in \mathcal{U}_0(\mathcal{A}) \cap D(\delta)$ in the EEB inequality. □

In order to have equivalence in the proposition above, one needs to require *complete passivity*, namely that $\otimes_{i=1}^N \omega$ is passive as a state on the tensored system $(\otimes_{i=1}^N \mathcal{A}, \otimes_{i=1}^N \tau_t)$ for all $N \in \mathbb{N}$.

Interpretation in the case $\dim(\mathcal{H}) < \infty$, where $\omega(A) = \mathrm{Tr}(\rho_\beta A)$ where $\rho_\beta = Z(\beta)^{-1}\exp(-\beta H)$ with $Z(\beta) = \mathrm{Tr}\exp(-\beta H)$, the Gibbs state. Consider a time dependent Hamiltonian $H(t) = H(t)^*, t \in [0, T]$ such that $H(0) = H(T) = H$, an let U be the associated unitary evolution on $[0, T]$. The change in energy between $t = 0$ and $t = T$ is given by

$$W_\beta := \mathrm{Tr}(U\rho_\beta U^* H) - \mathrm{Tr}(\rho_\beta H) = \mathrm{Tr}(\rho_\beta U^*[H, U]) = -\mathrm{i}\omega_\beta(U^*\delta(U)) \geq 0$$

since the KMS state is passive. Passivity expresses a basic thermodynamic fact: the total work done by the system on the environment in an arbitrary cyclic process, $-W_\beta$, is non-positive on average.

Definition 14. *Let $(\mathcal{A}, \tau_t)$ be a C^*-dynamical system with generator δ^0. A* local perturbation *δ^V of δ^0 is given by*

$$\delta^V = \delta^0 + \mathrm{i}[V, \cdot], \qquad D(\delta^V) = D(\delta^0),$$

for a $V = V^* \in \mathcal{A}$.

Using Thm 23, one can show that δ^V generates a strongly continuous group of automorphisms τ^V. Since $\frac{d}{ds}\tau_{-s}^0(\tau_s^V(A)) = \tau_{-s}^0(\mathrm{i}[V, \tau_s^V(A)])$, we have Duhamel's formula

$$\tau_t^V(A) = \tau_t^0(A) + \int_0^t \tau_{t-s}^0(\mathrm{i}[V, \tau_s^V(A)])ds,$$

[2] $\Lambda_n \subset \Lambda_m$ if $n \leq m$, and $\forall x \in \Gamma$, $\exists n_0 \in \mathbb{N}$ such that $x \in \Lambda_n$ $\forall n \geq n_0$

which can be solved iteratively yielding Dyson's expansion

$$\tau_t^V(A) = \tau_t^0(A) + \sum_{k=1}^{\infty} \int_{0 \le t_1 \le \cdots \le t_k \le t} \mathrm{i}[\tau_{t_1}^0(V), \mathrm{i}[\tau_{t_2}^0(V), \cdots \mathrm{i}[\tau_{t_k}^0(V), \tau_t^0(A)] \cdots]] dt_1 \cdots dt_k. \quad (3.5)$$

In fact, writing λV for $\lambda \in \mathbb{C}$, the series is norm convergent for all $\lambda \in \mathbb{C}, t \in \mathbb{R}, A \in \mathcal{A}$ and defines an analytic function $\lambda \mapsto \tau_t^{\lambda V}(A)$.

The unitary element solving

$$-\mathrm{i}\partial_t \Gamma_t^V = \Gamma_t^V \tau_t^0(V), \qquad \Gamma_0^V = 1,$$

has the following intertwining property $\tau_t^V(A)\Gamma_t^V = \Gamma_t^V \tau_t^0(A)$. Solving the differential equation iteratively again yields the expansion

$$\Gamma_t^V = 1 + \sum_{k=1}^{\infty} \mathrm{i}^k \int_{0 \le t_1 \le \cdots \le t_k \le t} \tau_{t_1}^0(V) \cdots \tau_{t_k}^0(V) dt_1 \cdots dt_k.$$

In fact, all above results continue to hold for a time dependent perturbation V_t. A cyclic perturbation of a C*-dynamical system is a norm-differentiable family $[0,T] \ni t \mapsto V_t = V_t^* \in \mathcal{A}$ such that $V_0 = V_T = 0$, $V_t \in D(\delta)$ and $\delta(dV_t/dt) = d\delta V_t/dt$.

Definition 15. *The* work performed on the system *along a cyclic V_t, $t \in [0,T]$ is*

$$W := \int_0^T \omega \circ \tau_t^V \left(\frac{dV_t}{dt} \right) dt.$$

where ω is the initial state of the system.

By the boundary condition, $0 = \int_0^T \partial_t \left(\omega \circ \tau_t^V (V_t) \right) dt$, so that

$$W = -\int_0^T \omega \circ \tau_t^V \left(\delta^0 (V_t) \right) dt \quad (3.6)$$

since $\delta^V(V_t) = \delta^0(V_t)$. Also note that by the first law of thermodynamics, this also equals the total heat given by the system to the environment.

Lemma 31. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system with generator δ^0 and $\mathbb{R} \ni t \mapsto V_t = V_t^* \in \mathcal{A}$ be a norm-differentiable local perturbation such that $V_t = 0$ if $t \in (-\infty, 0] \cup [T, \infty)$, $V_t \in D(\delta^0)$ and $\delta^0(dV_t/dt) = d\delta^0 V_t/dt$. Then $W = -\mathrm{i}\omega(\Gamma_T^V \delta^0(\Gamma_T^{V*}))$.*

Proof. Under the given assumption, $\Gamma_t^V \in D(\delta^0)$, and $\delta^0(\Gamma_t^V)$ is differentiable with $d\delta^0(\Gamma_t^V)/dt = \delta^0(d\Gamma_t^V/dt)$ (without proof). But then

$$\begin{aligned} -\mathrm{i}\omega(\Gamma_T^V \delta^0(\Gamma_T^{V*})) &= \int_0^T \omega \left(-\mathrm{i}\partial_t(\Gamma_t^V)\delta^0(\Gamma_t^{V*}) + \Gamma_t^V \delta^0 \left(-\mathrm{i}\partial_t(\Gamma_t^{V*}) \right) \right) \\ &= \int_0^T \omega \left(\Gamma_t^V \tau_t^0(V) \delta^0(\Gamma_t^{V*}) - \Gamma_t^V \delta^0(\tau_t^0(V)\Gamma_t^{V*}) \right) = -\int_0^T \omega \left(\Gamma_t^V \tau_t^0(\delta^0(V))\Gamma_t^{V*} \right). \end{aligned}$$

Conclude by (3.6). □

Theorem 32. *Under the assumptions of the previous lemma, if ω is a (τ, β)-KMS state for some β, then $W \geq 0$.*

Proof. By Lemma 31, $W = -\mathrm{i}\omega(\Gamma_T^V \delta^0(\Gamma_T^{V*}))$. Since Γ_T^V is unitary and ω is a (τ,β)-KMS state, $W \geq 0$ by passivity, Proposition 30. □

We now consider a cyclic machine working between two reservoirs at inverse temperature $\beta_1 \leq \beta_2$. The C*-dynamical system is given by $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ and $\tau^0 = \tau_1 \otimes \tau_2$, with generator $\delta^0 = \delta_1 \otimes 1 + 1 \otimes \delta_2$. The initial state is $\omega = \omega_1 \otimes \omega_2$ where ω_i is a (τ_i, β_i)-KMS state, and it is a $(\sigma, 1)$-KMS state for the dynamics $\sigma_t := \tau_{1,\beta_1 t} \otimes \tau_{2,\beta_2 t}$ with generator $\gamma = \beta_1\delta_1 \otimes 1 + 1 \otimes \beta_2\delta_2$. The machine is represented by a cyclic perturbation $V_t \in \mathcal{A}$ temporarily coupling the reservoirs. The total work on the system decomposes in $W = Q_1 + Q_2$ where

$$Q_1 = -\mathrm{i}\omega(\Gamma_T^V(\delta_1 \otimes 1)(\Gamma_T^{V*})), \qquad Q_2 = -\mathrm{i}\omega(\Gamma_T^V(1 \otimes \delta_2)(\Gamma_T^{V*}))$$

are the amounts of heat given to both reservoirs. Now,

$$\beta_1 Q_1 + \beta_2 Q_2 = -\mathrm{i}\omega(\Gamma_T^V(\beta_1\delta_1 \otimes 1 + 1 \otimes \beta_2\delta_2)(\Gamma_T^{V*})) = -\mathrm{i}\omega(\Gamma_T^V \gamma(\Gamma_T^{V*})) \geq 0,$$

or $Q_1(T_2 - T_1) \geq -WT_1$. Assuming now that $Q_1 < 0$ (heat pumped out of the hot reservoir)

$$\frac{-W}{-Q_1} \leq \frac{T_1 - T_2}{T_1}$$

which is Carnot's statement of the second law of thermodynamics, namely a bound on the efficiency of a cyclic machine initially at equilibrium (ratio of the work performed by the system to the heat pumped out of the hot reservoir).

Stability of the thermal equilibrium refers to a number of results revolving around the fact that the dynamics applied to a state 'close to thermal' drives the system back to equilibrium. In fact, under additional assumption, it can be shown that this property is equivalent to the KMS condition.

The first result is about *structural stability*, and can be proved by perturbation theory in the line of (3.5).

Proposition 33. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system, and ω a (τ,β)-KMS state on $\mathcal{A}$. Then, for every local perturbation V, there is a (τ^V,β)-KMS state ω^V and*

1. *ω^V is ω-normal*
2. *there is $C > 0$ such that $\|\omega - \omega^V\| \leq C\|V\|$*
3. *the map $\omega \mapsto \omega^V$ is a bijection from the set of (τ,β)-KMS states onto the set of (τ^V,β)-KMS states*

See exercises for a proof in the finite dimensional case. Note in particular that local perturbations cannot induce a phase transition.

Dynamical stability needs more assumptions to hold, usually in the form of asymptotic abelianness of the dynamical system, namely $[A, \tau_t(B)] \to 0$ in some sense.

Theorem 34. *Let $V = V^* \in \mathcal{A}$ and let ω be a (τ^V,β)-KMS state, and let $\tilde\omega$ be a weak-* accumulation point of $\omega \circ \tau_t^0$ as $t \to \infty$. If $\lim_{t\to\infty} \|[V, \tau_t^0(A)]\| = 0$ for all $A \in \mathcal{A}$, then $\tilde\omega$ is a (τ^0,β)-KMS state.*

Proof. By lower semicontinuity of $(u, v) \mapsto u \ln(u/v)$, we have

$$\begin{aligned}\tilde\omega(A^*A) \ln \frac{\tilde\omega(A^*A)}{\tilde\omega(AA^*)} &\leq \liminf_{t\to\infty} \omega \circ \tau_t^0(A^*A) \ln \frac{\omega \circ \tau_t^0(A^*A)}{\omega \circ \tau_t^0(AA^*)} \leq \liminf_{t\to\infty} -\mathrm{i}\beta\omega(\tau_t^0(A)^*\delta^V(\tau_t^0(A))) \\ &= -\mathrm{i}\beta\tilde\omega(A^*\delta^0(A)) + \beta \liminf_{t\to\infty} \omega(\tau_t^0(A)^*[V, \tau_t^0(A)]) = -\mathrm{i}\beta\tilde\omega(A^*\delta^0(A))\end{aligned}$$

by the EEB inequality, the decomposition $\delta^V = \delta^0 + \mathrm{i}[V, \cdot]$ and $\delta^0 \circ \tau^0 = \tau^0 \circ \delta^0$. □

Note that the theorem does not state whether the limit of $\omega \circ \tau_t^0$ exists. However, it does so in two simple cases. Firstly, if there is a unique (τ^0, β)-KMS state, since then all accumulation points of $\omega \circ \tau_t^0$ must be equal. Secondly, if $[V, \tau_t^0(A)]$ decays fast enough:

Proposition 35. *Let $V = V^* \in \mathcal{A}$ and let ω be a τ^V-invariant state. Then $\omega_\pm := \lim_{t\to\pm\infty} \omega\circ\tau_t^0$ exists (in the weak*-topology) if and only if $\mathbb{R} \ni t \mapsto \omega([V, \tau_t^0(A)])$ is integrable at $\pm\infty$ for all $A \in \mathcal{A}$.*

Proof. Integrating $\frac{d}{ds}\tau_{-s}^V(\tau_s^0(A)) = -\tau_{-s}^V(\mathrm{i}[V, \tau_s^0(A)])$ and using the invariance of ω yields

$$\omega(\tau_{t_2}^0(A)) - \omega(\tau_{t_1}^0(A)) = -\mathrm{i}\int_{t_1}^{t_2} \omega([V, \tau_s^0(A)])ds$$

for all $A \in \mathcal{A}$. □

In particular, a sufficient condition for the existence of the limit is the integrability of the map $\mathbb{R} \ni t \mapsto \|[V, \tau_t^0(A)]\|$. We finally state a sharp result. Let ω be an arbitrary reference state.

(A) For any self-adjoint element V of a norm-dense *-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, there is a $\lambda_V > 0$ such that

$$\int_{\mathbb{R}} \|[V, \tau_s^{\lambda V}(A)]\|ds < \infty, \qquad |\lambda| \le \lambda_V, A \in \mathcal{A}_0.$$

(S) For any self-adjoint element V of a norm-dense *-subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, there is a $\lambda_V > 0$ such that if $|\lambda| \le \lambda_V$, there exists a $\tau^{\lambda V}$-invariant, ω-normal state such that

$$\omega_+^{\lambda V} := \lim_{t\to\infty} \frac{1}{T}\int_0^T \omega \circ \tau_t^{\lambda V} dt \text{ exists,} \quad \text{and} \quad \lim_{\lambda\to 0} \|\omega - \omega^{\lambda V}\| = 0$$

Theorem 36. *Assume that ω is a factor state and that (A) holds. Then (S) holds if and only if ω is a (τ^0, β)-KMS state for some β.*

In that case, by a variant of Theorem (34), $\omega_+^{\lambda V}$ is a $(\tau^{\lambda V}, \beta)$-KMS state.

3.4 On the set of KMS states

Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system with an identity. For any $\beta > 0$, let $\mathcal{S}_\beta(\mathcal{A})$ be the set of all (τ, β)-KMS states. The physical intuition is as follows: for small β, there is a unique thermal state, corresponding to the high temperature phase. As β grows, the set of $\mathcal{S}_\beta(\mathcal{A})$ becomes non trivial, and any state can be decomposed into pure thermodynamic phases. This picture is made mathematically precise in the following theorem:

Theorem 37. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system with an identity, and let $\mathcal{S}_\beta(\mathcal{A})$ be the set of all (τ, β)-KMS states, for $\beta > 0$. Then,*

1. *$\mathcal{S}_\beta(\mathcal{A})$ is convex and weakly-* compact*
2. *The normal extension of ω to $\pi(\mathcal{A})''$ is a KMS state*
3. *$\omega \in \mathcal{S}_\beta(\mathcal{A})$ is an extremal point if and only if ω is a factor state, and if ω' is an ω-normal, extremal KMS state, then $\omega = \omega'$*
4. *$\pi(A)' \cap \pi(A)''$ consists of time-invariant elements*

5. *If $\omega \in \mathcal{S}_\beta(\mathcal{A})$ is such that the GNS Hilbert space is separable, there is a unique probability measure μ on $\mathcal{S}_\beta(\mathcal{A})$, which is concentrated on the extremal points, such that $\omega = \int_{\mathcal{S}_\beta(\mathcal{A})} \nu d\mu(\nu)$*

Proof. (Sketch, incomplete) (1). If $\omega_1, \omega_2 \in \mathcal{S}_\beta(\mathcal{A})$, $A, B \in \mathcal{A}$, with associated analytic functions $F_{\beta,1}(A, B, \cdot)$, $F_{\beta,1}(A, B, \cdot)$, then the analytic function $\lambda F_{\beta,1}(A, B, \cdot) + (1-\lambda)F_{\beta,2}(A, B, \cdot)$ has boundary values associated to $\lambda\omega_1 + (1-\lambda)\omega_2$ so that $\mathcal{S}_\beta(\mathcal{A})$ is convex. Moreover, the EEB inequality implies that $\mathcal{S}_\beta(\mathcal{A})$ is a weakly-* closed subset of the weakly-* compact set $\mathcal{E}(\mathcal{A})$, hence $\mathcal{S}_\beta(\mathcal{A})$ is weakly-* compact.

(2) Follows by density of $\pi(\mathcal{A})$ in $\pi(\mathcal{A})''$ in the weak topology, Corollary 7.

(3) If ω is not a factor state, then there exists a projection $1 \neq P \in \pi(A)' \cap \pi(A)''$. We first claim that $\omega(P) \neq 0$. Otherwise $0 = \omega(P) = \|P\Omega\|^2$ so that $P\Omega = 0$. But then, for any $A, B \in \mathcal{A}$, $\omega(A^*PB) = \langle \pi(A)\Omega, P\pi(B)\Omega\rangle = 0$ since $P \in \pi(A)'$, and hence $P = 0$ by cyclicity of Ω. Now, $\omega = \omega(P)\omega_1 + \omega(1-P)\omega_2$, where $\omega_1(A) = \omega(PA)/\omega(P)$ and $\omega_2(A) = \omega((1-P)A)/\omega(1-P)$, is a non-trivial decomposition of ω. Moreover, $\omega(P)\omega_1(BA) = \omega(PBA) = \omega(BPA) = \omega(PA\tau_{i\beta}(B)) = \omega(P)\omega_1(A\tau_{i\beta}(B))$ so that ω is not extremal in $\mathcal{S}_\beta(\mathcal{A})$.

(4). Let $C \in \pi(A)' \cap \pi(A)''$ and consider the normal extension of ω to $\pi(A)''$. Repeating the proof of Proposition 26, $t \mapsto \omega(\tau_t(A^*B)C)$ is constant. Hence, $t \mapsto \omega(\tau_t(A^*)C\tau_t(B)) = \langle \pi(A)\Omega, U_t^*\pi(C)U_t\pi(B)\Omega\rangle = \omega(A^*\tau_t(C)B)$ is constant for all $A, B \in \mathcal{A}$.

(5). That any KMS state can be decomposed into extremal KMS states follows from convexity and Krein-Milman's theorem. Uniqueness is more involved. □

The algebra $\pi(A)'' \supset \pi(A)$ contains both microscopic and macroscopic observables. Elements in the centre $\pi(A)' \cap \pi(A)''$ induce 'superselection rules': If $S = S^* \in \pi(A)' \cap \pi(A)''$ with $S \neq \lambda \cdot 1$, the Hilbert space decomposes into components on which S is a constant multiple of the identity, while these components with different 'quantum numbers' are not connected by any observable. In the case of KMS states, (3) above states that such observables associated with quantum numbers are constant in time. Furthermore, in a factor, any such S is a constant multiple of the identity. Hence, by (2) above, extremal KMS states associate fixed, non-fluctuating values to all quantum numbers: they are 'macroscopically pure' states.

Theorem 38. *Let $(\mathcal{A}, \tau_t)$ be a C*-dynamical system, and let ω be a faithful (τ, β)-KMS state, for $\beta > 0$. Let α be a *-automorphism of $\mathcal{A}$. Then,*

1. *$\omega \circ \alpha$ is a $(\alpha^{-1} \circ \tau \circ \alpha, \beta)$-KMS state*
2. *If $\omega \circ \alpha = \omega$, then $\alpha \circ \tau_t = \tau_t \circ \alpha$ for all $t \in \mathbb{R}$*
3. *If $\alpha \circ \tau_t = \tau_t \circ \alpha$ for all $t \in \mathbb{R}$, then $\omega \circ \alpha$ is a (τ, β)-KMS state*

Proof. Let F be the analytic function associated to ω. Then $F_\alpha(A, B; z) := F(\alpha(A), \alpha(B); z)$ is an analytic function in S_β, continuous on $\overline{S_\beta}$ and such that, for $t \in \mathbb{R}$,

$$F_\alpha(A, B; t) = \omega(\alpha(A)\tau_t(\alpha(B))) = (\omega \circ \alpha)(A(\alpha^{-1} \circ \tau_t \circ \alpha)(B))$$
$$F_\alpha(A, B; t + i\beta) = \omega(\tau_t(\alpha(B))\alpha(A)) = (\omega \circ \alpha)((\alpha^{-1} \circ \tau_t \circ \alpha)(B)A)$$

which shows that $\omega \circ \alpha$ is a $(\alpha^{-1} \circ \tau \circ \alpha, \beta)$-KMS state. In order to prove (2), we use the fact that the τ-group with respect to which a ω is a KMS state is unique[3]. But ω is simultaneously a (τ, β)-KMS state and by (1) a $(\alpha^{-1} \circ \tau \circ \alpha, \beta)$-KMS state, hence $\tau_t = \alpha^{-1} \circ \tau_t \circ \alpha$. Finally, (3) follows immediately from (1). □

[3] In the case $\dim(\mathcal{H}) < \infty$, a faithful state is given by a $\rho > 0$, which determines uniquely $H := -\beta^{-1} \ln \rho$ and hence the dynamics.

3.5 Symmetries

Definition 16. *Let $(\mathcal{A},\tau_t)$ be a C^*-dynamical system. A $*$-automorphism α of $\mathcal{A}$ is a* symmetry *if $\alpha\circ\tau_t=\tau_t\circ\alpha$ for all $t\in\mathbb{R}$.*

In this case, and by 1 above, if ω is a (τ,β)-KMS state, then so is $\omega\circ\alpha$. Hence, in the presence of a symmetry the set $\mathcal{S}_\beta(\mathcal{A})$ is invariant under τ for any fixed $\beta>0$. In particular, if there is a unique (τ,β)-KMS state, then it is itself invariant and one says that the symmetry is unbroken at β. If, on the other hand, there is a (τ,β)-KMS state which is not invariant, then the symmetry is said to be broken and there is more than one equilibrium state at β, indicating a phase transition. Examples are the breaking of rotational $SU(2)$-symmetry in magnetic transitions, translational $\mathbb{R}^d$ symmetry in liquid-solid transitions. Here is a general criterion for the absence of symmetry breaking:

(A) There is a sequence $U_n\in\mathcal{A}$ of unitary elements of the algebra such that $U_n\in D(\delta)$ and

$$\lim_{n\to\infty}\|\alpha(A)-U_n^*AU_n\|=0,\qquad A\in\mathcal{A}.$$

(Bi) There is M such that $\|\delta(U_n)\|\leq M$

(Bii) All (τ,β)-KMS states are α^2-invariant and there is M such that $\|U_n^*\delta(U_n)+U_n\delta(U_n^*)\|\leq M$

If (A) holds, one says that α is almost inner.

Theorem 39. *Let α be a symmetry of $(\mathcal{A},\tau_t)$. If (A) and either (Bi) or (Bii) are satisfied, then all (τ,β)-KMS states are α-invariant for all $\beta>0$.*

Note that the symmetry can still be broken in the ground state, $\beta=\infty$.

Proof. Let ω be a (τ,β)-KMS state, H the associated Hamiltonian such that $H\Omega=0$ and let $H=\int\lambda dP(\lambda)$. For any bounded interval $I\subset\mathbb{R}$, let $\{\check{h}_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued C_c^∞ functions supported on intervals $[a_n,b_n]$, with $|b_n-a_n|\leq1$ and such that $\sum_n\check{h}_n(\lambda)^2=1$ for all $\lambda\in I$. Let $A_n:=\tau_{h_n}(A)$, which is analytic for τ_t. First of all,

$$\omega(A_n^*A_n)=\int h_n(t)h_n(-s)\langle e^{iHs}\pi(A)\Omega,e^{iHt}\pi(A)\Omega\rangle dtds=\int_{a_n}^{b_n}\check{h}_n(\lambda)^2d\mu_A(\lambda)$$

as well as $\omega(A_nA_n^*)=\int_{a_n}^{b_n}\check{h}_n(\lambda)^2d\nu_A(\lambda)$. Hence, by the measure-theoretic KMS property, $\omega(A_n^*A_n)\geq\exp(\beta a_n)\omega(A_nA_n^*)$ and further

$$\omega(A_n^*A_n)\ln\frac{\omega(A_n^*A_n)}{\omega(A_nA_n^*)}\geq\beta a_n\omega(A_n^*A_n).$$

Similarly, $-i\omega(A_n^*\delta(A_n))=\int_{a_n}^{b_n}\lambda\check{h}_n(\lambda)^2d\mu_A(\lambda)$, and hence,

$$-i\omega(A_n^*\delta(A_n))\leq b_n\omega(A_n^*A_n)$$

We further write the EEB inequality for the observable $U_m^*A_n$, $n,m\in\mathbb{N}$, namely

$$\omega(A_n^*A_n)\ln\frac{\omega(A_n^*A_n)}{\omega(U_m^*A_nA_n^*U_m)}\leq-i\beta\omega(A_n^*U_m\delta(U_m^*)A_n)-i\beta\omega(A_n^*\delta(A_n))$$

and use the two inequalities above to obtain (note the position of * in the numerator!)

$$\omega(A_n^*A_n)\ln\frac{\omega(A_nA_n^*)}{\omega(U_m^*A_nA_n^*U_m)}\leq-i\beta\omega(A_n^*U_m\delta(U_m^*)A_n)+\beta(b_n-a_n)\omega(A_n^*A_n)\tag{3.7}$$

and $b_n - a_n \leq 1$.
Assumption (Bi). Since $|\omega(A_n^* U_m \delta(U_m^*) A_n)| \leq \|\delta(U_m)\| \omega(A_n^* A_n) \leq M\omega(A_n^* A_n)$, (3.7) yields

$$\omega(A_n A_n^*) \leq \mathrm{e}^{\beta(M+1)} \omega(U_m^* A_n A_n^* U_m)$$

and letting $m \to \infty$, $\omega(A_n A_n^*) \leq \mathrm{e}^{\beta(M+1)} (\omega \circ \alpha)(A_n A_n^*)$. Summing over n, we have proved that there exists a constant $C = C(\beta, M)$ such that

$$\omega(AA^*) \leq C(\omega \circ \alpha)(AA^*),$$

which extends to all $A \in \mathcal{A}$. By the remark after Lemma 9, there is a $T \in \pi_{\omega\circ\alpha}(\mathcal{A})'$ such that $\omega(A) = \langle T\Omega_{\omega\circ\alpha}, \pi_{\omega\circ\alpha}(A) T\Omega_{\omega\circ\alpha}\rangle$, which shows that ω is $(\omega \circ \alpha)$-normal. If ω is an extremal KMS state, then $\omega \circ \alpha$ is also extremal so that they must be equal by Theorem 37(3). Since this holds for all extremal KMS state, the general result holds by decomposition, Theorem 37(5).
Assumption (Bii). We repeat the procedure above with the state $\omega \circ \alpha$, sum (3.7) and the similar bound with $U_m \leftrightarrow U_m^*$, proceed as above and obtain

$$((\omega \circ \alpha)(A_n A_n^*))^2 \leq \mathrm{e}^{\beta(M+2)} \omega(A_n A_n^*)(\omega \circ \alpha^2)(A_n A_n^*). \tag{3.8}$$

Hence, $(\omega \circ \alpha)(A) \leq \tilde{C}\omega(A)$. Hence $(\omega \circ \alpha)$ is ω-normal and the conclusion holds as above. □

Chapter 4

Ideal quantum gases

4.1 The ideal Fermi gas

4.2 The ideal Bose gas & Bose-Einstein condensation

Chapter 5

Renormalisation

5.1 The renormalisation idea

5.2 Block spin transformation in the Ising model

Chapter 6

Phase transitions in quantum spin systems

Let $\mathcal{A} = \overline{\bigcup_{\Lambda \in \mathcal{F}(\Gamma)} \mathcal{A}_\Lambda}$ be the C*-algebra of a quantum spin system.

Definition 17. *An* interaction *on $\mathcal{A}$ is a map defined on $\mathcal{F}(\Gamma)$ such that for $X \in \mathcal{F}(\Gamma)$, $\Phi(X) = \Phi(X)^* \in \mathcal{A}_X$. Furthermore, for any non-negative function $\xi : \mathcal{F}(\Gamma) \to [0, \infty)$,*

$$\mathcal{B}_\xi := \left\{ \Phi : \|\Phi\|_\xi := \sup_{x \in \Gamma} \sum_{X \ni x} \|\Phi(X)\| \xi(X) < \infty \right\}.$$

*is a Banach space of interactions. Finally, an N-*body interaction *is defined by the condition $\Phi(X) = 0$ if $|X| \neq N$.*

In the case of a N body interaction, one writes $\Phi(x_1, \ldots, x_N), x_i \in \Gamma$. A simple example is $\xi(X) = 1$ implying an integrable decay. We shall use the following: Let D be the maximal degree in Γ and $\mathrm{diam}(X) := \max\{d(x, y) : x, y \in X\}$ for any $X \in \mathcal{F}(\Gamma)$. For any $\lambda > 0$, denote

$$\mathcal{B}_\lambda := \mathcal{B}_{\xi_\lambda}, \qquad \xi_\lambda(X) := |X| D^{2|X|} \mathrm{e}^{\lambda \mathrm{diam}(X)}.$$

Now, for $\Lambda \in \mathcal{F}(\Gamma)$, the Hamiltonian is the sum of interactions within Λ, namely

$$H_\Lambda := \sum_{X \subset \Lambda} \Phi(X)$$

and for $A \in \mathcal{A}$,

$$\tau_t^{\Phi,\Lambda}(A) = \mathrm{e}^{\mathrm{i}tH_\Lambda} A \mathrm{e}^{-\mathrm{i}tH_\Lambda}$$

is a strongly continuous one parameter group of *-automorphisms of $\mathcal{A}$. Let $\{\Lambda_n\}_{n \in \mathbb{N}}$ be a sequence in $\mathcal{F}(\Gamma)$ such that $\Lambda_n \subset \Lambda_m$ is $n \leq m$ and for any $x \in \Gamma$ there exists n_0 such that $x \in \Lambda_n$ for all $n \geq n_0$.

Theorem 40. *Let $\lambda > 0$ and $\Phi \in \mathcal{B}_\lambda$. There exists a strongly continuous one parameter group of *-automorphisms $\{\tau_t^\Phi : t \in \mathbb{R}\}$ of $\mathcal{A}$ such that, for any $A \in \mathcal{A}$,*

$$\lim_{n \to \infty} \|\tau_t^{\Phi,\Lambda_n}(A) - \tau_t^\Phi(A)\| = 0,$$

for all $t \in \mathbb{R}$. The convergence is uniform for t in a compact set and the limit is independent of the sequence $\{\Lambda_n\}_{n \in \mathbb{N}}$.

The theorem is an immediate consequence of the *Lieb-Robinson bound*: For $\Phi \in \mathcal{B}_\lambda$ there is $v_\lambda > 0$, such that for any $A \in \mathcal{A}_X, B \in \mathcal{A}_Y$, and $X \cup Y \in \Lambda$,

$$\|[\tau_t^{\Phi,\Lambda}(A), B]\| \le C\|A\|\|B\| \min\{|X|, |Y|\} \exp(-\lambda(d(X,Y) - v_\lambda|t|))$$

where the constant C is independent of Λ. This is a propagation estimate: up to exponentially small corrections, the support of A grows linearly with time, with velocity v_λ. Since, for $n \ge m$ with $A \in \mathcal{A}_\Lambda, \Lambda \subset \Lambda_m$,

$$\tau_t^{\Phi,\Lambda_n}(A) - \tau_t^{\Phi,\Lambda_m}(A) = \int_0^t \frac{d}{ds}\left(\tau_s^{\Phi,\Lambda_n} \circ \tau_{t-s}^{\Phi,\Lambda_m}(A)\right) ds = \int_0^t \tau_s^{\Phi,\Lambda_n}(\delta^{\Phi,\Lambda_n} - \delta^{\Phi,\Lambda_m})\tau_{t-s}^{\Phi,\Lambda_m}(A)ds$$

and $\delta^{\Phi,\Lambda_n} - \delta^{\Phi,\Lambda_m} = \sum_{X:X\cap(\Lambda_n\setminus\Lambda_m)\neq\emptyset}[\Phi(X), \cdot]$, we have

$$\begin{aligned}\|\tau_t^{\Phi,\Lambda_n}(A) - \tau_t^{\Phi,\Lambda_m}(A)\| &\le \int_0^t \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X\ni x} \|[\Phi(X), \tau_{t-s}^{\Phi,\Lambda_m}(A)]\|ds \\ &\le C|\Lambda|\|A\| \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X\ni x} \|\Phi(X)\| \exp(-\lambda(d(X,\Lambda) - v_\lambda|t|) \\ &\le \tilde{C}|\Lambda|\|A\|\|\Phi\|_\lambda \exp(-\lambda d(\Gamma\setminus\Lambda_m, \Lambda)) \exp(\lambda v_\lambda|t|).\end{aligned}$$

This vanishes uniformly as $m \to \infty$ for t in a compact interval, and $\{\tau_t^{\Phi,\Lambda_n}(A)\}_{n\in\mathbb{N}}$ is Cauchy.

A typical example is the Heisenberg models. Here $\Gamma = \mathbb{Z}^d$, and $\mathcal{H}_x = \mathbb{C}^{2s+1}$ is the representation space of $SU(2)$ with generator S^1, S^2, S^3. The Heisenberg Hamiltonian is given by

$$H_{\Lambda,J,h} = \sum_{\{x,y\}\in\Lambda\times\Lambda} \sum_{i=1}^3 J_{xy}^i S_x^i S_y^i - h\sum_{x\in\Lambda} S_x^3, \quad h > 0,$$

with some decay on $|J_{xy}^i|$ in $d(x,y)$. This defines a translation invariant Hamiltonian if $J_{xy}^i = J^i$ for all $\{x,y\} \in \Gamma\times\Gamma$ and an $SU(2)$-invariant interaction if $J_{xy}^i = J_{xy}$ for $i = 1, 2, 3$.

6.1 The theorem of Mermin & Wagner

We now apply Theorem 39 to the concrete case of low dimensional quantum spin systems and obtain a general form of the theorem of Mermin and Wagner. Note that this only one version of the theorem, namely about the absence of symmetry breaking, which does not necessarily exclude other types of phase transitions. The original proof in the generality given here is due to Fröhlich-Pfister. For simplicity, we consider $\mathcal{H}_x = \mathcal{H}$ for all $x \in \Gamma$.

Let G be a compact connected Lie group and let $G \ni g \mapsto U_g$ be a strongly continuous unitary representation of G on $\mathcal{H}$. This induces a group of *-automorphisms of $\mathcal{A}_{\{x\}}$ by $\alpha_g^{\{x\}}(A) = U_g^* A U_g$, and the tensor product representation $\otimes_{x\in\Lambda} U_g$ induces the tensor action α_g^Λ on $\mathcal{A}_\Lambda$, for any $\Lambda \in \mathcal{F}(\Gamma)$. Hence, this defines a strongly continuous group of *-automorphisms on $\mathcal{A}_{\text{loc}}$ which extends by continuity to $\{\alpha_g : g \in G\}$ on $\mathcal{A}$. Note that the complete system is rotated by the same element g, a 'global gauge transformation'. A typical example is $\mathcal{H} = \mathbb{C}^{2s+1}$ carrying the spin-s representation of $G = SU(2)$, namely $U_g = \exp(2\pi i g \cdot S)$, where g is an element of the unit ball and S is the vector of spin matrices.

Theorem 41. *Let $\mathcal{A}$ be as above with $\Gamma = \mathbb{Z}^2$, $\{\alpha_g : g \in G\}$ the action of the compact connected Lie group G, and Φ a G-invariant two-body interaction, namely*

$$\alpha_g(\Phi(x,y)) = \Phi(x,y), \quad \textit{for all } x, y \in \mathbb{Z}^2, g \in G.$$

If

$$\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}\|\Phi(x,y)\|d(x,y)^2<\infty$$

then for any $0<\beta<\infty$ *and any* (τ^Φ,β)*-KMS state* ω,

$$\omega\circ\alpha_g=\omega,\quad \textit{for all } g\in G.$$

For a translation and rotation invariant interaction, the sharp condition is $\|\Phi(x,y)\|\le Cd(x,y)^{-4}$: there are models with an interaction decaying as $d(x,y)^{-4+\epsilon}$ for which phase transitions are known to occur.

Proof. We consider a one dimensional subgroup H of G, and since G is compact, $H\simeq\mathbb{R}/\mathbb{Z}=\mathbb{S}^1$. We consider the generator $S=S^*\in\mathcal{L}(\mathcal{H})$, namely $U_\phi=\exp(\mathrm{i}\phi S)$ for $\phi\in[0,2\pi)$.

For $m\in\mathbb{N}$, let $\Lambda_m=[-m,m]\cap\mathbb{Z}^2$. Let ϕ be fixed, and let $\varphi_m:\mathbb{Z}^2\to[0,2\pi)$ be given by

$$\varphi_m(x)=\begin{cases}\phi & x\in\Lambda_m\\ \phi(2-\max\{|x_1|,|x_2|\}/m) & x=(x_1,x_2)\in\Lambda_{2m}\setminus\Lambda_m\\ 0 & \text{otherwise}\end{cases}$$

and finally

$$U_\phi(m):=\bigotimes_{x\in\Lambda_{2m}}U_x(\varphi_m(x))\in\mathcal{A}_{\Lambda_{2m}}\subset D(\delta),$$

which slowly interpolates between a full rotation on Λ_m and no rotation outside of Λ_{2m}.

Let $A\in\mathcal{A}_{\mathrm{loc}}$ and $m_0:=\min\{m\in\mathbb{N}:A\in\mathcal{A}_{\Lambda_m}\}$. We have $U_\phi(m)^*AU_\phi(m)=\alpha_\phi(A)$ for all $m\ge m_0$, so that Assumption (A) of Theorem 39 holds. We now claim that Assumption (Bii) also holds. Noting that for $A\in\mathcal{A}_\Lambda$, $\delta(A)=\mathrm{i}\sum_{\{x,y\}\cap\Lambda\neq\emptyset}[\Phi(x,y),A]$, we compute

$$U_\phi(m)^*\delta(U_\phi(m))=\mathrm{i}\sum_{\{x,y\}\in\mathbb{Z}^2\setminus\mathcal{N}_m}U_\phi(m)^*\Phi(x,y)U_\phi(m)-\Phi(x,y)$$

where $\mathcal{N}_m=\{\{x,y\}:x,y\in\Lambda_m \text{ or } x,y\in\mathbb{Z}^2\setminus\Lambda_{2m}\}$ by the symmetry of the interaction and the support of $U_m(\phi)$. Denote $U_\phi(m)^*\delta(U_\phi(m))+U_\phi(m)\delta(U_\phi(m)^*)=\mathrm{i}\sum_{x,y}\Delta_m(x,y)$. Note that

$$\varphi_m(x)S_x+\varphi_m(y)S_y=\frac{\varphi_m(x)+\varphi_m(y)}{2}(S_x+S_y)+\frac{\varphi_m(x)-\varphi_m(y)}{2}(S_x-S_y):=E_m(x,y)+O_m(x,y)$$

with $[E_m(x,y),O_m(x,y)]=0$ since $[S_x,S_y]=0$. Since, moreover, $E_m(x,y)$ generates the same rotation by $(\varphi_m(x)+\varphi_m(y))/2$ at both x and y, and by the symmetry of the interaction,

$$U_\phi(m)^*\Phi(x,y)U_\phi(m)=\mathrm{e}^{-\mathrm{i}O_m(x,y)}\mathrm{e}^{-\mathrm{i}E_m(x,y)}\Phi(x,y)\mathrm{e}^{\mathrm{i}E_m(x,y)}\mathrm{e}^{\mathrm{i}O_m(x,y)}=\mathrm{e}^{-\mathrm{i}O_m(x,y)}\Phi(x,y)\mathrm{e}^{\mathrm{i}O_m(x,y)}.$$

which has the commutator expansion

$$U_\phi(m)^*\Phi(x,y)U_\phi(m)-\Phi(x,y)=\sum_{k\ge1}\frac{i^k}{k!}\mathrm{ad}^k_{O_m(x,y)}(\Phi(x,y)).$$

Noting that $U_\phi(m)\delta(U_\phi(m)^*)=U_{-\phi}(m)^*\delta(U_{-\phi}(m))$ and $O_m(x,y)$ is odd under $\phi\to-\phi$, all odd terms in the series of $\Delta_m(x,y)$ cancel, yielding the estimate

$$\|\Delta_m(x,y)\|\le2\sum_{k\ge1}\frac{1}{(2k)!}\frac{1}{2^{2k}}|\varphi_m(x)-\varphi_m(y)|^{2k}\|\mathrm{ad}^{2k}_{S_x-S_y}(\Phi(x,y))\|.$$

It remains to observe that $|\varphi_m(x) - \varphi_m(y)| \leq |\phi| \min\{1, d(x,y)/m\}$, so that

$$|\varphi_m(x) - \varphi_m(y)|^{2k} \leq |\phi|^{2k} \left(\frac{d(x,y)}{m}\right)^2$$

and to carry out the spatial sum to obtain

$$\sum_{\{x,y\}\in\mathbb{Z}^2\setminus\mathcal{N}_m} \|\Delta_m(x,y)\| \leq \frac{4\mathrm{e}^{2\|S\||\phi|}}{m^2} \sum_{x\in\Lambda_{2m}} \sum_{y\in\mathbb{Z}^2} \|\Phi(x,y)\| d(x,y)^2 =: M < \infty$$

which is finite by assumption after estimating the sum by $(2m+1)^2 \sup_{x\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}^2}(\cdots)$.

For any $n \in \mathbb{N}$, we claim that $\omega \circ \alpha_{\pi/2^n} = \omega$. This follows from a recursive application of Theorem 39 starting with the observation that $\alpha_\pi^2 = \mathrm{id}$. Finally, the set $D := \{\phi \in \mathbb{S}^1 : \phi = \sum_{n=0}^N a_n(\pi/2^n), a_n \in \mathbb{Z}, N \in \mathbb{N}\}$ is dense in $\mathbb{S}^1$. For any $A \in \mathcal{A}$, the function $\phi \mapsto \xi_A(\phi) := \omega(\alpha_\phi(A) - A)$ is continuous and $\xi_A(\phi) = 0$ if $\phi \in D$. Hence $\xi_A(\phi) = 0$ for all $\phi \in \mathbb{S}^1$. □

Possible extensions following the same ideas with adapted assumptions include one-dimensional models, short range N-body interactions, and non-translation invariant models with possibly different representations of G at different points of Γ.

6.2 Existence of a phase transition in the Heisenberg model

In this section, we shall prove the existence of a phase transition at positive temperature for the antiferromagnetic Heisenberg model, following the original proof of Dyson-Lieb-Simon (1978). The proof relies on a spectral property of the Hamiltonian, *reflection positivity* which fails for the ferromagnetic model. Although a proof of phase transition in that case is still an open problem, recent progress has been made by Corregi, Giuliani and Seiringer (2013), who compute the free energy at low temperature.

We consider $\Lambda := \{-L/2, \cdots, L/2\}^d, L \in 2\mathbb{N}$ understood with periodic boundary conditions, and let E_Λ be the set of nearest neighbour pairs. The translation invariant, spin-S Heisenberg Hamiltonian is written as

$$H_\Lambda^{(u)} := -2 \sum_{\{x,y\}\in E_\Lambda} \left(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3\right), \qquad u \in [-1,1].$$

The case $u = 1$ is the ferromagnet, $u = 0$ the 'XY model', while $u = -1$ locally unitarily equivalent to the antiferromagnet on a bipartite lattice. Indeed, assume that $\Lambda = \Lambda_A \cup \Lambda_B$, with $\{x,y\} \in E_\Lambda$ implies $x \in \Lambda_A, y \in \Lambda_B$ or $x \in \Lambda_B, y \in \Lambda_A$ and note that local rotations by π along the 2 axis, generated by S_x^2, yield $\exp(-\mathrm{i}\pi S_x^2) S_x^j \exp(\mathrm{i}\pi S_x^2) = (-1)^j S_x^j$. It follows that conjugation $U_\Lambda^* H_\Lambda^{(-1)} U_\Lambda$ with the unitary $U_\Lambda := \prod_{x\in\Lambda_A} \exp(\mathrm{i}\pi S_x^2)$ yields the antiferromagnet.

Given the Gibbs state $\omega_{\beta,\Lambda}^{(u)}$, we are interested proving the existence of *long-range order*

$$\lim_{|x|\to\infty} \liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(S_0^3 S_x^3) > 0, \qquad d \geq 3, \tag{6.1}$$

for β sufficiently large[1]. This implies for the the magnetisation $M_\Lambda := |\Lambda|^{-1} \sum_{x\in\Lambda} S_x^3$ that

$$\liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(M_\Lambda^2) > 0, \qquad d \geq 3, \quad \beta \text{ sufficiently large}$$

[1] In the case of the antiferromagnet,

$$(-1)^{d(0,x)} \omega_{\beta,\Lambda}^{\mathrm{antiferro}}(S_0^3 S_x^3) = (-1)^{d(0,x)} \omega_{\beta,\Lambda}^{(-1)}(U_\Lambda^* S_0^3 S_x^3 U_\Lambda) = \omega_{\beta,\Lambda}^{(-1)}(S_0^3 S_x^3)$$

which remains uniformly bounded away from 0. This is called 'Néel ordering'.

which corresponds to the intuition of macroscopic fluctuations in the bulk magnetisation and the presence of multiple phases. In fact, a little more abstract nonsense would show that long-range order is inconsistent with the invariance of any extremal KMS state.

Theorem 42. *Consider $H_\Lambda^{(u)}$ for $u \in [-1,0]$ and spin S. Then,*

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega_{\beta,\Lambda}^{(u)}(S_0^3S_x^3) \geq \frac{1}{3}S(S+1) - \frac{1}{\sqrt{2}|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\sqrt{\frac{E^{(u)}(k)}{E(k)}} - \frac{1}{2\beta|\Lambda|}\sum_{k\in\Lambda^*\setminus\{0\}}\frac{1}{E(k)} \tag{6.2}$$

where Λ^* is the lattice dual to Λ and

$$E(k) := 2\sum_{i=1}^d(1-\cos(k_i)), \quad E^{(u)}(k) := \sum_{i=1}^d(1-u\cos(k_i))\omega_{\beta,\Lambda}^{(u)}(S_0^1S_{e_i}^1)+(u-\cos(k_i))\omega_{\beta,\Lambda}^{(u)}(S_0^2S_{e_i}^2).$$

Note that for any state ν, $|\nu(S_x^jS_y^j)| \leq \nu((S_x^j)^2)^{1/2}\nu((S_y^j)^2)^{1/2} \leq \nu(\vec{S}^2) = S(S+1)$ so that $|E^{(u)}(k)| \leq 4dS(S+1)$. Furthermore, $1-\cos(x) = (1/2)x^2 + O(x^4)$ as $x \to 0$ so that $E(k)^{-1}$ is integrable if $d \geq 3$, and $E(k)^{-1/2}$ is integrable if $d \geq 2$. Hence, if $d \geq 3$, there exist $0 < C_d, \kappa_d < \infty$ such that

$$\liminf_{\Lambda\to\mathbb{Z}^d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\omega_{\beta,\Lambda}^{(u)}(S_0^3S_x^3) \geq \frac{1}{3}S(S+1) - \kappa_d\sqrt{S(S+1)} - \frac{C_d}{\beta}$$

and the lower bound is strictly positive for S large enough and all $\beta \geq \beta_c = \beta_c(d,S)$. In turn, this implies long-range order, (6.1). Note that improved estimates allow to extend the statement to $d \geq 3$ and all $S \in (1/2)\mathbb{N}$.

Let $v : \mathbb{Z}^d \to \mathbb{R}$ and $h := \Delta v$, namely $h_x := \sum_{y:\{x,y\}\in E_\Lambda}(v_y - v_x)$. In $l^2(\Lambda)$,

$$\langle f, -\Delta g\rangle = \sum_{\{x,y\}\in E_\Lambda}(f_y - f_x)(g_y - g_x),$$

and in particular $\langle v, -\Delta v\rangle = \|h\|^2$.

Let

$$H_\Lambda^{(u)}(v) := H_\Lambda^{(u)} - \sum_{x\in\Lambda}h_xS_x^3,$$

to which we associate the partition function $Z_{\beta,\Lambda}^{(u)}(v) = \mathrm{Tr}\left(\exp(-\beta H_\Lambda^{(u)}(v))\right)$ and

$$\tilde{Z}_{\beta,\Lambda}^{(u)}(v) := Z_{\beta,\Lambda}^{(u)}(v)\mathrm{e}^{-\frac{1}{4}\beta\langle v,-\Delta v\rangle}$$

Let R be a reflection map of Λ and let $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_2 = R\Lambda_1$. Furthermore, $v_1 := v\restriction_{\Lambda_1}$, $v_2 := v\restriction_{\Lambda_2}$ and we shall write $v = v_1|v_2$.

We now exhibit the full structure of the proof.

Lemma 43. *If $u \leq 0$, then for any reflection R,*

$$\tilde{Z}_{\beta,\Lambda}^{(u)}(v_1|v_2)^2 \leq \tilde{Z}_{\beta,\Lambda}^{(u)}(v_1|Rv_1)\tilde{Z}_{\beta,\Lambda}^{(u)}(Rv_2|v_2)$$

Lemma 44. *If $u \leq 0$,*

$$Z_{\beta,\Lambda}^{(u)}(v) \leq Z_{\beta,\Lambda}^{(u)}(0)\mathrm{e}^{\frac{1}{4}\beta\langle v,-\Delta v\rangle}$$

Lemma 45. *If $u \le 0$, and for any $k \in \Lambda^* \setminus \{0\}$,*

$$\widehat{(S_0^3, S_\cdot^3)}_\beta^{(u)}(k) \le (2\beta E(k))^{-1}$$

where $(\cdot,\cdot)_\beta$ denotes Duhamel's two-point function,

$$(A,B)_\beta^{(u)} := \frac{1}{Z_{\beta,\Lambda}^{(u)}} \int_0^1 \mathrm{Tr}\left(\mathrm{e}^{-\beta s H_\Lambda^{(u)}} A \mathrm{e}^{-\beta(1-s)H_\Lambda^{(u)}} B\right) ds$$

Lemma 46. *For any $k \in \Lambda^* \setminus \{0\}$ such that $\widehat{(S_0^3, S_x^3)}_\beta^{(u)}(k) \le (2\beta E(k))^{-1}$,*

$$\omega_{\beta,\Lambda}^{(u)}\widehat{(S_0^3 S_\cdot^3)}(k) \le \sqrt{\frac{E^{(u)}(k)}{2E(k)}} + \frac{1}{2\beta E(k)}.$$

Proof of Theorem 42. Let $C_\Lambda(x) := \omega_{\beta,\Lambda}^{(u)}(S_0^3 S_x^3)$. We have

$$\frac{1}{|\Lambda|}\sum_{x\in\Lambda} \omega_{\beta,\Lambda}^{(u)}(S_0^3 S_x^3) = \widehat{C_\Lambda}(0) = C_\Lambda(0) - \sum_{k\in\Lambda^*\setminus\{0\}} \widehat{C_\Lambda}(k).$$

Furthermore, we note that $C_\Lambda(0) = \omega_{\beta,\Lambda}^{(u)}((S_0^3)^2) = (1/3)\omega_{\beta,\Lambda}^{(u)}(\vec{S}^2) = (1/3)S(S+1)$, which concludes the proof with Lemma 46. □

We should remark that in finite volume, the Gibbs state has all symmetries of the Hamiltonian since its density matrix is a function of the Hamitonian. In particular, $\omega_{\beta,\Lambda}^{(u)}(S_0^3) = 0$ or $\omega_{\beta,\Lambda}^{(u)}(M_\Lambda) = 0$ and their respective limits likewise. The limiting state must be a non-trivial superposition of extremal KMS states which break the $SU(2)$ symmetry. Here, we consider $m_{\mathrm{sp}} := \liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(|M_\Lambda|)$, namely the spontaneous magnetisation. One could also add a 'transverse magnetic field' to the Hamiltonian, namely $h\sum_{x\in\Lambda} S_x^3$ and study either the residual magnetisation, $m_{\mathrm{res}} := \lim_{h\to 0^+} \liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,h,\Lambda}^{(u)}(M_\Lambda)$, namely whether the system 'remembers' an external magnetic field which breaks the symmetry. It turns out that $m_{\mathrm{res}} \ge m_{\mathrm{sp}}$ and $m_{\mathrm{sp}} = 0$ if and only if $\liminf_{\Lambda\to\mathbb{Z}^d} \omega_{\beta,\Lambda}^{(u)}(M_\Lambda^2) = 0$, see exercises.

Proof of Lemma 43. Let $\mathcal{H} = \mathcal{K}\otimes\mathcal{K}$, with $\dim\mathcal{K} < \infty$, and let $A, B, C_1, \ldots C_l, D_1, \ldots D_l \in \mathcal{L}(\mathcal{K})$ be real matrices and $h_1, \ldots h_l \in \mathbb{R}$. Then,

$$\begin{aligned}\mathrm{Tr}&\left[\mathrm{e}^{A\otimes 1 + 1\otimes B - \sum_{k=1}^l (C_k\otimes 1 - 1\otimes D_k - h_k)^2}\right]^2 \\ &\le \mathrm{Tr}\left[\mathrm{e}^{A\otimes 1 + 1\otimes A - \sum_{k=1}^l (C_k\otimes 1 - 1\otimes C_k)^2}\right] \mathrm{Tr}\left[\mathrm{e}^{B\otimes 1 + 1\otimes B - \sum_{k=1}^l (D_k\otimes 1 - 1\otimes D_k)^2}\right] \quad (6.3)\end{aligned}$$

Indeed (in the case $l = 1$), we first apply Trotter's product formula

$$\mathrm{e}^{A\otimes 1 + 1\otimes B - (C\otimes 1 - 1\otimes D - h)^2} = \lim_{n\to\infty}\left(\mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\frac{1}{n}1\otimes B}\mathrm{e}^{-\frac{1}{n}(C\otimes 1 - 1\otimes D - h)^2}\right)^n$$

and the operator identity

$$\mathrm{e}^{-M^2} = (4\pi)^{-1/2}\int_{\mathbb{R}} \mathrm{e}^{-s^2/4}\mathrm{e}^{\mathrm{i}sM} ds$$

to write the trace as

$$(4\pi)^{-n/2}\int ds_1\cdots ds_n \mathrm{Tr}\left[\left(\mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\mathrm{i}\frac{s_1}{\sqrt{n}}C\otimes 1}\cdots\mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\mathrm{i}\frac{s_n}{\sqrt{n}}C\otimes 1}\right)\right]$$
$$\overline{\mathrm{Tr}\left[\left(\mathrm{e}^{\frac{1}{n}1\otimes B}\mathrm{e}^{\mathrm{i}\frac{s_1}{\sqrt{n}}1\otimes D}\cdots\mathrm{e}^{\frac{1}{n}1\otimes B}\mathrm{e}^{\mathrm{i}\frac{s_n}{\sqrt{n}}1\otimes D}\right)\right]}\mathrm{e}^{\mathrm{i}\frac{h\sum_{i=1}^n s_i}{\sqrt{n}}}\mathrm{e}^{-\frac{\sum_{i=1}^n s_i^2}{4}}$$

where we noted that matrices acting on different factors commute, that $\mathrm{Tr}(M\otimes 1)(1\otimes N) = \mathrm{Tr}(M\otimes 1)\mathrm{Tr}(1\otimes N)$, and the reality of the matrices to take the complex conjugate (not the adjoint) without reversing the order of the matrices. Cauchy-Schwarz's inequality for the s-integrals now yields

$$|\cdot|^2 \le \frac{1}{(4\pi)^{n/2}}\int \mathrm{Tr}\prod_{i=1}^n \mathrm{e}^{\frac{1}{n}A\otimes 1}\mathrm{e}^{\mathrm{i}\frac{s_i}{\sqrt{n}}C\otimes 1}\overline{\mathrm{Tr}\prod_{i=1}^n \mathrm{e}^{\frac{1}{n}1\otimes A}\mathrm{e}^{\mathrm{i}\frac{s_i}{\sqrt{n}}1\otimes C}}\mathrm{e}^{-\frac{\sum_{i=1}^n s_i^2}{4}}\cdot\frac{1}{(4\pi)^{n/2}}\int(A\leftrightarrow B).$$

Reversing the above steps yields the claim.

We now write the Heisenberg Hamiltonian as

$$\begin{aligned}H_\Lambda^{(u)}(v) = &\sum_{\{x,y\}\in E_\Lambda}\left((S_x^1-S_y^1)^2+(\sqrt{u}S_x^2-\sqrt{u}S_y^2)^2+((S_x^3+v_x/2)-(S_y^3+v_y/2))^2\right)\\&+E_\Lambda-\frac{1}{4}\sum_{\{x,y\}\in E_\Lambda}(v_x-v_y)^2.\end{aligned}$$

where $E_\Lambda = -d\sum_{x\in\Lambda}\left((S_x^1)^2+u(S_x^1)^2+(S_x^3)^2\right)$, and the remaing term is removed in the definition of $\tilde{Z}_{\beta,\Lambda}^{(u)}(v)$. The lemma now follows from (6.3) with $\mathcal{H}_\Lambda = \mathcal{H}_{\Lambda_1}\otimes\mathcal{H}_{\Lambda_2}$ and

$$\begin{aligned}A &= -\beta\sum_{\{x,y\}\in E_{\Lambda_1}}\left((S_x^1-S_y^1)^2+(\sqrt{u}S_x^2-\sqrt{u}S_y^2)^2+((S_x^3+v_x/2)-(S_y^3+v_y/2))^2\right)-\beta E_{\Lambda_1}\\B &= -\beta\sum_{\{x,y\}\in E_{\Lambda_2}}\left((S_x^1-S_y^1)^2+(\sqrt{u}S_x^2-\sqrt{u}S_y^2)^2+((S_x^3+v_x/2)-(S_y^3+v_y/2))^2\right)-\beta E_{\Lambda_2}\\C_i^1 &= \sqrt{\beta}S_{x_i}^1,\quad D_i^1=\sqrt{\beta}S_{y_i}^1,\quad C_i^2=\sqrt{\beta u}S_{x_i}^2,\quad D_i^1=\sqrt{\beta u}S_{y_i}^2,\\C_i^3 &= \sqrt{\beta}(S_{x_i}^3+v_{x_i}/2),\quad D_i^3=\sqrt{\beta}(S_{y_i}^3+v_{y_i}/2)\end{aligned}$$

where $\{x_i, y_i\}$ denote the edges crossing the boundary between Λ_1 and Λ_2. Note that S^1, S^3 are real and S^2 is imaginary, so that the above matrices are real for $u \le 0$. □

Proof of Lemma 44. We prove the equivalent statement $\tilde{Z}_{\beta,\Lambda}^{(u)}(v) \le \tilde{Z}_{\beta,\Lambda}^{(u)}(0)$, which can be interpreted as a variational problem, namely $v = 0$ is a maximiser of the functional $v \mapsto \tilde{Z}_{\beta,\Lambda}^{(u)}(v)$. Since $\tilde{Z}_{\beta,\Lambda}^{(u)} : l^\infty(\Lambda) \to \mathbb{R}$ is continuous, bounded and $\lim_{\|v\|_\infty\to\infty}\tilde{Z}_{\beta,\Lambda}^{(u)}(v) = 0$, there is a maximiser. Let $\bar{v}$ be a maximiser and $\bar{Z} = \tilde{Z}_{\beta,\Lambda}^{(u)}(\bar{v})$. If $\tilde{Z}_{\beta,\Lambda}^{(u)}(\bar{v}_1|R\bar{v}_1) < \bar{Z}$, then Lemma 43 yields $\bar{Z}^2 < \bar{Z}\tilde{Z}_{\beta,\Lambda}^{(u)}(R\bar{v}_2|\bar{v}_2)$, namely $\tilde{Z}_{\beta,\Lambda}^{(u)}(R\bar{v}_2|\bar{v}_2) > \bar{Z}$, which is a contradiction. Hence, if $\bar{v}$ is a maximiser, so is $\bar{v}_1|R\bar{v}_1$. Since this holds for any reflection R, this implies inductively that the constant field is a maximiser, and in fact any constant field is so, since $\tilde{Z}_{\beta,\Lambda}^{(u)}(v+\text{const}) = \tilde{Z}_{\beta,\Lambda}^{(u)}(v)$. □

Proof of Lemma 45. Lemma 44 implies that $\partial^2/\partial\lambda^2\tilde{Z}_{\beta,\Lambda}^{(u)}(\lambda v)|_{\lambda=0} \le 0$, or equivalently

$$\left(Z_{\beta,\Lambda}^{(u)}(0)\right)^{-1}\frac{\partial^2}{\partial\lambda^2}Z_{\beta,\Lambda}^{(u)}(\lambda v)\bigg|_{\lambda=0} \le \frac{\beta}{2}\langle v, -\Delta v\rangle.$$

Since $Z^{(u)}_{\beta,\Lambda}(\lambda v) = \operatorname{Tr}\exp(-\beta(H^{(u)}_\Lambda - \lambda\langle S^3, \Delta v\rangle))$, Duhamel's formula $(\exp(F(t))' = \int_0^1 \exp(sF)F'\exp((1-s)F)ds$ yields

$$\frac{\partial^2}{\partial\lambda^2} Z^{(u)}_{\beta,\Lambda}(\lambda v)\Big|_{\lambda=0} = \beta^2 \int_0^1 \operatorname{Tr}\left(\mathrm{e}^{-\beta s H^{(u)}_\Lambda}\langle S^3, \Delta v\rangle \mathrm{e}^{-\beta(1-s)H^{(u)}_\Lambda}\langle S^3, \Delta v\rangle\right) ds,$$

namely

$$2\beta\left(\langle S^3, -\Delta v\rangle, \langle S^3, -\Delta v\rangle\right)^{(u)}_\beta \leq \langle v, -\Delta v\rangle,$$

for any field v. Let $v_x(k) = \cos(kx)$, $k \in \Lambda^* \setminus \{0\}$ for which $-\Delta v(k) = E(k)v(k)$. Hence,

$$2\beta E(k) \sum_{x,y\in\Lambda} \cos(kx)\cos(ky)(S^3_x, S^3_y)^{(u)}_\beta \leq \sum_{x\in\Lambda} \cos^2(kx).$$

It remains to use the translation invariance of $(S^3_x, S^3_y)^{(u)}_\beta$ to express the right hand side as

$$\sum_x \cos^2(kx) \sum_z \cos(kz)(S^3_0, S^3_z)^{(u)}_\beta + \sum_x \cos(kx)\sin(kz) \sum_z \sin(kz)(S^3_0, S^3_z)^{(u)}_\beta.$$

The second term vanishes as $(S^3_0, S^3_z)^{(u)}_\beta = (S^3_z, S^3_0)^{(u)}_\beta$. Similarly, the sum over z in the first one equals the Fourier transform of $(S^3_0, S^3_\cdot)^{(u)}_\beta$, which yields the claim. □

Proof of Lemma 46. This follows from 'Falk-Bruch's inequality', namely

$$\omega^{(u)}_{\beta,\Lambda}(A^*A + AA^*) \leq \sqrt{(A^*, A)^{(u)}_\beta \omega^{(u)}_{\beta,\Lambda}([A^*, [H, A]])} + 2(A^*, A)^{(u)}_\beta$$

applied to $A = |\Lambda|^{-1}\sum_x \exp(-\mathrm{i}kx)S^3_x$. Indeed

$$\begin{aligned}
\omega^{(u)}_{\beta,\Lambda}(A^*A + AA^*) &= 2\omega^{(u)}_{\beta,\Lambda}\widehat{(S^3_0, S^3_\cdot)}^{(u)}_\beta(k) \\
\omega^{(u)}_{\beta,\Lambda}([A^*[H, A]) &= 4\beta E^{(u)}(k) \\
(A^*, A)^{(u)}_\beta &= \widehat{(S^3_0, S^3_\cdot)}^{(u)}_\beta(k)
\end{aligned}$$

and we conclude by the bound on $\widehat{(S^3_0, S^3_\cdot)}^{(u)}_\beta(k)$. □

Chapter 7

Computer simulations