On the affine homotopy invariance of G-torsors

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Serre's problem on projective modules

Question (Serre, 1955)

Let k be a field. Let $n \geq 0$ be an integer. Let P be a finitely generated projective module on $k[t_1, \dots, t_n]$. Is P free?

- Quillen - Suslin (1976) Yes.

Notation. k : field;
A : finite-type regular k-algebra;
P : finitely generated projective module on $A[t_1, \dots, t_n]$.

Question

Does there exist a projective A-module P' such that

$$P \simeq P' \otimes_A A[t_1, \dots, t_n]?$$

- Lindel (1981) Yes.

G-torsors

G: Reductive algebraic group over a field k;
A: Smooth, finite-type k-algebra; write $U = \mathrm{Spec}\ A$.
The projection map $pr : U \times \mathbb{A}^1 \to U$ induces a map

$$pr^* : H^1_{\mathrm{Zar}}(U, G) \to H^1_{\mathrm{Zar}}(U \times \mathbb{A}^1, G).$$

Question

When is pr^* an isomorphism for every smooth affine k-scheme U?

- It is classically known that one has to put a suitable isotropy hypothesis on G for the above question to have an affirmative answer.
- First counterexamples when G is anisotropic were due to Ojanguren-Sridharan (1971) and Parimala (1978).

Isotropy hypothesis

$(*)$ Every almost k-simple component of the derived group G_{der} of G contains a k-subgroup scheme isomorphic to $\mathbb{G}_m$.

- Asok-Hoyois-Wendt (2015) - Let k be an infinite field. If G satisfies $(*)$, then the map

$$pr^* : H^1_{\mathrm{Zar}}(U, G) \to H^1_{\mathrm{Zar}}(U \times \mathbb{A}^1, G)$$

  is a bijection, for every smooth affine scheme U over k.
- The case when U is the spectrum of a field was proved by Raghunathan (1989).
- Raghunathan also obtained counterexamples when G is anisotropic, absolutely almost simple, not of type F_4 or G_2 and such that in the central isogeny class of G there exists a group G' embedded in a reductive group H whose underlying variety is rational such that H/G' is a torus.

Main theorem

Theorem (Joint with Chetan Balwe, 2016)

Let k be an infinite perfect field and let G be a reductive algebraic group over k, which does not satisfy the isotropy hypothesis $(*)$. Then the natural map

$$H^1_{\mathrm{Zar}}(U, G) \to H^1_{\mathrm{Zar}}(U \times \mathbb{A}^1, G)$$

cannot be a bijection for all smooth affine schemes U over k.

- Grothendieck-Serre conjecture (Colliot-Thélène-Ojanguren, Raghunathan, Fedorov-Panin) $\Rightarrow$ We can replace Zar by Nis.
- Asok-Hoyois-Wendt: affine homotopy invariance of $H^1_{\mathrm{Nis}}(-, G) \Rightarrow \mathrm{Sing}_*^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-local.
- We will show that if G does not satisfy $(*)$, then $\mathrm{Sing}_*^{\mathbb{A}^1} G$ is not $\mathbb{A}^1$-local.

$\mathbb{A}^1$-homotopy theory

- $\mathbb{A}^1$-homotopy theory of schemes was developed by Morel and Voevodsky in the 1990's in which the role of the unit interval $[0, 1]$ in homotopy theory is played by the affine line $\mathbb{A}^1$.
- One enlarges the category Sm/k of smooth schemes over a field k to $\triangle^{op}Sh(Sm/k)$, whose objects are simplicial sheaves of sets over the big Nisnevich site $(Sm/k)_{\mathrm{Nis}}$.
- Localizing $\triangle^{op}Sh(Sm/k)$ at stalk-wise weak equivalences yields the simplicial homotopy category $\mathcal{H}_s(k)$.
- Its Bousfield localization with respect to the collection of maps of the form $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ is called the $\mathbb{A}^1$-homotopy category over k and is denoted by $\mathcal{H}(k)$.

The $\mathrm{Sing}_*^{\mathbb{A}^1}$ construction

For a scheme X over k, define $\mathrm{Sing}_*^{\mathbb{A}^1} X$ to be the simplicial sheaf given by

$$(\mathrm{Sing}_*^{\mathbb{A}^1} X)_n(U) = \mathrm{Hom}(U \times \mathbb{A}^n, X).$$

The "simplicial data" is given by the face and degeneracy maps on

$$\mathbb{A}^n \simeq \Delta_n = \mathrm{Spec}\left(\frac{k[x_0, \ldots, x_n]}{(\sum_i x_i = 1)}\right).$$

There is a canonical map

$$X \to \mathrm{Sing}_*^{\mathbb{A}^1} X.$$

$\mathbb{A}^1$-connected components

Let X be a scheme over k.

- The sheaf of $\mathbb{A}^1$-chain connected components of X is defined by

$$\mathcal{S}(X) := a_{\mathrm{Nis}}\left(U \mapsto \pi_0(\mathrm{Sing}_*^{\mathbb{A}^1} X(U))\right) = a_{\mathrm{Nis}}\left(U \mapsto X(U)/\sim\right),$$

  where $\sim$ denotes the equivalence relation generated by naive $\mathbb{A}^1$-homotopies.

- The sheaf of $\mathbb{A}^1$-connected components of X is defined by

$$\pi_0^{\mathbb{A}^1}(X) := a_{\mathrm{Nis}}\left(U \mapsto \pi_0(L_{\mathbb{A}^1} X(U))\right) = a_{\mathrm{Nis}}\left(U \mapsto \mathrm{Hom}_{\mathcal{H}(k)}(U, X)\right),$$

  where $L_{\mathbb{A}^1}$ denotes the $\mathbb{A}^1$-localization functor on $\mathcal{H}(k)$.

There are canonical epimorphisms $X \twoheadrightarrow \mathcal{S}(X) \twoheadrightarrow \pi_0^{\mathbb{A}^1}(X)$.

$\mathbb{A}^1$-locality

A simplicial sheaf $\mathcal{X}$ on Sm/k is said to be $\mathbb{A}^1$-local if for any $U \in Sm/k$, the projection map $U \times \mathbb{A}^1 \to U$ induces a bijection

$$\mathrm{Hom}_{\mathcal{H}_s(k)}(U, \mathcal{X}) \to \mathrm{Hom}_{\mathcal{H}_s(k)}(U \times \mathbb{A}^1, \mathcal{X}).$$

- Examples of $\mathbb{A}^1$-local objects: $\mathbb{G}_m$, algebraic tori, abelian varieties, curves of genus ≥ 1, $\mathrm{Sing}_*^{\mathbb{A}^1} G$ if G satisfies $(*)$ etc.
- If $\mathrm{Sing}_*^{\mathbb{A}^1} X$ is $\mathbb{A}^1$-local, then the natural map

$$\mathcal{S}(X) \to \pi_0^{\mathbb{A}^1}(X)$$

  is an isomorphism.
- However, $\mathrm{Sing}_*^{\mathbb{A}^1} X$ is not $\mathbb{A}^1$-local in general.

A method of showing non-$\mathbb{A}^1$-locality of $\mathrm{Sing}_*^{\mathbb{A}^1} X$

One can iterate the $\mathcal{S}$-construction infinitely often:

$$X \twoheadrightarrow \mathcal{S}(X) \twoheadrightarrow \mathcal{S}^2(X) \twoheadrightarrow \cdots \twoheadrightarrow \varinjlim_n \mathcal{S}^n(X).$$

- The sheaf $\varinjlim_n \mathcal{S}^n(X)$ is always $\mathbb{A}^1$-invariant.
- If $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant, then $\pi_0^{\mathbb{A}^1}(X) \simeq \varinjlim_n \mathcal{S}^n(X)$.
- For an algebraic group G, we know that $\pi_0^{\mathbb{A}^1}(G)$ is $\mathbb{A}^1$-invariant Choudhury (2014).
- If we show that $\mathcal{S}(G) \twoheadrightarrow \mathcal{S}^2(G)$ is not an isomorphism, then we can conclude that $\mathrm{Sing}_*^{\mathbb{A}^1} G$ is not $\mathbb{A}^1$-local.

Reductive groups

- All algebraic groups will be assumed to be connected over k.
- Every reductive group G over k admits a central isogeny

$$G_{\mathrm{der}} \times \mathrm{rad}(G) \to G,$$

  where G_{der} is a semisimple group, called the derived group of G and $\mathrm{rad}(G)$ is a torus, called the radical of G.
- There exist almost k-simple groups $G_1, \ldots, G_n$ and a central isogeny

$$G_1 \times \cdots \times G_n \to G_{\mathrm{der}}.$$

- An algebraic group is said to be almost k-simple if it is smooth and admits no infinite normal k-subgroup.

Strategy to prove the main theorem

Theorem

Let k be an infinite perfect field and let G be a reductive algebraic group over k, which does not satisfy the isotropy hypothesis $(*)$. Then $\mathrm{Sing}_*^{\mathbb{A}^1} G$ is not $\mathbb{A}^1$-local.

$(*)$ Every almost k-simple component of the derived group G_{der} of G contains a k-subgroup scheme isomorphic to $\mathbb{G}_m$.

- Step 1. Prove the theorem for semisimple, simply connected, almost k-simple and anisotropic groups.
- Step 2. Show that if we have a central isogeny $G' \to G$ of reductive groups and if $\mathrm{Sing}_*^{\mathbb{A}^1} G'$ is not $\mathbb{A}^1$-local, then $\mathrm{Sing}_*^{\mathbb{A}^1} G$ is not $\mathbb{A}^1$-local.

Ingredients in the proof of Step 1

Theorem (Borel-Tits)

Let G be a smooth affine group scheme over a perfect field k. Then the following are equivalent:
(1) G admits no k-subgroup isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$.
(2) G admits a G-equivariant compactification $\bar{G}$ such that $G(k) = \bar{G}(k)$.

- As a consequence, anisotropic groups admit no nonconstant maps from $\mathbb{A}^1$.
- In other words, we have $\mathcal{S}(G)(k) = G(k)$, for anisotropic G.
- Therefore, it suffices to obtain a pair of distinct k-points of G that map to the same element in $\mathcal{S}^2(G)(k)$.

Ingredients in the proof of Step 1

Let F/k be a field extension.
Let $G(F)^+ :=$ subgroup of $G(F)$ generated by the subsets $U(F)$, where U varies over all F-subgroups of G which are isomorphic to the additive group $\mathbb{G}_a$.
The group

$$W(F,G) := G(F)/G(F)^+$$

is called the Whitehead group of G over F.

Theorem (Monastyrnii-Platonov-Yanchevskii, P. Gille)

Suppose that G is a semisimple, simply connected, almost k-simple and isotropic group over k. Then $W(k,G) \simeq W(k(t),G)$.

Proof of Step 1

Recall: We want to show that $G(k) \simeq \mathcal{S}(G)(k) \to \mathcal{S}^2(G)(k)$ is not an isomorphism.

- Grothendieck: Every reductive group over a perfect field is unirational; that is, admits a dominant rational map from a projective space.
- Therefore, there exist two distinct points $x, y \in G(k)$ that are R-equivalent, that is, there exists a rational map $h : \mathbb{A}^1 \dashrightarrow X$ such that $h(0) = x$ and $h(1) = y$.
- Let U be the largest open subscheme of $\mathbb{A}^1$ on which h is defined; let $\mathbb{A}^1 \setminus U = \{p_1, \ldots, p_n\}$.
- Let $k_i := k(p_i)$. Borel-Tits $\Rightarrow$ G_{k_i} is isotropic, for every i. Reason: Because of the compactification $G \hookrightarrow \bar{G}$ such that $G(k) = \bar{G}(k)$, every rational map $h : \mathbb{P}^1 \dashrightarrow G$ is defined at all the k-rational points.

Proof of Step 1, continued...

- Consider the following Nisnevich cover of $\mathbb{A}^1$:

$$\begin{tikzcd} & \coprod_i \mathbb{A}^1_{k_i} \arrow[d] \\ U \arrow[r] & \mathbb{A}^1_k \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r] & \mathcal{S}(G) \end{tikzcd}$$

- In order to show that x and y map to the same element in $\mathcal{S}^2(G)(k)$, that is, "connected" by an element of $\mathcal{S}(G)(\mathbb{A}^1_k)$, we need to define maps $U \to \mathcal{S}(G)$ and $V := \coprod_i \mathbb{A}^1_{k_i} \to \mathcal{S}(G)$, which agree after restriction to $U \times_{\mathbb{A}^1} V$.
- It suffices to define maps $U \to G$ and $V \to G$ that are naively $\mathbb{A}^1$-homotopic after restriction to $U \times_{\mathbb{A}^1} V$.

Proof of Step 1, continued...

$$\begin{tikzcd} & V = \coprod_i \mathbb{A}^1_{k_i} \arrow[d] & \\ U \arrow[r] & \mathbb{A}^1_k & G \end{tikzcd}$$

- The map $U \to G$ is just h. Since h is a rational map $\mathbb{P}^1 \dashrightarrow G$, it corresponds to a $k(t)$-valued point of G, which we call η.
- Since G_{k_i} is isotropic, we have $W(k_i, G) \simeq W(k_i(t), G)$. Thus, the image of η in $G(k_i(t))$ can be connected to a k_i-rational point, say q_i, by naive $\mathbb{A}^1$-homotopies. We choose the map $V \to G$ to be the disjoint unioin of constant maps $\coprod_i q_i$.
- After replacing V by a suitable open subscheme, we can ensure that the two restrictions to $U \times_{\mathbb{A}^1} V$ are naively $\mathbb{A}^1$-homotopic.

This completes the proof of Step 1.

Ingredients in the proof of Step 2

Classifying spaces of groups of multiplicative type: $B_{fppf}G$.

- For a group sheaf G, we will denote by BG the pointed simplicial sheaf on the big *fppf* site of schemes over k, whose n-simplices are G^n with usual face and degeneracy maps.
- We define $B_{fppf}G$ to be the simplicial Nisnevich sheaf of sets on Sm/k defined by

$$B_{\mathrm{fppf}}(G) := i^* \mathbf{R}_{\mathrm{fppf}}(BG),$$

where $\mathbf{R}_{\mathrm{fppf}}$ denotes the fibrant replacement functor for the Čech injective *fppf*-local model structure and $i : Sm/k \to Sch/k$ denotes the inclusion functor.

Lemma

Let G be an algebraic group over a field k of multiplicative type. Then $B_{\mathrm{fppf}}G$ is $\mathbb{A}^1$-local.

Proof of Step 2

Let $G' \to G$ be a central isogeny of reductive groups with kernel μ. Since the center of a reductive group is of multiplicative type, it follows that $B_{\mathrm{fppf}}\mu$ is $\mathbb{A}^1$-local.
Facts:

- Classification of *fppf*-locally trivial torsors:

$$\pi_0(B_{\mathrm{fppf}}\mu)(-) = H^1_{\mathrm{fppf}}(-, \mu).$$

- The simplicial fiber sequence

$$G' \to G \to B_{\mathrm{fppf}}\mu$$

  is an $\mathbb{A}^1$-fiber sequence.

Proof of Step 2, continued...

- Suppose, if possible, that $\mathrm{Sing}_*^{\mathbb{A}^1} G$ is $\mathbb{A}^1$-local.
- The long exact sequence of homotopy groups associated to the fiber sequence $G' \to G \to B_{\mathrm{fppf}}\mu$ gives us the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\pi_{i+1}^s(\mathrm{Sing}_*^{\mathbb{A}^1} G) & \longrightarrow & \pi_{i+1}^s(\mathrm{Sing}_*^{\mathbb{A}^1} B_{\mathrm{fppf}}\mu) & \longrightarrow & \pi_i^s(\mathrm{Sing}_*^{\mathbb{A}^1} G') & \longrightarrow & \pi_i^s(\mathrm{Sing}_*^{\mathbb{A}^1} G) & \longrightarrow & \pi_i^s(\mathrm{Sing}_*^{\mathbb{A}^1} B_{\mathrm{fppf}}\mu) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
\pi_{i+1}^{\mathbb{A}^1}(G) & \longrightarrow & \pi_{i+1}^{\mathbb{A}^1}(B_{\mathrm{fppf}}\mu) & \longrightarrow & \pi_i^{\mathbb{A}^1}(G') & \longrightarrow & \pi_i^{\mathbb{A}^1}(G) & \longrightarrow & \pi_i^{\mathbb{A}^1}(B_{\mathrm{fppf}}\mu).
\end{array}$$

- By the five lemma and the $\mathbb{A}^1$-Whitehead theorem, this implies that $\mathrm{Sing}_*^{\mathbb{A}^1} G'$ is $\mathbb{A}^1$-local, a contradiction.

Some remarks

- The sheaf $\pi_0^{\mathbb{A}^1}(G)$ of $\mathbb{A}^1$-connected components of an algebraic group G is closely related to the group of R-equivalence classes of G.
- (Balwe - S , 2015) If G is a semisimple, simply connected, almost k-simple algebraic group, then we have $\pi_0^{\mathbb{A}^1}(G)(k) \simeq G(k)/R$.
- (Balwe - S , 2016) This is no longer true if you drop the hypothesis of simple-connectedness. However, $G(k)/R$ is always a sub-quotient of $\pi_0^{\mathbb{A}^1}(G)(k)$, for any reductive group G over k.
- We expect this fact to be helpful in answering questions about $G(k)/R$; in particular, about its abelian-ness.

Thank you very much!