ON THE LAGRANGIAN STRUCTURE
OF QUANTUM FLUID MODELS

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ABSTRACT. Some quantum fluid models are written as the Lagrangian flow of mass distributions and their geometric properties are explored. The first model includes magnetic effects and leads, via the Madelung transform, to the electromagnetic Schrödinger equation in the Madelung representation. It is shown that the Madelung transform is a symplectic map between Hamiltonian systems. The second model is obtained from the Euler-Lagrange equations with friction induced from a quadratic dissipative potential. This model corresponds to the quantum Navier-Stokes equations with density-dependent viscosity. The fact that this model possesses two different energy-dissipation identities is explained by the definition of the Noether currents.

1. INTRODUCTION

This work is concerned with the derivation of known and new quantum fluid models using a Lagrangian method on the space of mass distributions (or probability measures). The Lagrangian representation of the Schrödinger equation in the Madelung picture is well known in the literature. In fact, Dirac presented already in 1933 the Lagrangian approach as an alternative formulation of the Hamiltonian theory in quantum mechanics. He expressed the Schrödinger equation as a critical point of a suitable action functional [6]. Feynman developed in the 1940s the path-integral formulation extending the principle of least action to quantum mechanics [10]. Later, Schrödinger's equation was derived from Newton's third law using Nelson's stochastic mechanics [23], which has been put into the mathematical framework of stochastic processes by Lafferty [18, Corollary 2.8]. The Schrödinger equation in its Madelung representation was shown in [26] to be a lift of Newton's law using Otto's Riemannian calculus for optimal transportation of probability measures. In this paper, we will extend this approach in two ways.

Before we explain our main results, we recall some basic elements of classical Lagrangian mechanics. The motion of a particle system in $\mathbb{R}^d$ ($d \geq 1$) is described by the trajectory $q(t)$ in the configuration space $M \subset \mathbb{R}^d$, where M is a manifold, with the velocity $\dot{q}(t)$. The Lagrangian $L(q, \dot{q})$ defines the dynamics of the system. For example, $L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \Phi(x)$ is the difference of the kinetic and potential energies for some given potential $\Phi : \mathbb{R}^d \to \mathbb{R}$. The variable q is

1

an element of M, whereas $\dot{q}$ lies in the tangent space $T_q M$. Hence, the Lagrangian is defined on the tangent bundle $TM = \{(q, \dot{q}) : \dot{q} \in T_q M\}$. We refer to, e.g., [11, 28] for details of geometric mechanics. The equations of motion are obtained from the principle of least action by calculating the critical points of the action functional

$$A(x) = \int_0^T L(x(t), \dot{x}(t))dt.$$

Criticality of the curve $\gamma : [0, T] \to \mathbb{R}^d$ is (formally) equivalent to the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(\gamma, \dot{\gamma}) - \frac{\partial L}{\partial x}(\gamma, \dot{\gamma}) = 0, \quad t \in (0, T).$$

Friction can be included by means of a dissipative potential $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ (see, e.g., [7]):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(\gamma, \dot{\gamma}) - \frac{\partial L}{\partial x}(\gamma, \dot{\gamma}) + \frac{\partial D}{\partial \dot{x}}(\gamma, \dot{\gamma}) = 0 \quad t \in (0, T).$$

An example is linear friction which is given by the quadratic potential $D(x, \dot{x}) = \alpha|\dot{x}|^2$ with $\alpha > 0$.

Following [18, 26], we consider in this work *a lift of this formalism on the space of probability measures* and *derive novel Navier-Stokes equations with quantum corrections*. We will recall the basic setup in Section 2; here we sketch only our main results.

First, we propose a lifted Lagrangian, defined on the tangent bundle of the set of probability measures, including the magnetic vector potential A and the Fisher information (defined in (6) below). Then the Euler-Lagrange equations are given by the continuity equation for the particle density μ and the Hamilton-Jacobi equation for the velocity potential S (see Theorem 1)

$$\partial_t \mu + \mathrm{div}\,(\mu(\nabla S - A)) = 0,$$

$$\partial_t S + \frac{1}{2}|\nabla S - A|^2 + \Phi(x) - \frac{\hbar^2}{2}\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} = 0 \quad \text{in } \mathbb{R}^d, \ t > 0,$$

where, with a slight abuse of notation, $\hbar$ is the *scaled* Planck constant. Introduce the wave function $\Psi = \sqrt{\mu}\exp(iS/\hbar)$ via the so-called Madelung transform, for smooth solutions (μ, S) with positive density (or mass distribution) μ. Then Ψ solves the magnetic Schrödinger equation

$$i\hbar\partial_t\Psi = \frac{1}{2}\left(\frac{\hbar}{i}\nabla - A\right)^2 \Psi + \Phi(x)\Psi \quad \text{in } \mathbb{R}^d, \ t > 0.$$

We give a systematic analysis of the Madelung transform as a symplectic map between Hamiltonian systems, preserving the magnetic Schrödinger Hamiltonian (see Theorem 10).

Second, we show that the lifted Euler-Lagrange equation with linear friction leads to the quantum Navier-Stokes equations. After identifying vector fields modulo rotational components, these equations read as (see Theorem 11)

$$(1) \qquad \partial_t\mu + \mathrm{div}(\mu v) = 0,$$

$$(2) \qquad \partial_t(\mu v) + \mathrm{div}(\mu v \otimes v) + \nabla p(\mu) + \mu\nabla\Phi(x) - \frac{\hbar^2}{2}\mu\nabla\left(\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}}\right) = \alpha\,\mathrm{div}(\mu D(v)),$$

where the velocity is given by $v = \nabla S$, $v \otimes v$ is a matrix with components $v_j v_k$, $p(\mu)$ is the pressure, and $D(v) = \frac{1}{2}(\nabla v + \nabla v^\top) = \nabla v$ is the symmetric velocity gradient. This system was first derived by Brull and Méhats [4] from the Wigner-BGK equation (named after Bhatnagar, Gross, and Krook) using a Chapman-Enskog expansion. An alternative derivation from the Wigner-Fokker-Planck model by just applying a moment method was proposed in [16]. For systems including the energy equation, we refer to [16, 17]. Our approach yields a third way to derive the quantum Navier-Stokes equations. An advantage of our method is that we can propose more general friction terms, leading to a variety of nonlinear viscosities (see Remark 12). The selection of quantum mechanically correct dissipation terms remains a research topic for the future (see [1] for a Lindblad equation approach).

Surprisingly, system (1)-(2) allows for *two* different energies, as observed in [15]. Indeed, a formal computation shows that the Hamiltonian

$$\mathscr{H}_Q = \int_{\mathbb{R}^d} \left(\frac{1}{2}|v|^2 + U(\mu) + \Phi(x) + \frac{\hbar^2}{8}|\nabla \log \mu|^2 \right) \mu \, dx$$

is a Lyapunov functional along the solutions to (1)-(2), see Proposition 13. Here, the internal energy U relates to the pressure p by $p'(s) = sU''(s)$, $s > 0$. Furthermore, the energy

$$\mathscr{H}_Q^* = \int_{\mathbb{R}^d} \left(\frac{1}{2}|v + v_{\mathrm{os}}|^2 + U(\mu) + \Phi(x) + \left(\frac{\hbar^2}{8} - \alpha^2 \right)|\nabla \log \mu|^2 \right) \mu \, dx,$$

where $v_{\mathrm{os}} = \alpha \nabla \log \mu$ is the osmotic velocity, is another Lyapunov functional. We will explain this fact by a variant of the Noether theory. Indeed, time invariance of the system leads to dissipation of the Hamiltonian $\mathscr{H}_Q$ (since we have friction, the energy is not a constant of motion). Interestingly, a special transformation of the variables (t, μ) leads to a Noether current which equals $\mathscr{H}_Q^*$ (see Theorem 15). Thus, the existence of the second energy functional is a consequence of a "Noether symmetry", showing that the quantum Navier-Stokes equations exhibit a certain geometric structure.

The originality of the present work is twofold. First, we exploit the Lagrangian approach on the space of probability measures in a systematic way and show its flexibility by deriving various model equations. Second, we suggest an alternative way to include dissipative effects in quantum models by using Euler-Lagrange equations with friction. The calculations are formal but they can be made rigorous under suitable regularity assumptions, as pointed out in [18]. In particular, we provide a consistent extension not only of classical mechanics but also of optimal transport theory towards quantum mechanics, which related to the Lagrangian formulation in Bohmian mechanics, cf. Markowich et al. [21].

The paper is organized as follows. The basic setup of Lagrangian mechanics on the set of probability measures is introduced in Section 2. The following sections are concerned with three applications of the Lagrangian method. For the particle motion in a potential field, we recover the usual flow equations, showing that our approach includes the classical case (Section 3). The Euler-Lagrange equation for a charged particle in a magnetic field is computed in Section 4, and the symplectic structure of the flow equations is analyzed. Section 5 is devoted to the derivation of the quantum Navier-Stokes equations and the relation between energy functionals and the Noether theory.

2. Basic Setup

In this section, we extend the classical Lagrangian mechanics to a configuration space consisting of probability measures. A similar approach is contained in the work of Lafferty [18]. We recall the definition of the phase space, introduce the Lagrangians considered in this paper, and formulate the (dissipative) Euler-Lagrange equations.

2.1. **Phase space.** Let $\mathscr{P}(\mathbb{R}^d)$ $(d \geq 1)$ be the set of probability measures on $\mathbb{R}^d$. Obviously, the space $\mathbb{R}^d$ is embedded in $\mathscr{P}(\mathbb{R}^d)$ via the Dirac masses $x \mapsto \delta_x$. A physical interpretation of $\mu \in \mathbb{R}^d$ is that μ represents a (possibly diffuse) distribution of mass with fixed total amount. The following arguments may be made rigorous on the set $\mathscr{P}^\infty(\mathbb{R}^d)$ of absolutely continuous probability measures with smooth positive density and finite exponential moments, as pointed out by Lott [19]. However, similarly to the previous works [19, 24, 25, 26], we shall not try to find the maximal subset of $\mathscr{P}(\mathbb{R}^d)$ on which our formulas remain valid, and therefore, we assume that the measures $\mu \in \mathscr{P}(\mathbb{R}^d)$ are sufficiently smooth for the formulas to hold. In the following, we often identify the measure $\mu \in \mathscr{P}(\mathbb{R}^d)$ with its density $d\mu/dx \simeq \mu$ and we write $\mathscr{P}$ instead of $\mathscr{P}(\mathbb{R}^d)$.

Given $\mu \in \mathscr{P}$ we introduce the tangent space of $\mathscr{P}$ at μ by

$$T_\mu \mathscr{P} = \{\eta \in \mathscr{S}'(\mathbb{R}^d) : \exists v \in C^\infty(\mathbb{R}^d; \mathbb{R}^d),\ \eta + \operatorname{div}(\mu v) = 0\},$$

where $\mathscr{S}'(\mathbb{R}^d)$ is the dual of the Schwartz space, which is the collection of infinitesimal variations of μ by smooth flows. The tangent bundle

$$T\mathscr{P} = \bigcup_{\mu \in \mathscr{P}} T_\mu \mathscr{P}$$

serves as the physical phase space for our Lagrangian mechanics of mass distributions. We remark that the motion of a single particle with velocity u is included in our formalism by means of the representation $\eta = -\operatorname{div}(\delta_x v)$, where v is any vector field on $\mathbb{R}^d$ satisfying $v(x) = u$. We also notice that in Hamiltonian mechanics, the phase space is defined by the pairs of generalized coordinates in $T\mathscr{P}$ and generalized momenta in the dual space $T^*\mathscr{P}$. We refer to Section 4.2 for details.

2.2. **Lagrangians.** A function $\mathscr{L} : T\mathscr{P} \to \mathbb{R}$ is called a Lagrangian. Below, we shall mostly be concerned with Lagrangians $\mathscr{L}$, which are obtained as lifts from classic Lagrange functions $L : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, defined by

$$(3) \qquad \mathscr{L}(\mu, \eta) = \inf \left\{ \int_{\mathbb{R}^d} L(x, v(x)) \mu(dx) : v \in C^\infty(\mathbb{R}^d; \mathbb{R}^d),\ \eta + \operatorname{div}(\mu v) = 0 \right\},$$

where $\mu \in \mathscr{P}$ and $\eta \in T_\mu \mathscr{P}$. The infimum is necessary since the map $v \mapsto -\operatorname{div}(\mu v) \in T_\mu \mathscr{P}$ is generally not injective. We prefer the notation $\mathscr{L}(\mu, \eta)$ instead of the simpler (and geometrically more consistent) notation $\mathscr{L}(\eta)$ in order to emphasize the importance of the referring base point for η in $T_\mu \mathscr{P}$. Notice that the classical case is embedded in this situation since

$$\mathscr{L}(\delta_x, -\operatorname{div}(\delta_x v)) = \int_{\mathbb{R}^d} L(x, v(x)) \delta(dx) = L(x, v(x)).$$

We present some examples studied in this paper.

2.2.1. *Single-particle dynamics.* The kinetic energy $L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2$ is well known from optimal transport theory [2, 3, 24]. A standard duality argument shows that the infimum in (3) is attained. Indeed, we compute formally, for $\mu \in \mathscr{P}$ and $\eta \in T_\mu \mathscr{P}$:

$$\mathscr{L}(\mu, \eta) = \inf_v \sup_\chi \int_{\mathbb{R}^d} \left(\frac{1}{2}|v|^2 + \eta\chi - v \cdot \nabla\chi \right) \mu(dx)$$

$$= \sup_\chi \inf_v \int_{\mathbb{R}^d} \left(\frac{1}{2}|v - \nabla\chi|^2 - \frac{1}{2}|\nabla\chi|^2 + \eta\chi \right) \mu(dx).$$

The infimum is realized at $v = \nabla\chi$:

$$\mathscr{L}^* = \mathscr{L}(\mu, \eta) = \sup_\chi \int_{\mathbb{R}^d} \left(\eta\chi - \frac{1}{2}|\nabla\chi|^2 \right) \mu(dx).$$

Defining $S = \text{argsup}\,\mathscr{L}^*$ and inserting $v = \nabla S, \chi = S$ into $\mathscr{L}$, we find that

$$\mathscr{L}(\mu, \eta) = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla S|^2 \mu(dx).$$

We recall that $S : \mathbb{R}^d \to \mathbb{R}$ is the (up to constants) unique solution to $-\text{div}(\mu\nabla S) = \eta$ in $\mathbb{R}^d$. The function S is called the velocity potential of the variation η with respect to the state μ. We introduce the notation

(4) $$\Delta_\mu S = \text{div}(\mu\nabla S) \quad \text{in } \mathbb{R}^d.$$

The minimizer defines a quadratic form on the tangent space $T_\mu \mathscr{P}$:

$$\|\eta\|^2_{T_\mu \mathscr{P}} = \int_{\mathbb{R}^d} |\nabla S(x)|^2 \mu(dx).$$

This is Otto's Riemannian (weighted $H^{-1}(\mathbb{R}^d)$) tensor on $T\mathscr{P}$ inducing the L^2-Wasserstein metric on $\mathscr{P}$ as an intrinsic distance [24] and to the square of the Kantorovich distance [3, Prop. 1.1] (also see [22, Theorem 9]).

2.2.2. *Charged particles in a magnetic field.* The Lagrange function $L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 + \dot{q} \cdot A - \Phi(x)$ models the motion of a charged particle in a magnetic field, where $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic vector potential [28, Section 12.6] and $\Phi : \mathbb{R}^d \to \mathbb{R}$ is the electric potential. By a similar computation as in the previous example, for $\mu \in \mathscr{P}$ and $\eta \in T_\mu \mathscr{P}$,

$$\mathscr{L}(\mu, \eta) = \sup_\chi \inf_v \int_{\mathbb{R}^d} \left(\frac{1}{2}|v + (A - \nabla\chi)|^2 - \frac{1}{2}|A - \nabla\chi|^2 + \eta\chi - \Phi(x) \right) \mu(dx).$$

Then, taking $v^* = \nabla\chi - A$ to realize the infimum and $S = \text{argsup}\,\mathscr{L}, \chi = S$, it holds that

$$\mathscr{L}(\mu, \eta) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla S - A|^2 + (\nabla S - A) \cdot A - \Phi(x) \right) \mu(dx)$$

(5) $$= \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla S|^2 - \frac{1}{2}|A|^2 - \Phi(x) \right) \mu(dx),$$

and $S : \mathbb{R}^d \to \mathbb{R}$ is the (up to constants) unique solution to

$$\eta = -\operatorname{div}(\mu v^*) = -\operatorname{div}(\mu(\nabla S - A)) = -\operatorname{div}(\mu \nabla S) + \operatorname{div}(\mu A) \quad \text{in } \mathbb{R}^d.$$

With the notation (4), we have $S = -\Delta_\mu^{-1}(\eta - \operatorname{div}(\mu A))$ in $\mathbb{R}^d$.

2.2.3. *Charged quantum particles.* Substracting from the kinetic energy of the previous example the Fisher information $I(\mu)$, defined by

$$\tag{6} I(\mu) = \int_{\mathbb{R}^d} |\nabla \log \mu|^2 \mu(dx),$$

the lifted Lagrangian

$$\tag{7} \mathscr{L}(\mu, \eta) = \|\eta\|^2_{T_\mu \mathscr{P}} - V(\mu) - \frac{\hbar^2}{8} I(\mu)$$

was considered by Lafferty [18] and von Renesse [26] to formulate the Schrödinger equation by means of the Madelung equations. We remark that Feng and Nguyen [9] employed $-I(\mu)$ instead of $I(\mu)$ to derive compressible Euler-type equations from minimizers of an action functional defined on probability measure-valued paths. One may augment $\mathscr{L}$ also by the internal energy term

$$\tag{8} -\int_{\mathbb{R}^d} U(\mu)\mu(dx),$$

where $U : \mathbb{R} \to \mathbb{R}$ is the (smooth) internal energy potential.

2.3. **Smooth curves in $\mathscr{P}$.** Let $\mu : [0, T] \to \mathscr{P}$ be a smooth curve, i.e., its time derivative $\dot{\mu}_t := \partial_t \mu(t)$ exists in the distributional sense and $\dot{\mu}_t \in T_{\mu_t} \mathscr{P}$ for all $t \in [0, T]$. For instance, $\dot{\mu}_t \in \mathscr{S}'(\mathbb{R}^d)$ may be defined for each $t \in [0, T]$ by

$$\partial_t \langle \mu_t, \xi \rangle = \langle \dot{\mu}_t, \xi \rangle \quad \text{for all } \xi \in \mathscr{S}(\mathbb{R}^d),$$

where $\langle \cdot, \cdot \rangle$ is the dual product between $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d)$.

Let $\mu : [0, T] \to \mathscr{P}$ be a smooth curve. If $\dot{\mu}_t \in \mathscr{S}'(\mathbb{R}^d)$ is regular and $\mu_t \in \mathscr{P}^\infty(\mathbb{R}^d)$ (see Section 2.1 for the definition of $\mathscr{P}^\infty(\mathbb{R}^d)$), standard elliptic theory provides the existence of (up to an additive constant) unique smooth solution $S_t : \mathbb{R}^d \to \mathbb{R}$ to the problem

$$-\operatorname{div}(\mu_t \nabla S_t) = \dot{\mu}_t \quad \text{in } \mathbb{R}^d.$$

In particular, the curve $\dot{\mu} : (0, T) \to T\mathscr{P}$, $t \mapsto \eta_t := \dot{\mu}_t = -\operatorname{div}(\mu_t \nabla S_t)$ is well defined and, by definition of the tangent space, $\eta_t \in T_{\mu_t} \mathscr{P}$. Again, the single-particle motion $c : [0, T] \to \mathbb{R}^d$ is included by taking $\gamma_t = \delta_{c(t)}$ and $\eta_t = -\operatorname{div}(v_t \delta_{c(t)}) \in \mathscr{S}'(\mathbb{R}^d)$, where v_t is some vector field such that $v_t(x) = \dot{c}(t)$ for $x \in \mathbb{R}^d$.

2.4. **Action functional and critical points.** Given a Lagrangian $\mathscr{L}$ on $\mathscr{P}$ (see Section 2.2), we define the action functional on smooth curves $\gamma : [0, T] \to \mathscr{P}$ by

$$\mathscr{A}(\gamma) = \int_0^T \mathscr{L}(\gamma_t, \dot{\gamma}_t) dt.$$

A critical point of $\mathscr{A}$ is a curve γ which satisfies

$$\frac{d}{ds} \mathscr{A}(\gamma^s)\Big|_{s=0} = 0$$

for all smooth variations $\gamma : [-\varepsilon, \varepsilon] \times [0, T] \to \mathscr{P}$, $(s, t) \mapsto \gamma_t^s$, satisfying $\gamma_t^0 = \gamma_t$ for $t \in [0, T]$. Hence, assuming differentiability of $\mathscr{L}$, a curve is a critical point if and only if it satisfies the Euler-Lagrange equation

$$(9) \qquad \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \eta}(\gamma, \dot{\gamma}) - \frac{\partial \mathscr{L}}{\partial \mu}(\gamma, \dot{\gamma}) = 0.$$

A Lagrangian system on $\mathscr{P}$ with friction is modeled by means of a dissipative potential $\mathscr{D} : T\mathscr{P} \to \mathbb{R}$:

$$(10) \qquad \frac{d}{dt} \frac{\partial \mathscr{L}}{\partial \eta}(\gamma, \dot{\gamma}) - \frac{\partial \mathscr{L}}{\partial \mu}(\gamma, \dot{\gamma}) + \frac{\partial \mathscr{D}}{\partial \eta}(\gamma, \dot{\gamma}) = 0.$$

Renesse identified in [26] the flow (9), with $\mathscr{L}$ given by (7), with the Schrödinger equation in its Madelung representation. We extend this concept in the following sections for more general Lagrangians.

3. Example 1: Particle Motion in a Potential Field

We show that the formalism of Section 2 includes as a special case the motion of a single particle in a potential $\Phi(x)$. Indeed, choosing the Lagrangian as the lift of the classical Lagrangian $L(q, \dot{q}) = \frac{1}{2}|\dot{q}|^2 - \Phi(x)$, the arguments in Section 2.2 yield, for vector fields $v \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$,

$$\mathscr{L}(\delta_x, -\operatorname{div}(\delta_x v)) = L(x, v(x)) = \frac{1}{2}|v(x)|^2 - \Phi(x).$$

Elementary computations show that curves $\gamma_t = \delta_{x_t}$ with $\ddot{x}_t = -\nabla\Phi(x_t)$ are critical flows for the corresponding lifted action functional $\mathscr{A}$, i.e., γ_t is a critical point for $\mathscr{A}$ (see Section 2.4).

Clearly, the case of a collection of point masses moving in a joint potential is more interesting. When the particle system is coalescing (corresponding to inelastic particle collisions), the system may eventually collapse to single Dirac measures moving along a classical particle trajectory. This situation is described by the above Lagrangian. An example is the chemotactic movement of cells modeled by a Keller-Segel system, which may exhibit finite-time blow-up. After blow-up, collapsed parts seems to consist of evolving Dirac measures.

4. Example 2: The Magnetic Schrödinger Equation

We consider the motion of a charged quantum particle in a magnetic field with magnetic vector potential A. According to Section 2.2, the Lagrangian reads as

$$(11) \qquad \mathscr{L}_{\mathrm{M}}(\mu, \eta) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla S|^2 - \frac{1}{2} |A|^2 - \Phi(x) - \frac{\hbar^2}{8} |\nabla \log \mu|^2 \right) \mu(dx),$$

where $\mu \in \mathscr{P}$, $\eta \in T_\mu \mathscr{P}$, and $S = -\Delta_\mu^{-1}(\eta - \operatorname{div}(\mu A))$. The corresponding action functional becomes

$$(12) \qquad \mathscr{A}_{\mathrm{M}}(\gamma) = \int_0^T \mathscr{L}_{\mathrm{M}}(\gamma_t, \dot{\gamma}_t) dt,$$

where $\gamma : [0, T] \to \mathscr{P}$ is a smooth curve.

4.1. Magnetic Madelung equations.
We show that the critical points for $\mathscr{A}_{\mathrm{M}}$ solve Madelung-type and quantum hydrodynamic equations.

Theorem 1 (Magnetic Madelung equations). *A smooth curve $\mu : [0, T] \to \mathscr{P}$ is a critical point for $\mathscr{A}_{\mathrm{M}}$, i.e.*

$$(13) \qquad \frac{d}{dt} \frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial \eta} - \frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial \mu} = 0,$$

if and only if the flow of the generalized momenta $S_t : \mathbb{R}^d \to \mathbb{R}$, $t \in [0, T]$, of

$$(14) \qquad \partial_t \mu + \operatorname{div}\left(\mu(\nabla S - A) \right) = 0 \quad in\ \mathbb{R}^d$$

solves the Hamilton-Jacobi equation

$$(15) \qquad \partial_t S + \frac{1}{2} |\nabla S - A|^2 + \Phi(x) - \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} = 0 \quad in\ \mathbb{R}^d.$$

For the proof of the above theorem, we need an auxiliary result. Let denote

$$\mathscr{M} = \left\{ \xi \text{ smooth signed measure on } \mathbb{R}^d : \langle \xi, 1 \rangle = 0, \int_{\mathbb{R}^d} e^{\alpha |x|} |\xi|(dx) < \infty \text{ for all } \alpha > 0 \right\}$$

the set of smooth signed measures with zero mean and finite exponential absolute moments. Here, $\langle \cdot, \cdot \rangle$ denotes the dual product between the space of finitely additive measures on $\mathbb{R}^d$ and the space $L^\infty(\mathbb{R}^d)$. Then, for $\mu \in \mathscr{M}$ and $S \in \mathscr{S}(\mathbb{R}^d)$, the differential operator $\Delta_\mu(S) = \operatorname{div}(\mu \nabla S)$ is well defined. Furthermore, we write

$$\delta_* F(\mu, \xi) = \frac{d}{d\varepsilon} F(\mu + \varepsilon \xi) \Big|_{\varepsilon = 0}$$

for the first variation of F at μ in the direction of ξ. If $\delta_* F(\mu, \xi) = \int_{\mathbb{R}^d} G \xi dx$, we set $G = \partial F / \partial \mu$, the variational derivative of F with respect to μ.

Lemma 2. *For smooth measures $\mu \in \mathscr{P}$, the operator-valued functions $\mu \mapsto \Delta_\mu$ and $\mu \mapsto \Delta_\mu^{-1}$ are differentiable in the direction of $\xi \in \mathscr{M}$, and their first variations are given by*

$$\delta_* \Delta_{(\mu, \xi)} = \Delta_\xi, \quad \delta_* \Delta_{(\mu, \xi)}^{-1} = -\Delta_\mu^{-1} \Delta_\xi \Delta_\mu^{-1}.$$

Proof. The first claim follows from

$$\delta_* \Delta_{(\mu,\xi)}(S) = \frac{d}{d\varepsilon} \operatorname{div}\left((\mu + \varepsilon\xi)\nabla S\right)\Big|_{\varepsilon=0} = \operatorname{div}(\xi\nabla S) = \Delta_\xi S.$$

To prove the second claim, we notice that $\Delta_{(\mu,\xi)}\Delta_{(\mu,\xi)}^{-1}(S) = S$ implies, by the Leibniz rule, that

$$0 = \delta_*(\Delta_{(\mu,\xi)}\Delta_{(\mu,\xi)}^{-1})(S) = \delta_*\Delta_{(\mu,\xi)}(\Delta_\mu^{-1}S) + \Delta_\mu \delta_* \Delta_{(\mu,\xi)}^{-1}(S).$$

By the first claim, this can be written as

$$0 = \Delta_\xi \Delta_\mu^{-1} S + \Delta_\mu \delta_* \Delta_{(\mu,\xi)}^{-1}(S),$$

and multiplication by Δ_μ^{-1} from the left shows the result. $\square$

Proof of Theorem 1. The theorem is proved by calculating the derivatives in the Euler-Lagrange equation (13). To this aim, we set $\mathscr{L} = \mathscr{T} - \mathscr{V}$, where

$$\mathscr{T}(\mu,\eta) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla S|^2 \mu(dx) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla \Delta_\mu^{-1}(\eta - \operatorname{div}(\mu A))|^2 \mu(dx),$$

$$\mathscr{V}(\mu) = \int_{\mathbb{R}^d}\left(\frac{1}{2}|A|^2 + \Phi(x) + \frac{\hbar^2}{8}|\nabla \log \mu|^2\right)\mu(dx)$$

are the "kinetic energy" and "potential energy" terms. First, we find that, for fixed $\mu \in \mathscr{P}$ and for any $\xi \in \mathscr{M}$,

$$\delta_* \mathscr{T}(\eta,\xi) = \frac{1}{2}\frac{d}{d\varepsilon}\int_{\mathbb{R}^d} |\nabla \Delta_\mu^{-1}(\eta - \operatorname{div}(\mu A) + \varepsilon\xi)|^2 \mu(dx)\Big|_{\varepsilon=0}$$

$$= \int_{\mathbb{R}^d} \nabla \Delta_\mu^{-1}(\eta - \operatorname{div}(\mu A)) \cdot \nabla \Delta_\mu^{-1}\xi\mu(dx)$$

$$= -\int_{\mathbb{R}^d} \nabla S \cdot \nabla \Delta_\mu^{-1}\xi\mu(dx) = -\int_{\mathbb{R}^d} \mu\nabla S \cdot \nabla \Delta_\mu^{-1}\xi dx.$$

Then, by integrating by parts and using the definition of Δ_μ,

$$\delta_* \mathscr{T}(\eta,\xi) = \int_{\mathbb{R}^d} \Delta_\mu^{-1}\operatorname{div}(\mu\nabla S)\xi dx = \int_{\mathbb{R}^d} \Delta_\mu^{-1}(\Delta_\mu S)\xi dx = \int_{\mathbb{R}^d} S\xi dx,$$

showing that $\partial\mathscr{T}/\partial\eta = S$. The expression $\mathscr{V}$ does not depend on η, and hence, $\partial\mathscr{V}/\partial\eta = 0$. Thus,

(16) $$\frac{\partial\mathscr{L}_{\mathrm{M}}}{\partial\eta} = S.$$

Next, we compute $\partial\mathscr{T}/\partial\mu$. We observe that $\mathscr{T}$ can be reformulated as

$$\mathscr{T}(\mu,\eta) = \frac{1}{2}\int_{\mathbb{R}^d} \mu\nabla S \cdot \nabla S dx = -\frac{1}{2}\int_{\mathbb{R}^d} \operatorname{div}(\mu\nabla S)S dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}^d} S\Delta_\mu S dx = -\frac{1}{2}\int_{\mathbb{R}^d} (\eta - \operatorname{div}(\mu A))\Delta_\mu^{-1}(\eta - \operatorname{div}(\mu A))dx,$$

using $S = -\Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A))$. Hence, the first variation reads as

$$\delta_* \mathcal{T}(\mu, \xi) = -\frac{1}{2} \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} (\eta - \mathrm{div}((\mu + \varepsilon\xi)A)) \Delta_{\mu+\varepsilon\xi}^{-1} (\eta - \mathrm{div}((\mu + \varepsilon\xi)A)) dx \Big|_{\varepsilon=0}.$$

We employ the product rule and Lemma 2 to compute the first variation of Δ_μ^{-1}:

$$\delta_* \mathcal{T}(\mu, \xi) = \int_{\mathbb{R}^d} \mathrm{div}(\xi A) \Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A)) dx$$
$$+ \frac{1}{2} \int_{\mathbb{R}^d} (\eta - \mathrm{div}(\mu A)) \Delta_\mu^{-1} \Delta_\xi \Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A)) dx.$$

The first term becomes, after an integration by parts,

$$\int_{\mathbb{R}^d} \mathrm{div}(\xi A) \Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A)) dx = -\int_{\mathbb{R}^d} \mathrm{div}(\xi A) S \, dx = \int_{\mathbb{R}^d} (A \cdot \nabla S) \xi dx.$$

For the second term, we find that, by the definition of Δ_ξ,

$$\frac{1}{2} \int_{\mathbb{R}^d} (\eta - \mathrm{div}(\mu A)) \Delta_\mu^{-1} \Delta_\xi \Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A)) dx$$
$$= \frac{1}{2} \int_{\mathbb{R}^d} \Delta_\mu^{-1}(\eta - \mathrm{div}\,\mu A) \, \mathrm{div}\,(\xi \nabla \Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A))) dx$$
$$= -\frac{1}{2} \int_{\mathbb{R}^d} \nabla \Delta_\mu^{-1}(\eta - \mathrm{div}\,\mu A) \cdot (\xi \nabla \Delta_\mu^{-1}(\eta - \mathrm{div}(\mu A))) dx$$
$$= -\frac{1}{2} \int_{\mathbb{R}^d} |\nabla S|^2 \xi dx.$$

We conclude that

$$\delta_* \mathcal{T}(\mu, \xi) = \int_{\mathbb{R}^d} \left((A \cdot \nabla S) - \frac{1}{2} |\nabla S|^2 \right) \xi dx$$

and therefore, the variational derivative equals

(17)
$$\frac{\partial \mathcal{T}}{\partial \mu} = A \cdot \nabla S - \frac{1}{2} |\nabla S|^2.$$

It remains to calculate $\partial \mathcal{V}/\partial \mu$. The first two terms in the integral of $\mathcal{V}$ depend only linearly on μ which shows that

$$\frac{\partial}{\partial \mu} \int_{\mathbb{R}^d} \left(\frac{1}{2} |A|^2 + \Phi(x) \right) \mu dx = \frac{1}{2} |A|^2 + \Phi(x).$$

The first variation of the Fisher information becomes

$$\delta_* \left(\int_{\mathbb{R}^d} |\nabla \log \mu|^2 \mu dx \right)(\mu, \xi) = \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} |\nabla \log(\mu + \varepsilon \xi)|^2 (\mu + \varepsilon \xi) dx \Big|_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} |\nabla \log \mu|^2 (\mu + \varepsilon \xi) dx \Big|_{\varepsilon=0} + \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} |\nabla \log(\mu + \varepsilon \xi)|^2 \mu dx \Big|_{\varepsilon=0}$$

$$= \int_{\mathbb{R}^d} |\nabla \log \mu|^2 \xi dx + 2 \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} \nabla \log(\mu + \varepsilon \xi) \cdot \nabla(\log \mu) \mu dx \Big|_{\varepsilon=0}$$

$$= \int_{\mathbb{R}^d} \frac{|\nabla \mu|^2}{\mu^2} \xi dx + 2 \int_{\mathbb{R}^d} \nabla \frac{\xi}{\mu} \cdot \nabla \mu dx$$

$$= \int_{\mathbb{R}^d} \left(\frac{|\nabla \mu|^2}{\mu} - 2\Delta \mu \right) \frac{\xi}{\mu} dx = -4 \int_{\mathbb{R}^d} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \xi dx.$$

We infer that

$$\frac{\partial}{\partial \mu} \frac{\hbar^2}{8} \int_{\mathbb{R}^d} |\nabla \log \mu|^2 \mu dx = -\frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}}.$$

Summarizing, we conclude that

$$(18) \qquad \frac{\partial \mathscr{V}}{\partial \mu} = \frac{1}{2}|A|^2 + \Phi(x) - \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}}$$

and for the Lagrangian

$$\frac{\partial \mathscr{L}_{\mathrm{M}}}{\partial \mu} = A \cdot \nabla S - \frac{1}{2}|\nabla S|^2 - \frac{1}{2}|A|^2 - \Phi(x) + \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}}$$

$$= -\frac{1}{2}|\nabla S - A|^2 - \Phi(x) + \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}},$$

which finishes the proof. $\qquad \square$

We call (14)-(15) the magnetic Madelung equations. The expression $(\hbar^2/2)\Delta \sqrt{\mu}/\sqrt{\mu}$ is referred to as the Bohm potential. It is the quantum correction to the (magnetic) hydrodynamic equations. Via the Madelung transformation $\Psi = \sqrt{\mu} \exp(iS/\hbar)$, smooth solutions (μ, S) to (14)-(15) with initial data $\mu(\cdot, 0) = \mu_0$, $S(\cdot, 0) = S_0$ in $\mathbb{R}^d$ yield solutions to the magnetic Schrödinger equation

$$(19) \qquad i\hbar \partial_t \Psi = \frac{1}{2}\left(\frac{\hbar}{i}\nabla - A \right)^2 \Psi + \Phi(x)\Psi, \quad t > 0, \quad \Psi(\cdot, 0) = \sqrt{\mu_0} \exp(iS_0/\hbar) \quad \text{in } \mathbb{R}^d.$$

Remark 3. Taking the gradient of (15), multiplying the resulting equation by μ and employing (14) similarly as in the proof of Theorem 14.1 in [14], we find the quantum hydrodynamic

equations

$$\partial_t \mu + \mathrm{div}(\mu v) = 0,$$

$$\partial_t(\mu v) + \mathrm{div}(\mu v \otimes v) - \frac{\hbar^2}{2}\mu\nabla\left(\frac{\Delta\sqrt{\mu}}{\sqrt{\mu}}\right) + \mu\nabla\Phi(x) = 0, \quad t > 0,$$

$$\mu(\cdot, 0) = \mu_0, \quad (\mu v)(\cdot, 0) = \mu_0(\nabla S_0 - A) \quad \text{in } \mathbb{R}^d,$$

where $v = \nabla S - A$ and $v \otimes v$ denotes the matrix with components $v_j v_k$. Here, we have used the fact that A does not depend on time. Thus the dynamics of a charged particle in an electromagnetic field is formally the same as that of a charged particle in an electric field, with different initial conditions and a different velocity function v.   □

Remark 4. Including the internal energy (8) into the Lagrangian (5), without magnetic field,

$$\mathscr{L}(\mu, \eta) = \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla S|^2 - U(\mu) - \Phi(x) - \frac{\hbar^2}{8}|\nabla\log\mu|^2\right)\mu(dx), \quad S = \nabla\Delta_\mu^{-1}\eta,$$

we can derive the nonlinear Schrödinger equation. Indeed, curves of the corresponding action functional are critical if and only if (μ, S) solves

$$\partial_t \mu + \mathrm{div}(\mu\nabla S) = 0,$$

$$\partial_t S + \frac{1}{2}|\nabla S|^2 + \Phi(x) + U'(\mu) - \frac{\hbar^2}{2}\frac{\Delta\sqrt{\mu}}{\sqrt{\mu}} = 0.$$

Taking the gradient, multiplying the equation by μ, and setting $\Psi = \sqrt{\mu}\exp(iS/\hbar)$, we arrive at the nonlinear Schrödinger equation

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2}\Delta\Psi + f(|\Psi|^2)\Psi + \Phi(x)\Psi,$$

where f is defined by $f(s) = s^{-1/2}U'(s)$ ($s > 0$).   □

4.2. **Almost symplectic equivalence of measure and wave function dynamics.** We have mentioned in Section 4.1 that solutions (μ, S) to (14)-(15) yield solutions to the magnetic Schrödinger equation (19) via the Madelung transform $(\mu, S) \mapsto \Psi = \sqrt{\mu}\exp(iS/\hbar)$. Similarly to the treatment of the standard Schrödinger case in [26], we shall now give a systematic analysis of this transformation as a symplectic map between two Hamiltonian systems, which turn out to be almost equivalent, as specified in Theorem 10 below.

4.2.1. *Hamiltonian Formulation of magnetic Madelung flow.* The first step is to identifiy the Hamiltonian description of the Lagrangian flow $(14) - (15)$ by means of the Legendre transform on $T\mathscr{P}$ induced by the lifted Lagrangian (11). Since in the current situation, $\mathscr{L}_M$ is no longer quadratic in $\eta \in T_\mu\mathscr{P}$, its induced Legendre transform is not a simple Riesz isomorphism on the Hilbert space $(T_\mu\mathscr{P}, \|\cdot\|_{T_\mu\mathscr{P}})$. As a consequence, the distinct roles played by tangent space $T\mathscr{P}$ of generalized coordinates and its dual space $T^*\mathscr{P}$ of generalized momenta become apparent.

We recall that the cotangent bundle $T^*\mathscr{P}$ consists of all pairs (μ, F), where $\mu \in \mathscr{P}$ and $F : T_\mu\mathscr{P} \to \mathbb{R}$ is linear. From the definition of the tangent space $T_\mu\mathscr{P}$ follows that any distribution

η in $T\mathscr{P}$ annihilates the constant functions. Therefore, in our situation, $T^*\mathscr{P}$ can be defined by

$$T^*\mathscr{P} = \{(\mu, f) : \mu \in \mathscr{P}, \ f \in \mathscr{S}_0(\mathbb{R}^d)\},$$

where

$$\mathscr{S}_0 = \{f = \phi + c : \phi \in \mathscr{S}, c \in \mathbb{R}\}/\sim$$

is the space of equivalence classes of shifted Schwartz functions, with $f \sim g$ if and only if $f - g = \text{const.}$

In analogy to the classical approach, one defines the Hamiltonian $\mathscr{H}_M : T^*\mathscr{P} \to \mathbb{R}$ associated to the Lagrangian $\mathscr{L}_M : T\mathscr{P} \to \mathbb{R}$ as its Legendre transform, i.e.

$$\mathscr{H}_M(\mu, f) = \sup_{\eta \in T_\mu \mathscr{P}} (\langle \eta, f \rangle - \mathscr{L}_M(\mu, \eta)),$$

where $(\mu, f) \in \mathscr{P} \times \mathscr{S}_0(\mathbb{R}^d)$ and $\langle \cdot, \cdot \rangle$ denotes the dual bracket in $\mathscr{S}'(\mathbb{R}^d)$ and $\mathscr{S}(\mathbb{R}^d)$. Thanks to the strict convexity of $\mathscr{L}_M$, the supremum is attained at $\eta^* \in T_\mu \mathscr{P}$ which is the unique solution to $f = (\partial \mathscr{L}_M/\partial \eta)(\mu, \eta^*)$, and hence,

$$\mathscr{H}_M(\mu, f) = \langle \eta^*, f \rangle - \mathscr{L}_M(\mu, \eta^*).$$

Now, the variational derivative $\partial \mathscr{L}_M/\partial \eta$ has been computed in Section 4.1, see formula (16). Therefore, $f = (\partial \mathscr{L}_M/\partial \eta)(\mu, \eta^*) = S^*$, where $S^* = -\Delta_\mu^{-1}(\eta^* - \text{div}(\mu A))$, and S^* is unique as a solution in $\mathscr{S}_0(\mathbb{R}^d)$. As a result, we have identified the change of coordinates

$$T\mathscr{P} \to T^*\mathscr{P}, \quad (\mu, \eta) \mapsto (\mu, S), \quad S = -\Delta_\mu^{-1}(\eta - \text{div}(\mu A)),$$

as the Legendre transform from the physical phase space of variations $T\mathscr{P}$ to the space of generalized momenta $T^*\mathscr{P}$.

Inserting the identification $\eta^* = -\Delta_\mu S^* + \text{div}(\mu A)$ into the Hamiltonian gives an explicit expression for $\mathscr{H}_M$:

$$\mathscr{H}_M(\mu, S^*) = \langle -\Delta_\mu S^* + \text{div}(\mu A), S^* \rangle - \mathscr{L}_M(\eta^*, \mu)$$

$$= -\int_{\mathbb{R}^d} \text{div}\,(\mu(\nabla S^* - A))S^* dx - \mathscr{L}_M(\eta^*, \mu).$$

Integrating by parts in the first integral and using the definition of $\mathscr{L}_M$ gives

$$\mathscr{H}_M(\mu, S^*) = \int_{\mathbb{R}^d} \mu |\nabla S^*|^2 dx - \int_{\mathbb{R}^d} \mu A \cdot \nabla S^* dx$$

$$- \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla S^*|^2 - \frac{1}{2}|A|^2 - \Phi(x) - \frac{\hbar^2}{8}|\nabla \log \mu|^2 \right) \mu dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} |\nabla S^* - A|^2 \mu dx + \int_{\mathbb{R}^d} \Phi(x) \mu dx + \frac{\hbar^2}{8} \int_{\mathbb{R}^d} |\nabla \log \mu|^2 \mu dx.$$

We see that the Hamiltonian is, as expected, the sum of the magnetic, potential, and quantum energies, respectively. Indeed, the classical magnetic Hamiltonian is $H_M = \frac{1}{2}|p - A|^2 + \Phi(x)$, where p is the momentum. In the lifted version, the momentum becomes ∇S, and therefore, $\mathscr{H}_{M,mag} = \frac{1}{2} \int_{\mathbb{R}^d} |\nabla S - A|^2 \mu(dx)$, which is the above expression.

As a second ingredient for a Hamiltonian description of the associated flow of generalized momenta on $T^*\mathscr{P}$, we introduce a symplectic form on $T^*\mathscr{P}$, similarly as in [26] on the physical phase space $T\mathscr{P}$. We recall that a symplectic form ω on a vector space is a skew-symmetric, non-degenerate, bilinear form, i.e. $\omega(v, w) = -\omega(w, v)$ for all u, v and $\omega(v, w) = 0$ for all w implies that $v = 0$.

Lemma 5 (Symplectic form on $T^*\mathscr{P}$). *Each pair $(\phi, \psi) \in \mathscr{S}_0(\mathbb{R}^d) \times \mathscr{S}_0(\mathbb{R}^d)$ induces a vector field $V_{\phi,\psi} : T^*\mathscr{P} \to TT^*\mathscr{P}$ via*

$$V_{\phi,\psi}(\mu, f) = (-\operatorname{div}(\mu\nabla\psi), \phi) \in T_{(\mu,f)}T^*\mathscr{P}, \quad (\mu, f) \in T^*\mathscr{P}.$$

Furthermore, $T^*\mathscr{P}$ is endowed with a unique symplectic form ω, defined on the above vector fields by

$$(20) \qquad \omega(V_{\phi_1,\psi_1}, V_{\phi_2,\psi_2}) = \int_{\mathbb{R}^d} (\nabla\phi_1 \cdot \nabla\psi_2 - \nabla\phi_2 \cdot \nabla\psi_1)\mu(dx),$$

where $(\phi_j, \psi_j) \in \mathscr{S}_0(\mathbb{R}^d) \times \mathscr{S}_0(\mathbb{R}^d)$, $j = 1, 2$.

Proof. Expression (20) clearly defines a skew-symmetric bilinear form. Furthermore, an elementary calculation shows that ω is non-degenerate. Uniqueness follows from the fact that for given $(\mu, f) \in T^*\mathscr{P}$, the set of tangent vectors $\{V_{\phi,\psi}(\mu, f) : \phi, \psi \in \mathscr{D}_0(\mathbb{R}^d)\}$ is total in $T_{(\mu,f)}T^*\mathscr{P}$. $\qquad\square$

Recall that a Hamiltonian flow on a manifold M with symplectic form ω is induced by an energy function $\varphi : M \to \mathbb{R}$ via the integral curves of the corresponding Hamiltonian vector field X_φ on M. The latter is uniquely defined by the requirement that in any $p \in M$, it holds that

$$\omega(X_\varphi, Z) = d\varphi(Z) \quad \text{for all } Z \in T_p M.$$

The form (20) for $M = T^*\mathscr{P}$ allows us to study Hamiltonian flows for various energy functions φ on $T^*\mathscr{P}$. For $\varphi = \mathscr{H}_{\mathrm{M}}$, we arrive at the following statement, which is the analogue of Proposition 3.4 in [26] (also see Corollary 3.5 in that paper).

Theorem 6 (Critical points and Hamiltonian flow). *A smooth curve of measures $\gamma : [0, T] \to \mathscr{P}$, $t \mapsto \gamma_t$, is a critical point of the action functional $\mathscr{A}_{\mathrm{M}}$, defined in (12), if and only if the corresponding curve $(\gamma_t, S_t) \in T^*\mathscr{P}$ in the space of generalized momenta, where $S_t = \Delta_{\gamma_t}^{-1}(\dot\gamma_t - \operatorname{div}(\gamma_t A))$, is a Hamiltonian flow on $(T^*\mathscr{P}, \omega)$ associated to the Hamiltonian $\mathscr{H}_{\mathrm{M}}$.*

Proof. It suffices to compute the corresponding Hamiltonian vector field $X_{\mathscr{H}_{\mathrm{M}}}$ on $M := T^*\mathscr{P}$. To this aim, fix $p = (\mu, f) \in T^*M$ and choose $Z = V_{\phi,\psi}(\mu, f) \in TT^*\mathscr{P}$ as in Definition 5. Then

$$\begin{aligned}
d\mathscr{H}_{\mathrm{M}}(V_{\phi,\psi}(\mu, f)) &= \frac{d}{d\varepsilon}\mathscr{H}_{\mathrm{M}}(V_{\phi,\psi}(\mu - \varepsilon\operatorname{div}(\mu\nabla\psi), f + \varepsilon\phi))\Big|_{\varepsilon=0} \\
&= \int_{\mathbb{R}^d}(\nabla f - A) \cdot \nabla\phi\mu dx + \frac{1}{2}\int_{\mathbb{R}^d}\nabla|\nabla f - A|^2 \cdot \nabla\psi\mu dx \\
&\quad + \int_{\mathbb{R}^d}\nabla\Phi \cdot \nabla\psi\mu dx - \frac{\hbar^2}{2}\int_{\mathbb{R}^d}\nabla\frac{\Delta\sqrt{\mu}}{\sqrt{\mu}} \cdot \nabla\psi\mu dx.
\end{aligned}$$

Comparing with (20), we find that

$$X_{\mathscr{H}_{\mathrm{M}}}(\mu, f) = \left(-\Delta_\mu^{-1} \operatorname{div}((\nabla f - A)\mu), \frac{1}{2}|\nabla f - A|^2 + \Phi - \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \right).$$

Hence, a smooth curve $t \mapsto (\mu_t, S_t) \in T^*\mathscr{P}$ is an integral curve for $X_{\mathscr{H}_{\mathrm{M}}}$ if and only if the corresponding flow of variations $t \mapsto \dot{\mu}_t \in T\mathscr{P}$ solves (14)-(15). $\qquad\square$

4.2.2. *Hamiltonian Structure of the magnetic Schrödinger flow.* Let us recall the basic fact that the magnetic Schrödinger equation has a Hamiltonian structure, too. Indeed, denoting by $\mathscr{C} = C^\infty(\mathbb{R}^d; \mathbb{C})$ the linear space of smooth complex-valued functions on $\mathbb{R}^d$ and identifying as usual the tangent space over an element $\Psi \in \mathscr{C}$ with the space $\mathscr{C}$, the tangent bundle $T\mathscr{C}$ is naturally equipped with the symplectic form

$$\omega_{\mathrm{C}}(F, G) = -2 \int_{\mathbb{R}^d} \mathfrak{I}(F \cdot \overline{G})(x)dx,$$

where $\mathfrak{I}(z)$ is the imaginary part of $z \in \mathbb{C}$ and $\overline{z}$ is its complex conjugate. This way $(\mathscr{C}, \hbar\omega_{\mathrm{C}})$ becomes a symplectic space. On $\mathscr{C}$ we define the energy function $\mathscr{H}_{\mathrm{C}} : \mathscr{C} \mapsto \mathbb{R}$ by

$$\mathscr{H}_{\mathrm{C}}(\Psi) = \frac{1}{2} \int_{\mathbb{R}^d} \left| \left(\frac{\hbar}{i}\nabla - A \right) \Psi \right|^2 dx, + \int_{\mathbb{R}^d} \Phi(x) |\Psi|^2(x) \, dx,$$

which is the magnetic Schrödinger Hamiltonian.

Proposition 7. *A smooth flow of wave functions $t \mapsto \Psi_t \in \mathscr{C}$ solves the magnetic Schrödinger equation* (19) *if and only if it is a Hamiltonian flow induced from the energy function $\mathscr{H}_{\mathrm{C}}$ on the symplectic space $(\mathscr{C}, \hbar\omega_{\mathrm{C}})$.*

Proof. We only sketch the proof of this classical but mostly forgotten fact. For $\Psi, \zeta \in \mathscr{C}$, we find by a straightforward computation that

$$\begin{aligned}
\frac{d}{d\epsilon} \mathscr{H}_{\mathrm{C}}(\Psi + \epsilon\zeta)\Big|_{\varepsilon=0} &= \mathfrak{R} \int_{\mathbb{R}^d} \left(\left(\frac{\hbar}{i}\nabla - A \right)^2 + 2\Phi \right) \Psi \cdot \overline{\zeta} dx \\
&= \mathfrak{I} \int_{\mathbb{R}^d} i \left(\left(\frac{\hbar}{i}\nabla - A \right)^2 + 2\Phi \right) \Psi \cdot \overline{\zeta} dx \\
&= \omega_{\mathrm{C}} \left(-\frac{i}{\hbar} \left(\frac{1}{2} \left(\frac{\hbar}{i}\nabla - A \right)^2 + \Phi \right) \Psi, \zeta \right).
\end{aligned}$$

This shows that the Hamiltonian vector field $X_{\mathscr{H}_{\mathrm{C}}}$ associated to $\mathscr{H}_{\mathrm{C}}$ on $(\mathscr{C}, \omega_{\mathrm{C}})$ is

$$X_{\mathscr{H}_{\mathrm{C}}}(\Psi) = -\frac{i}{\hbar} \left(\frac{1}{2} \left(\frac{\hbar}{i}\nabla - A \right)^2 + \Phi \right) \Psi.$$

Hence, solutions to the magnetic Schrödinger equation (19) are precisely the integral curves of the Hamiltonian vector field $X_{\mathscr{H}_{\mathrm{C}}}$. $\qquad\square$

4.2.3. *Madelung transform: precise definition and symplectic properties.* Let $\mathscr{C}_* = \{\Psi \in \mathscr{C} : \int_{\mathbb{R}^d} |\Psi|^2 dx = 1, \Psi(x) \neq 0 \text{ for all } x \in \mathbb{R}^d\}$ be the set of smooth nowhere vanishing normalized wave functions. Each $\Psi \in \mathscr{C}_*$ admits a decomposition $\Psi = |\Psi| \exp(iS/\hbar)$, where the smooth function $S : \mathbb{R}^d \to \mathbb{R}$ is uniquely defined up to an additive constant of the form $2\pi\hbar k$, $k \in \mathbb{N}$. In particular, the Madelung transform is well defined

$$(21) \qquad \sigma : \mathscr{C}_* \to T^*\mathscr{P}, \quad \sigma(\Psi) = (|\Psi(x)|^2 dx, S) \in \mathscr{P} \times \mathscr{S}_0(\mathbb{R}^d).$$

Recall that by the definition of $\mathscr{S}_0(\mathbb{R}^d)$ as the space of equivalence classes of shifted Schwartz functions, the map σ is not injective. However, we may apply the abstract notion of a symplectic submersion (see [26]) which is a generalization of a symplectic isomorphism where the injectivity assumption is dropped.

Definition 8 (Symplectic submersion on manifolds)**.** *Let (M, ω_M), (N, ω_N) be symplectic manifolds equipped with the symplectic forms ω_M, ω_N, respectively, and let $s : M \to N$ be a smooth map. Then s is called a* symplectic submersion *if its differential $s_* : TM \to TN$ is surjective and satisfies $\omega_N(s_* X, s_* Y) = \omega_M(X, Y)$ for all $X, Y \in TM$.*

Similarly to the isomorphism case one may easily see that Hamiltonian flows are stable under symplectic submersions. This is stated in the following proposition, cf. [26, Prop. 4.2].

Proposition 9 (Submersions between Hamiltonian flows)**.** *Let M, N be symplectic manifolds equipped with the symplectic forms ω_M, ω_N, respectively, and let $s : M \to N$ be a symplectic submersion. If the Hamiltonians $F \in C^\infty(M)$ and $G \in C^\infty(N)$ are related by $F = G \circ s$, the submersion s maps Hamiltonian flows associated to F on (M, ω_M) to Hamiltonian flows associated to G on (N, ω_N).*

We are now ready to state the main result of this section which asserts that the Madelung transform is a symplectic submersion from $\mathscr{C}_*$ to $T^*\mathscr{P}$.

Theorem 10 (Madelung transform as a symplectic submersion)**.** *The Madelung transform $\sigma : \mathscr{C}_* \to T^*\mathscr{P}$, defined in (21), is a symplectic submersion from $(\mathscr{C}_*, \hbar\omega_C)$ to $(T^*\mathscr{P}, \omega)$, preserving the magnetic Schrödinger Hamiltonian,*

$$\mathscr{H}_C = \mathscr{H}_M \circ \sigma.$$

Proof. Since the proof is very similar to the proof of Theorem 4.3 in [26], we give only a sketch. First, we restrict the phase $S/\hbar$ in $|\Psi| \exp(iS/\hbar)$ to the interval $[0, 2\pi\hbar)$ by defining an appropriate bijection. We can prove that the differential s_* is surjective. A calculation shows that $\omega_{T^*\mathscr{P}}(s_* V_{\phi_1,\psi_1}, s_* V_{\phi_2,\psi_2}) = \hbar\omega_C(V_{\phi_1,\psi_1}, V_{\phi_2,\psi_2})$ for all vector fields $V_{\phi_1,\psi_1}, V_{\phi_2,\psi_2}$. Thus, s is a symplectic submersion. The remaining part $\mathscr{H}_C = \mathscr{H}_M \circ \sigma$ is a computation; see [26, Section 4] for details.   □

In light of Proposition 9 and Theorem 10, the magnetic Schrödinger equation (19) for wave functions can be interpreted as the lift of the physically intuitive Lagrangian flow on probability measures (or mass distributions) (15) to the larger space of complex wave functions. The lifted Hamiltonian system is the familiar magentic Schrödinger equation for wave functions and has the advantage that it is linear. However, a disadvantage is that a new and unphysical degree of

freedom, incorporated in the constant phase shift for wave functions and describing the same physical state, is introduced.

5. Example 3: Quantum Navier-Stokes Equations

In this section, we consider the quantum Lagrangian

$$(22) \qquad \mathscr{L}_Q(\mu, \eta) = \int_{\mathbb{R}^d} \left(\frac{1}{2} |\nabla S|^2 - U(\mu) - \Phi(x) - \frac{\hbar^2}{8} |\nabla \log \mu|^2 \right) \mu(dx),$$

where $\mu \in \mathscr{P}$, $\eta \in T_\mu \mathscr{P}$, $S = -\Delta_\mu^{-1} \eta$, and $U(\mu)$ denotes the internal energy which is assumed to be a smooth function. Here, we are interested in the Lagrangian flow with dissipation

$$\mathscr{D}(\mu, \eta) = \frac{\alpha}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \mu(dx),$$

where $\alpha \geq 0$, and $v = \nabla S$ is the unique potential velocity field inducing the variation η of the state μ.

5.1. Quantum Navier-Stokes equations.

We show that the dissipative Lagrangian flow on $\mathscr{P}$ can be related to the Navier-Stokes equations including the Bohm potential and a density-depending viscosity. Our result reads as follows.

Theorem 11 (Quantum Navier-Stokes equations). *A smooth curve $\mu : [0, T] \to \mathscr{P}$ satisfies*

$$(23) \qquad \frac{d}{dt} \frac{\partial \mathscr{L}_Q}{\partial \eta}(\mu, \dot{\mu}) - \frac{\partial \mathscr{L}_Q}{\partial \mu}(\mu, \dot{\mu}) + \frac{\partial \mathscr{D}}{\partial \eta}(\mu, \dot{\mu}) = 0$$

if and only if the mass flux $t \mapsto \mu_t v_t$ with $v = -\nabla \Delta_\mu^{-1} \mu$ solves the quantum Navier-Stokes equation

$$(24) \qquad \partial_t(\mu v) + \mathrm{div}(\mu v \otimes v) + \nabla p(\mu) + \mu \nabla \Phi(x) - \frac{\hbar^2}{2} \mu \nabla \left(\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \right) = \alpha \mu \nabla \Delta_\mu^{-1}(\nabla^2 : (\mu \nabla v)).$$

Here, $v \otimes v$ is a tensor with components $v_j v_k$; the pressure function $p(\mu)$ is defined through $p'(s) = s U''(s)$ for $s \geq 0$; and the product ":" signifies summation over both indices. Identifying vector fields modulo rotational components, we can write this equation as

$$(25) \qquad \partial_t(\mu v) + \mathrm{div}(\mu v \otimes v) + \nabla p(\mu) + \mu \nabla \Phi(x) - \frac{\hbar^2}{2} \mu \nabla \left(\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \right) \equiv \alpha \, \mathrm{div}(\mu D(v)),$$

where $A \equiv B$ if and only if $\mathrm{div}(A - B) = 0$, and $D(v) = \frac{1}{2}(\nabla v + \nabla v^\top) = \nabla v$ is the symmetric velocity gradient.

The system of quantum Navier-Stokes equations is given by (24) and the continuity equation

$$(26) \qquad \qquad \partial_t \mu + \mathrm{div}(\mu v) = 0.$$

In this model, the viscous stress tensor is $\mathbb{S} = \nu D(v)$, where the viscosity $\nu = \alpha \mu$ depends on the particle density μ. For variants of the stress tensor, see Remark 12.

Proof. We write $\mathscr{L}_Q = \mathscr{T} - \mathscr{V}$, where

$$\mathscr{T}(\mu, \eta) = \|\eta\|^2_{T_\mu \mathscr{P}} = \int_{\mathbb{R}^d} |\nabla \Delta_\mu^{-1} \eta|^2 \mu(dx)$$

corresponds to the "kinetic energy" and

$$(27) \qquad \mathscr{V}(\mu, \eta) = \int_{\mathbb{R}^d} \left(\Phi(x) + U(\mu) + \frac{\hbar^2}{8} |\nabla \log \mu|^2 \right) \mu(dx)$$

corresponds to the "potential energy". By the proof of Theorem 1 (see (18) with $A = 0$), we have

$$(28) \qquad \frac{\partial \mathscr{V}}{\partial \mu} = \Phi(x) + U'(\mu) - \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}}.$$

Since $\mathscr{V}$ does not depend on η, it follows that $\partial \mathscr{V} / \partial \eta = 0$. Furthermore, by (16) and (17) (with $A = 0$),

$$(29) \qquad \frac{\mathscr{L}_Q}{\partial \eta} = \frac{\partial \mathscr{T}}{\partial \eta} = S, \quad \frac{\partial \mathscr{T}}{\partial \mu} = -\frac{1}{2} |\nabla S|^2.$$

It remains to compute $\partial \mathscr{D} / \partial \eta$. To this end, let $\xi \in \mathscr{M}$ and set $\zeta = \Delta_\mu^{-1} \xi$. Since $v = \nabla S = -\nabla \Delta_\mu^{-1} \eta$, we infer that

$$\begin{aligned} \delta_* \mathscr{D}(\eta, \xi) &= \frac{\alpha}{2} \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} |\nabla^2 \Delta_\mu^{-1}(\eta + \varepsilon \xi)|^2 \mu(dx) \Big|_{\varepsilon=0} \\ &= \frac{\alpha}{2} \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} |\nabla^2 (\Delta_\mu^{-1} \eta + \varepsilon \Delta_\mu^{-1} \xi)|^2 \mu(dx) \Big|_{\varepsilon=0} \\ &= \frac{\alpha}{2} \frac{d}{d\varepsilon} \int_{\mathbb{R}^d} \nabla^2 (-S + \varepsilon \zeta) : \nabla^2 (-S + \varepsilon \zeta) \mu(dx) \Big|_{\varepsilon=0} \\ &= -\alpha \int_{\mathbb{R}^d} \nabla^2 S : \nabla^2 \zeta \mu dx = -\alpha \int_{\mathbb{R}^d} \Delta_\mu^{-1} (\nabla^2 : (\mu \nabla^2 S)) \xi dx. \end{aligned}$$

This implies that

$$(30) \qquad \frac{\partial \mathscr{D}}{\partial \eta} = -\alpha \Delta_\mu^{-1} (\nabla^2 : (\mu \nabla^2 S)).$$

Inserting this expression as well as (28) and (29) into (23) gives

$$\partial_t S + \frac{1}{2} |\nabla S|^2 + \Phi(x) + U'(\mu) - \frac{\hbar^2}{2} \frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} = \alpha \Delta_\mu^{-1} \nabla^2 : (\mu \nabla^2 S).$$

We take the gradient, multiply this equation by μ, and replace $\nabla S = v$:

$$\mu \partial_t v + \frac{1}{2} \mu \nabla |v|^2 + \mu \nabla \Phi(x) + \mu U''(\mu) \nabla \mu - \frac{\hbar^2}{2} \mu \nabla \left(\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \right) = \alpha \mu \nabla \Delta_\mu^{-1} \nabla^2 : (\mu \nabla v).$$

Then, employing the continuity equation $v \partial_t \mu + v \operatorname{div}(\mu v) = 0$ and rearranging terms, we obtain

$$\partial_t(\mu v) + \operatorname{div}(\mu v \otimes v) + \mu \nabla \Phi(x) + \nabla p(\mu) - \frac{\hbar^2}{2} \mu \nabla \left(\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \right) = \alpha \mu \nabla \Delta_\mu^{-1} \nabla^2 : (\mu D(v)),$$

which equals (24). The final step is the projection on the space of curl-free fields by taking the divergence which leads to (25). Indeed, observing that

$$\text{div}\,(\mu\nabla\Delta_\mu^{-1}\nabla^2 : (\mu D(v))) = \Delta_\mu\Delta_\mu^{-1}(\nabla^2 : (\mu D(v)) = \text{div}\,(\,\text{div}(\mu D(v))),$$

we conclude the proof. □

Remark 12. The Lagrangian approach allows us to choose other dissipation terms. We consider two simple examples:

$$\mathscr{D}_1(\mu, \eta) = \frac{\alpha}{p}\int_{\mathbb{R}^d} |\nabla v|^p \mu(dx), \quad p \geq 2,$$

$$\mathscr{D}_2(\mu, \eta) = \frac{1}{2}\int_{\mathbb{R}^d} g(\mu)(\nu_1|\nabla v|^2 + \nu_2(\text{div}\,v)^2\mathbb{I})\mu(dx),$$

where $g : \mathbb{R} \to [0, \infty)$ is some function and ν_1, $\nu_2 > 0$. The variational derivatives are computed similarly as in the proof of Theorem 11. The results are as follows:

$$\frac{\partial \mathscr{D}_1}{\partial \eta} = -\alpha\Delta_\mu^{-1}\nabla^2 : (\mu|D(v)|^{p-2}D(v)),$$

$$\frac{\partial \mathscr{D}_2}{\partial \eta} = -\Delta_\mu^{-1}\nabla^2 : (\mu g(\mu)(\nu_1 D(v) + \nu_2(\text{div}\,v)\mathbb{I})).$$

The viscous term in the quantum Navier-Stokes equations is obtained after taking the gradient, multiplying by μ, and projecting it on the space of curl-free vectors:

$$\text{div}\left(\mu\nabla\frac{\partial\mathscr{D}_1}{\partial\eta}\right) = -\alpha\,\text{div}\,(\mu\nabla\Delta_\mu^{-1}\nabla^2 : (\mu|D(v)|^{p-2}D(v)))$$

$$= -\alpha\,\text{div}\,(\,\text{div}(\mu|D(v)|^{p-2}D(v))),$$

and similarly for the second expression. The viscous stress tensors become

$$\mathbb{S}_1 = \alpha\mu|D(v)|^{p-2}D(v), \quad \mathbb{S}_2 = \mu g(\mu)(\nu_1 D(v) + \nu_2(\text{div}\,v)\mathbb{I}).$$

The viscosity $\nu_1 = \alpha\mu|D(v)|^{p-2}$ depends not only on the particle density but also on the velocity gradient. When we choose $g(\mu) = 1/\mu$, the viscosities are constant, which corresponds to the case of Newtonian fluids (see, e.g., [8, Formula (1.16)]). □

5.2. Energy-dissipation identities and Noether currents.

According to Section 4.2, the Hamiltonian $\mathscr{H}_Q : T^*\mathscr{P} \to \mathbb{R}$ associated to the Lagrangian $\mathscr{L}_Q : T\mathscr{P} \to \mathbb{R}$, defined in (22), is given by

$$\mathscr{H}_Q(\mu, S) = \langle\eta, S\rangle - \mathscr{L}_Q(\mu, \eta),$$

where $S = (\partial\mathscr{L}_Q/\partial\eta)(\mu, \eta) = -\Delta_\mu^{-1}\eta$. Inserting $\eta = -\Delta_\mu S$ and the definition (22) of $\mathscr{L}_Q$ into this expression, we find that

$$\mathscr{H}_Q(\mu, S) = \int_{\mathbb{R}^d} |\nabla S|^2\mu dx - \mathscr{L}_Q(\mu, \eta)$$

(31)
$$= \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla S|^2 + U(\mu) + \Phi(x) + \frac{\hbar^2}{8}|\nabla\log\mu|^2\right)\mu dx,$$

which is the sum of the kinetic, internal, potential, and quantum energies. In this section, we will derive energy-dissipation identities for smooth solutions to the quantum Navier-Stokes equations (24) and (26).

Proposition 13 (Energy-dissipation identity). *Let (μ, v) be a smooth solution to* (24) *and* (26). *Then*

$$(32) \qquad \frac{d\mathcal{H}_Q}{dt} + \alpha \int_{\mathbb{R}^d} \mu |\nabla v|^2 dx = 0.$$

Proof. Multiplying (24) by v and (26) by $-\frac{1}{2}|v|^2 + U'(\mu) + \Phi(x) + (\hbar^2/2)(\Delta \sqrt{\mu}/\sqrt{\mu})$ and adding the resulting equations, a straightforward computation yields

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left(\frac{1}{2}|v|^2 + U(\mu) + \mu\Phi(x) + \frac{\hbar^2}{8}\mu|\nabla \log \mu|^2 \right) dx = \alpha \int_{\mathbb{R}^d} \mu v \cdot \nabla \Delta_\mu^{-1}(\nabla^2 : (\mu v)) dx.$$

The left-hand side equals $d\mathcal{H}_Q/dt$. The right-hand side can be rewritten, using $v = \nabla S$ and integration by parts, as

$$-\alpha \int_{\mathbb{R}^d} \operatorname{div}(\mu \nabla S) \Delta_\mu^{-1}(\nabla^2 : (\mu \nabla^2 S)) dx = -\alpha \int_{\mathbb{R}^d} \Delta_\mu S \Delta_\mu^{-1}(\nabla^2 : (\mu \nabla^2 S)) dx$$

$$= -\alpha \int_{\mathbb{R}^d} \Delta_\mu^{-1} \Delta_\mu S (\nabla^2 : (\mu \nabla^2 S)) dx$$

$$= -\alpha \int_{\mathbb{R}^d} \nabla^2 S : (\mu \nabla^2 S) dx = -\alpha \int_{\mathbb{R}^d} \mu |\nabla v|^2 dx,$$

proving the claim. $\qquad\square$

Remark 14. Proposition 13 is the counterpart of the energy dissipation law for classical damped Lagrangian systems in $\mathbb{R}^n$ in which case the analogue of (23) reads as

$$(33) \qquad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) - \frac{\partial L}{\partial \mu}(q, \dot{q}) + \frac{\partial D}{\partial \dot{q}}(q, \dot{q}) = 0.$$

Writing the dynamics in Hamiltonian coordinates $t \mapsto (q(t), p(t))$ via the Legendre transform, i.e. $p = p(q, \dot{q}) = (\partial L/\partial \dot{q})(q, \dot{q})$, for the Hamiltonian we obtain

$$H(q, p(q, \dot{q})) = \langle \dot{q}, \frac{\partial L}{\partial \dot{q}}(q, \dot{q}) \rangle - L(q, \dot{q}),$$

which yields, after differentiation with respect to t and inserting (33),

$$\frac{dH}{dt}(q(t), p(t)) = -\langle \dot{q}, \frac{\partial D}{\partial \dot{q}}(q, \dot{q}) \rangle.$$

In our case, by the same computation and using (30), it follows that

$$\frac{d\mathcal{H}_Q}{dt} = -\langle \eta, \frac{\partial \mathcal{D}}{\partial \eta} \rangle = -\alpha \langle \Delta_\mu S, \Delta_\mu^{-1}(\nabla^2 : (\mu \nabla^2 S)) \rangle = -\alpha \int_{\mathbb{R}^d} |\nabla^2 S|^2 d\mu,$$

which equals (32). $\qquad\square$

It has been shown in [15] that the *projected* system (25)-(26) possesses a second energy functional,

$$(34) \qquad \mathcal{H}_Q^*(\mu, S) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|w|^2 + U(\mu) + \Phi(x) + \left(\frac{\hbar^2}{8} - \alpha^2 \right) |\nabla \log \mu|^2 \right) \mu \, dx,$$

where $w = v + v_{os}$ and $v_{os} = \alpha \nabla \log \mu$ is the osmotic velocity first introduced by Nelson [23, Formula (26)]. More precisely, let (μ, v) with $v = \nabla S = -\nabla \Delta_\mu^{-1} \eta$ be a smooth solution to

$$(35) \qquad \partial_t \mu + \mathrm{div}(nv) = 0, \quad x \in \mathbb{R}^d, \; t > 0,$$

$$(36) \qquad \partial_t(\mu v) + \mathrm{div}(\mu v \otimes v) + \nabla p(\mu) + \mu \nabla \Phi(x) - \frac{\hbar^2}{2} \mu \nabla \left(\frac{\Delta \sqrt{\mu}}{\sqrt{\mu}} \right) = \alpha \, \mathrm{div}(\mu D(v)).$$

Then a formal computation [15] shows that

$$\frac{d \mathcal{H}_Q^*}{dt} + \alpha \int_{\mathbb{R}^d} \left(\mu |\nabla w|^2 + U''(\mu)|\nabla \mu|^2 + \left(\frac{\hbar^2}{8} - \alpha^2 \right) \mu |\nabla^2 \log \mu|^2 \right) dx = 0,$$

which provides additional estimates for the solutions if $\hbar^2/8 > \alpha^2$. We wish to understand why system (35)-(36) possesses *two* dissipative laws.

A first partial answer was given in [16]. There it was shown that the osmotic velocity emerges from gauge field theory by introducing the local gauge transformation $\psi \mapsto \phi = \exp(-i\alpha \log \mu)\psi$, where ψ is a given quantum state. This transformation leaves the particle density invariant but it changes the mass flux $nv = -\Im(\overline{\psi} \nabla \psi)$ according to

$$nw = -\Im(\overline{\phi} \nabla \phi) = -\Im(\overline{\psi} \nabla \psi - i\alpha \mu \nabla \log \mu) = \mu(v + \alpha \nabla \log \mu).$$

Our goal is to show that the new velocity w can be interpreted as a special transformation of (t, μ) and that the Hamiltonian $\mathcal{H}_Q^*$ can be interpreted as the Noether current associated to this transformation.

To this end, we recall some basic facts from classical Noether theory (see, e.g., [5, Chapter 9]). Let a Lagrangian $L(t, q, \dot{q})$ be given. We introduce the transformations $T(t, q; s)$ and $Q(t, q; s)$, where $s > 0$ is a parameter, such that $t = T(t, q; 0)$ and $q = Q(t, q; 0)$. Setting

$$\delta t = \frac{\partial T}{\partial s}(t, q; 0), \quad \delta q = \frac{\partial Q}{\partial s}(t, q; 0),$$

Taylor's expansion gives $T(t, q) = t + s\delta t + O(s^2)$ and $Q(t, q) = q + s\delta q + O(s^2)$ as $\gamma \to 0$. For infinitesimal small $s > 0$, we can formulate the transformation as $t \mapsto t + \delta t$ and $q \mapsto q + \delta q$. Now, the *Noether current* is defined as

$$J = \delta t \left(\frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) - \delta q \frac{\partial L}{\partial \dot{q}}.$$

If the Lagrangian density $L(t, q, \dot{q})$ is invariant under the above transformation, Noether's theorem states that the Noether current is constant along any extremal of the action integral over L.

On the space of probability measures, we define the lifted Noether current as

$$\mathcal{J}(\mu, \eta) = \delta t \left\langle \frac{\partial \mathcal{L}}{\partial \eta}(\mu, \eta), \eta \right\rangle - \left\langle \frac{\partial \mathcal{L}}{\partial \eta}(\mu, \eta), \delta \mu \right\rangle, \quad (\mu, \eta) \in T\mathcal{P},$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product in suitable spaces. We prove the following result.

Theorem 15 (Noether currents)**.** *Let the Lagrangian $\mathscr{L}_Q$ be given by* (22)*. Then*

- $\delta t = 1$, $\delta \mu = 0$*:* $\mathscr{J} = \mathscr{H}_Q$*, defined in* (31)*;*
- $\delta t = 1$, $\delta \mu = \alpha \Delta \mu$*:* $\mathscr{J} = \mathscr{H}_Q^*$*, defined in* (34)*.*

Proof. The theorem follows by inserting the transformations into the definition of the Noether current. We recall from (29) that $\partial \mathscr{L}_Q / \partial \eta = S$, where $S = -\Delta_\mu^{-1} \eta$. Then, if $\delta t = 1$, $\delta \mu = 0$, we find that

$$\mathscr{J} = \int_{\mathbb{R}^d} S \eta \, dx - \mathscr{L}_Q = \int_{\mathbb{R}^d} \mu |\nabla S|^2 dx - \mathscr{L}_Q = \mathscr{H}_Q.$$

Next, if $\delta t = 1$, $\delta \mu = \alpha \Delta \mu$, we compute

$$\begin{aligned}
\mathscr{J} &= \int_{\mathbb{R}^d} (S \eta - \alpha \Delta \mu S) dx - \mathscr{L}_Q \\
&= \int_{\mathbb{R}^d} \left(\frac{1}{2} \mu |\nabla S|^2 + U(\mu) + \psi(x) + \frac{\hbar^2}{8} \mu |\nabla \log \mu|^2 + \alpha \nabla \mu \cdot \nabla S \right) dx \\
&= \int_{\mathbb{R}^d} \left(\frac{1}{2} \mu |\nabla (S + \alpha \log \mu)|^2 + \left(\frac{\hbar^2}{8} - \alpha^2 \right) \mu |\nabla \log \mu|^2 + U(\mu) + \psi(x) \right) dx \\
&= \mathscr{H}_Q^*,
\end{aligned}$$

completing the proof. ☐

Notice that Noether's theorem, which yields energy conservation, can be applied only if $\alpha = 0$, otherwise we have dissipation of energy. For a classical Noether theory including dissipative terms, we refer to [7, 27] or the more recent works [12, 13]. The extension of this theory to our context is an open question.

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