

HARNACK INEQUALITY FOR FUNCTIONAL SDEs WITH BOUNDED MEMORY

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ABSTRACT. We use a coupling method for functional stochastic differential equations with bounded memory to establish an analogue of Wang's dimension-free Harnack inequality [13]. The strong Feller property for the corresponding segment process is also obtained.

1. INTRODUCTION AND STATEMENT OF RESULTS

Harnack inequalities are known to hold for a wide range of Markov processes such as diffusions or symmetric jump processes on manifolds, graphs, fractals or even more general metric measure spaces, where they are a fundamental tool for the analysis of the corresponding transition semigroups, cf. [6]. In most cases, a Harnack inequality is established under appropriate ellipticity conditions by harmonic analysis arguments. Such arguments are typically not applicable in non-Markovian or infinite dimensional set-ups. However, as shown in [1] for finite dimensional diffusions, the dimension-free Harnack inequality of Wang [13] may be proved by a purely probabilistic coupling technique which was recently adapted to the infinite dimensional case of monotone SPDEs [14, 9, 2].

In this note we show that the coupling method works well also in the non-Markovian case of stochastic functional equations with additive noise and Lipschitz drift with bounded delay. Moreover, the strong Feller property is obtained for the corresponding infinite dimensional segment process. The main additional difficulty compared to the diffusive case [1] is the necessity to couple together two solutions including their pasts at a given time. It turns out that this can be done simply by driving the second process with the drift induced from the segment process of the first.

For a precise statement of our results, fix $r > 0$ and let \mathcal{C} denote the space of continuous \mathbb{R}^d -valued functions on $[-r, 0]$ endowed with the sup-norm $\|\cdot\|$. For a function or a process X defined on $[t-r, t]$ we write $X_t(s) := X(t+s)$, $s \in [-r, 0]$. Consider the stochastic functional differential equation

$$\begin{cases} dX(t) = V(X_t) dt + dW(t), \\ X_0 = \varphi, \end{cases} \quad (1.1)$$

where W is an \mathbb{R}^d -valued Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the augmented Brownian filtration $\mathcal{F}_t^W = \sigma(W(u), 0 \leq u \leq t) \vee \mathcal{N} \subset \mathcal{F}$, where \mathcal{N} denotes the null-sets in \mathcal{F} , φ is an (\mathcal{F}_t^W) -independent \mathcal{C} -valued random variable and $V : \mathcal{C} \rightarrow \mathbb{R}^d$ is a measurable map.

Below we assume that V admits a decomposition

$$V(x) = v(x(0)) + Z(x), \quad (1.2)$$

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where $v \in C(\mathbb{R}^d; \mathbb{R}^d)$ is a dissipative vector field on \mathbb{R}^d , i.e.

$$\langle v(a) - v(b), a - b \rangle \leq 0 \quad \forall a, b \in \mathbb{R}^d \quad (1.3)$$

and Z is globally Lipschitz on \mathcal{C} , i.e. for some $L > 0$,

$$|Z(x) - Z(y)| \leq L \|x - y\| \quad \forall x, y \in \mathcal{C}. \quad (1.4)$$

Global existence and uniqueness for (1.1) hold under much weaker conditions, c.f. [10]. In particular, the corresponding segment process $\{X_t^\varphi \in \mathcal{C} \mid t \geq 0, \varphi \in \mathcal{C}, t \mapsto X_t^\varphi\}$ solves (1.1) and induces a Markov semigroup (P_t) on \mathcal{C} via $\varphi \mapsto P_t f(\varphi) = \mathbb{E}(f(X_t^\varphi))$, for bounded measurable $f : \mathcal{C} \rightarrow \mathbb{R}$.

Now our main result is the following version of Wang's dimension free Harnack inequality for the semigroup (P_t) .

Theorem 1.1. *Assume that $V = v + Z$ as in (1.2) with dissipative v and $\text{Lip}_{\mathcal{C}}(Z) \leq L$, then for any $p > 1, T > r$ and any bounded measurable $f : \mathcal{C} \rightarrow \mathbb{R}$,*

$$(P_T f(y))^p \leq P_T(f^p)(x) \exp\left(\frac{p}{p-1} \rho_T^2(x, y)\right) \quad \forall x, y \in \mathcal{C}, \quad (1.5)$$

where

$$\rho_T^2(x, y) = \inf_{s \in [r, T]} \left\{ \frac{|x(0) - y(0)|^2}{s - r} + s L^2 \|x - y\|^2 \right\}.$$

Remark 1.2. Elementary computation yields

$$\rho_T^2(x, y) = \begin{cases} \frac{|x(0) - y(0)|^2}{T - r} + T L^2 \|x - y\|^2 & \text{for } T \leq r + \frac{|x(0) - y(0)|}{L \|x - y\|} \\ 2L |x(0) - y(0)| \cdot \|x - y\| + r L^2 \|x - y\|^2 & \text{for } T \geq r + \frac{|x(0) - y(0)|}{L \|x - y\|}. \end{cases}$$

Moreover, (X_t) exhibits the following strong Feller property on the infinite dimensional state space \mathcal{C} . Since the driving noise for (X_t) is only d -dimensional, this is a noticeable result.

Corollary 1.3. *Under (1.2)-(1.4) the segment process (X_t^x) is eventually strong Feller, i.e. let $f : \mathcal{C} \rightarrow \mathbb{R}$ be bounded measurable, then for $t > r$ the map $x \mapsto P_t f(x) \in \mathbb{R}$ is continuous on \mathcal{C} .*

Remark 1.4. Our proofs below can easily be modified to include the case of random V and random, strictly elliptic diffusion coefficient $\sigma = \sigma(t) \in \mathbb{R}^{d \times d}$ in front of $dW(t)$ in (1.1). However, (X_t) can generally not be expected to be strong Feller in case σ depends on the segment X_t , i.e. $\sigma = \sigma(X_t)$. If, for example, $d = 1$ and the diffusion part in (1.1) is of the form $g(X(t-1))$ or $g(\int_{t-r}^t X(s) ds)$ with smooth, strictly increasing and positive g , then the transition probabilities $P_t(x, dy)$ and $P_t(z, dy)$ are mutually singular for all $t > 0$ whenever $x \neq z$, since the initial condition can perfectly recovered from X_t , using the law of the iterated logarithm c.f. [11].

Another straightforward consequence of Theorem 1.1 is the following smoothing property of (P_t) . For more on this we refer to [14, 2].

Corollary 1.5. *Assume that (X_t) admits an invariant measure $\mu \in \mathcal{M}(\mathcal{C})$ such that*

$$\int_{\mathcal{C}} e^{\lambda \|x\|^2} \mu(dx) < \infty \text{ for some } \lambda > 4(2L + rL^2), \quad (1.6)$$

then for $t > r + L^{-1}$, P_t is μ -hyperbounded i.e. P_t is a bounded operator from $L^2(\mathcal{C}, \mu)$ to $L^4(\mathcal{C}, \mu)$.

In the following we give an example when (1.6) holds. For $z \in \mathbb{R}^d$ we set $v(z) = -\lambda_0 z$ for some $\lambda_0 > 4(2L + rL^2)$. Furthermore, assume that $\sup_{x \in \mathbb{R}^d} \|Z(x)\| \leq M$ for some $M \geq 0$. Clearly in this case the solution $(X(t))_{t \geq 0}$ of (1.1) solves the following integral equation

$$X(t) = e^{-\lambda_0 t} X_0 + \int_0^t e^{-\lambda_0(t-s)} Z(X_s) \, ds + J^{\lambda_0}(t), \quad (1.7)$$

where $J^{\lambda_0}(t) := \int_0^t e^{-\lambda_0(t-s)} \, dW_s$ is the Ornstein-Uhlenbeck process solving

$$\begin{cases} du(t) = v(u(t)) \, dt + dW_t, & t \geq 0 \\ u(0) = 0. \end{cases}$$

Let $\delta > 0$. By using (1.7) there exists a positive constant $c_\delta > 1$ such that for $t \geq 0$,

$$|X(t)|^2 \leq c_\delta e^{-2\lambda_0 t} |X_0|^2 + c_\delta \frac{M^2}{\lambda_0^2} + (1 + \delta) |J^{\lambda_0}(t)|^2 \quad (1.8)$$

On the other hand we know that $\sup_{t \geq 0} \mathbb{E} \left(e^{\varepsilon |J^{\lambda_0}(t)|^2} \right) < +\infty$ whenever $\varepsilon < \lambda_0$. Now Theorem 12.1 in [8] implies that

$$\sup_{t \geq 0} \mathbb{E} \left(e^{\varepsilon \|J_t^{\lambda_0}\|^2} \right) < +\infty \quad \text{for } \varepsilon < \lambda_0.$$

Therefore, from (1.8) and by assuming that the initial condition X_0 is deterministic we have

$$\sup_{t \geq 0} \mathbb{E} \left(e^{\varepsilon \|X_t\|^2} \right) < +\infty \quad \text{for } \varepsilon(1 + \delta) < \lambda_0.$$

From the arbitrariness of δ we obtain

$$\sup_{t \geq 0} \mathbb{E} \left(e^{\varepsilon \|X_t\|^2} \right) < +\infty \quad \text{for } \varepsilon < \lambda_0. \quad (1.9)$$

This implies in particular tightness of the family $\{X_t : t \geq 0\}$ and hence by using a similar argument as in [4] we deduce the existence of an invariant measure μ for the segment process $(X_t)_{t \geq 0}$ on the space \mathcal{C} . Moreover by [12, Theorem 3] we have $\mathcal{L}(X_t)$ converges to μ in total variation as $t \rightarrow +\infty$. Hence inequality (1.9) yields

$$\int_{\mathcal{C}} e^{\varepsilon \|x\|^2} \mu(dx) < +\infty \quad \text{for } \varepsilon < \lambda_0.$$

Thus, choosing λ such that $4(2L + rL^2) < \lambda < \lambda_0$ yields the integrability condition (1.6). It is classical that the Harnack inequality (1.5) implies that the semigroup $(P_t)_{t \geq 0}$ is strong Feller and irreducible, hence uniqueness of μ follows from the classical Doob's Theorem [3, Theorem 4.2.1]. Alternatively, uniqueness follows from [5] and [12, Theorem 3].

2. PROOFS

Proof of Theorem 1.1. As in [1] we shall employ a coupling argument. Let $x, y \in \mathcal{C}$ be given and let X denote the solution of (1.1) starting from initial condition $x \in \mathcal{C}$. Fix $1 > \epsilon > 0$ and define $H : \mathbb{R}^d \rightarrow \mathbb{R}^d$,

$$H(x) = \begin{cases} \frac{x}{|x|} |x|^\epsilon, & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

H is continuous and the gradient of the convex function $\frac{1}{1+\epsilon} |x|^{1+\epsilon}$ on \mathbb{R}^d , hence it is also monotone, i.e.

$$\langle H(x) - H(y), x - y \rangle \geq 0 \quad \forall x, y \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^d . Thus, for fixed $\gamma > 0$, by the general existence and uniqueness result for monotone SDEs, c.f. e.g [7], there exists a unique process $(\tilde{Y}(t))_{t \geq 0}$ solving

$$\begin{cases} d\tilde{Y}(t) = v(\tilde{Y}(t)) dt + Z(X_t) dt - \gamma \cdot H(\tilde{Y}(t) - X(t)) dt + dW(t), \\ \tilde{Y}(0) = y(0), \end{cases} \quad (2.1)$$

and which we extend by $\tilde{Y}(t) = y(t)$ for $t \in [-r, 0]$.

In particular, in view of (1.3), for $R(t) = X(t) - \tilde{Y}(t)$ and $t \geq 0$,

$$d|R|^2(t) \leq -2\gamma \cdot |R|^{1+\epsilon}(t) dt,$$

i.e.

$$|R(t)|^2 \leq (|R(0)|^{1-\epsilon} - \gamma(1-\epsilon) \cdot t)_+^{2/(1-\epsilon)}. \quad (2.2)$$

such that $R(t) = 0$ for $t \geq |R(0)|^{1-\epsilon}/(\gamma(1-\epsilon))$. Hence, for $s \in]r, T]$, choosing $\gamma = \gamma_s = |R(0)|^{1-\epsilon}/((s-r)(1-\epsilon))$ implies $X_t = \tilde{Y}_t$ in \mathcal{C} for all $t \geq s$.

On the other hand we may rewrite equation (2.1) with $\gamma = \gamma_s$ as

$$\begin{cases} d\tilde{Y}(t) = V(\tilde{Y}_t) dt + d\tilde{W}(t), \\ \tilde{Y}_0 = y, \end{cases} \quad (2.3)$$

where $d\tilde{W}(t) = dW(t) - \zeta(t) dt$ with $\zeta(t) = \gamma_s \cdot H(\tilde{Y}(t) - X(t)) - (Z(X_t) - Z(\tilde{Y}_t))$.

Now, due to (2.2) and the Lipschitz bound on Z it holds that

$$\begin{aligned} \frac{1}{2} \int_0^T |\zeta(u)|^2 du &\leq \gamma_s^2 \int_0^s |R(u)|^{2\epsilon} du + L^2 \int_0^s \|R_u\|^2 du \\ &\leq \frac{|R(0)|^2}{(1-\epsilon^2)(s-r)} + L^2 \int_0^s \|R_u\|^2 du \\ &\leq \frac{|R(0)|^2}{(1-\epsilon^2)(s-r)} + L^2 s \|R_0\|^2 \quad \mathbb{P}\text{-a.s.}, \end{aligned} \quad (2.4)$$

where in the last step we used the a.s. monotonicity of $u \mapsto |R(u)|$ for $u \geq 0$.

In particular, the Novikov condition is satisfied for the exponential martingale $\mathcal{E}(\xi)$ with $\xi(t) = \int_0^t \zeta(s) dW(s)$, $t \in [0, T]$, and by the Girsanov theorem $(\tilde{W}_t)_{t \in [0, T]}$ is a Brownian motion under the probability measure $d\mathbb{Q} = D d\mathbb{P}$ with $D = \exp(\int_0^T \langle \zeta(u), dW(u) \rangle - \frac{1}{2} \int_0^T |\zeta(u)|^2 du)$, i.e. (\mathbb{Q}, \tilde{Y}) is weak solution to (1.1) starting from y .

Finally, for $p > 1$ and $q = p/(p-1)$

$$\begin{aligned} P_T f(y) &= \mathbb{E}_{\mathbb{Q}}[f(\tilde{Y}_T)] = \mathbb{E}_{\mathbb{P}}[D \cdot f(\tilde{Y}_T)] = \mathbb{E}_{\mathbb{P}}[D \cdot f(X_T)] \\ &\leq (\mathbb{E}_{\mathbb{P}}[D^q])^{\frac{1}{q}} (\mathbb{E}_{\mathbb{P}} f^p(X_T))^{\frac{1}{p}} = (\mathbb{E}_{\mathbb{P}}[D^q])^{\frac{1}{q}} \cdot (P_T(f^p)(x))^{\frac{1}{p}}, \end{aligned}$$

with

$$\mathbb{E}_{\mathbb{P}}(D^q) = \mathbb{E}_{\mathbb{P}}\left(\exp\left(q \int_0^T \zeta(u) dW(u) - \frac{q}{2} \int_0^T |\zeta(u)|^2 du\right)\right) \leq \left\| \exp\left(\frac{q(q-1)}{2} \int_0^T |\zeta(u)|^2 du\right) \right\|_{L^\infty(\mathbb{P})}.$$

Hence, due to (2.4), we arrive at

$$(P_T f(y))^p \leq P_T(f^p)(x) \exp\left(\frac{p}{p-1} \left(\frac{|x(0) - y(0)|^2}{(s-r)(1-\epsilon^2)} + sL^2\|x - y\|^2\right)\right),$$

such that the claim follows by letting $\epsilon \rightarrow 0$ and optimizing over $s \in]r, T]$. \square

Proof of Corollary 1.3. For $T > r$ and $x, y \in \mathcal{C}$, proceed as in the previous proof by choosing $\epsilon > 0$ and $s = T$. Then for $f : \mathcal{C} \rightarrow \mathbb{R}$ bounded measurable

$$\begin{aligned} |P_T f(x) - P_T f(y)| &= |\mathbb{E}_{\mathbb{Q}} f(\tilde{Y}_T) - \mathbb{E}_{\mathbb{P}} f(X_T)| = |\mathbb{E}_{\mathbb{P}}((1-D)f(X_T))| \\ &\leq \|f\|_{\infty} \sqrt{\mathbb{E}_{\mathbb{P}}(1-D)^2} = \|f\|_{\infty} \sqrt{\mathbb{E}_{\mathbb{P}}(D^2) - 1} \\ &\leq \|f\|_{\infty} \sqrt{\exp\left(2\left(\frac{|x(0) - y(0)|}{(T-r)(1-\epsilon^2)} + T L^2\|x - y\|^2\right)\right) - 1}, \end{aligned}$$

which tends to zero, even uniformly, for $x \rightarrow y$ in \mathcal{C} . \square

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