ON AN EVI CURVE CHARACTERIZATION OF HILBERT SPACES

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Abstract. It is shown that the existence of a large enough collection of EVI (evolution variational inequality) curves for a lower semi-continuous functional on a Banach space implies that the space is in fact a Hilbert space. The main result is exemplified by the $p$-Laplacian evolution in Banach space.

1. Introduction

In the theory of gradient flows developed in [AGS05] one may find two competing weak formulations of the basic equation

$$\dot{u} = -\nabla \varphi(u)$$

on e.g. a Riemannian manifold, both of which prompt for generalizations to more general metric spaces $(X, d)$. The first is the notion of a curve $(a, b) \ni t \mapsto u_t \in X$ of maximal slope w.r.t. the potential function $\varphi : X \rightarrow (-\infty, \infty]$ satisfying

$$\varphi(u_t)' \leq -\frac{1}{2}|u_t'|^2 - \frac{1}{2}g^2(u_t) \quad \text{for a.e. } t \in (a, b)$$

where $g : X \rightarrow [0, \infty]$ is an upper gradient of $\varphi$ and

$$|u_t'| = \limsup_{\varepsilon \to 0} \frac{d(u_{t+\varepsilon}, u_t)}{\varepsilon}$$

is the metric derivative of the curve $u$, cf. [AGS05, Definition 1.3]. The second notion applies when e.g. $\varphi$ is geodesically $\alpha$-convex. In this case a curve $(a, b) \ni t \mapsto u_t \in X$ is called a solution to the evolution variational inequality (EVI) (cf. [AGS05, eq. (11.0.3)]) if for all $v \in \text{dom}(\varphi)$

$$\frac{1}{2} \frac{d}{dt} d^2(u_t, v) + \frac{\alpha}{2} d^2(u_t, v) \leq \varphi(v) - \varphi(u_t) \quad \text{for a.e. } t \in (a, b),$$

and it is a simple exercise to check that in the smooth Riemannian situation all three notions coincide, cf. [AGS05, Theorem 11.1.4].

Formulation (1.2) has the obvious advantage over (1.3) that no convexity assumption on the potential $\varphi$ is needed. In this note we want to point out that the validity of (1.3) for a large enough collection of curves may impose strong regularity conditions also on the base space $(X, d)$. This is formulated as Theorem 3.6 below, treating the simple case when $(X, d)$ is assumed a Banach space which then must be Hilbertian provided we find sufficiently many EVI curves. Seen as a negative result this suggests that (1.3) does not really make much sense except in a Hilbertian or Riemannian situation, whereas (1.2) is known to induce a much richer theory. In

2000 Mathematics Subject Classification. 46C15, 47J20, 49J40, 58E35.

Key words and phrases. evolution variational inequality, gradient flow, characterizations of Hilbert spaces, quasi-convex function, Gâteaux subdifferentiable function, duality map.

As an example consider $X = \mathbb{R}^d$ equipped with the standard $p$-norm $\|x\|_p = (\sum |x_i|^p)^{1/p}$ and $\varphi(x) = \frac{1}{2} \sum x_i^2$, where (1.2) is equivalent to $\dot{u} = -\|u\|^{2-p} \left( |u_1|^{p-1} \text{sgn}(u_1), \ldots, |u_d|^{p-1} \text{sgn}(u_d) \right)$ and (1.3) admits only the trivial solution $u = 0$. 

1
this context, we mention the so-called doubly nonlinear evolution inclusions, see [AGS05, Proposition 1.4.1] and [RMS08].

2. EVI CURVES

Let \((X, \|\|)\) be a Banach space. Let \(\varphi : X \to (-\infty, +\infty]\) be a proper, lower semi-continuous function on \(X\). As usually, \(\text{dom}(\varphi) := \{x \in X \mid \varphi(x) < +\infty\}\).

**Definition 2.1.** Let \(\alpha \in \mathbb{R}\). We say that \(\varphi\) is \(\alpha\)-convex if
\[
x \mapsto \varphi(x) - \frac{\alpha}{2} \|x\|^2
\]
is convex.

**Definition 2.2.** We say that a function \(u : I \to X\) (where \(I\) is a real interval) satisfies an evolution variational inequality (EVI) with respect to \(\varphi\) and \(\alpha \in \mathbb{R}\) if \(u_t \in \text{dom}(\varphi)\) for every \(t \in I\) and
\[
\frac{1}{2} \frac{d^+}{dt} \|u_t - z\|^2 - \frac{1}{2} e^{\alpha t} \|u_s - z\|^2 \leq \left( \int_s^t e^{\alpha r} dr \right) [\varphi(z) - \varphi(u_r)]
\]
for all \(z \in X\) and for all \(s, t \in I\) with \(s < t\).

We denote the set of curves that satisfy an evolution variational inequality with respect to \(\varphi\) and \(\alpha\) by \(\text{Flow}_\alpha(\varphi)\).

We note that by [DS08, Section 3], (2.1) is equivalent to
\[
\frac{1}{2} \frac{d^+}{dt} \|u_t - z\|^2 + \frac{\alpha}{2} \|u_t - z\|^2 \leq \varphi(z) - \varphi(u_t) \quad \forall t \in I, \forall z,
\]
where
\[
\frac{d^+}{dt} f(t) := \limsup_{h \to 0^+} \frac{1}{h} (f(t + h) - f(t)), \quad t \in I,
\]
for any function \(f : I \to \mathbb{R}\).

According to [CD10b, Proposition 1.1], each \(u \in \text{Flow}_\alpha(\varphi)\) satisfies that
\[
t \mapsto \varphi(u_t)
\]
is nonincreasing,
\[
t \mapsto \varphi(u_t)
\]
and, under our lower semi-continuity hypothesis for \(\varphi\), \(u \in \text{Flow}_\alpha(\varphi)\) is equivalent to \(\varphi \circ u \in L^1_{\text{loc}}(I)\) and
\[
\frac{1}{2} \int_s^t \frac{\alpha}{2} \|u_r - z\|^2 \, dr \leq \int_s^t [\varphi(z) - \varphi(u_r)] \, dr
\]
for all \(z \in X\) and for all \(s, t \in I\) with \(s < t\).

If \(X\) has the Radon-Nikodým property (e.g. if it is reflexive), then it follows from absolute continuity that there exists a negligible set \(N \subset I\) such that \(u\) is differentiable in \(I \setminus N\) with derivative in \(L^1_{\text{loc}}(I; X)\), see e.g. [AGS05, Remark 1.1.3]. Let \(s_0 \in I\) such that \(u\) is differentiable in \(s_0\). Let \(J : X \to 2^{X^*}\) be the normalized duality map, where \((X^*, \|\|_{X^*})\) is the Banach space dual of \(X\). In other words,
\[
y \in J(x) \iff x^* (y, x) = \|y\|^2_{X^*} = \|x\|^2.
\]

Note that the normalized duality map is homogeneous and norm-to-weak* upper semi-continuous (in the sense of graphs), see [Cio00, Proposition 1.4.7, Theorem 1.4.12]. It follows from (2.4) and the subdifferentiability of \(\frac{1}{2} \|\cdot\|^2\) that for all \(w \in J(u_{s_0} - z)\)
\[
\frac{1}{2} \int_{s_0}^t \frac{\alpha}{2} \|u_r - z\|^2 \, dr \leq \int_{s_0}^t [\varphi(z) - \varphi(u_r)] \, dr \quad \forall z
\]
Dividing by $t - s_0$ and passing to the limit $t \to s_0^+$ gives for any $w \in J(u_{s_0} - z)$
\begin{equation}
\chi_x(w, u_{s_0})_X + \frac{\alpha}{2} \|u_{s_0} - z\|^2 \leq \varphi(z) - \varphi(u_{s_0}) \quad \forall z.
\end{equation}

The limit
\[ \lim_{t \to s_0^+} \frac{1}{t - s_0} \int_{s_0}^t [\varphi(z) - \varphi(u_r)] \, dr \]
exists because
\[ \lim_{r \to s_0^+} \varphi(u_r) = \varphi(u_{s_0}) \]
by the lower semi-continuity of $\varphi$ and (2.3).

\section{The main result}

\begin{definition}
Let $D \subset X$. Let $x \in \text{dom}(\varphi)$. The left $D$-Gâteaux subgradient $\partial^L_D\varphi(x) \subset X^*$ is defined by
\[ y \in \partial^L_D\varphi(x) \iff \liminf_{t \to 0^+} \frac{\varphi(x + th) - \varphi(x)}{t} \geq \chi_x'(y, h)_X \quad \forall h \in D. \]
\end{definition}

If $D = \{ x \in X \mid |x| \leq 1 \}$, then $\partial^L_D\varphi(x)$ is the local subgradient which coincides with the usual subgradient $\partial\varphi(x)$ whenever $\varphi$ is convex and Gâteaux differentiable in $x$. We note that our results below remain true, if one replaces “$\liminf_{t \to 0^+}$” in Definition 3.1 by “$\limsup_{t \to 0^-}$”.

\begin{definition}
Define $\Gamma_\alpha(\varphi)$ by
\[ (s, u) \in \Gamma_\alpha(\varphi) \iff s \in \text{dom}(u), u \in \text{Flow}_\alpha(\varphi) \text{ and } u_s \text{ is differentiable at } s. \]
\end{definition}

\begin{hypothesis}
There exist sets $D, U \subset X$ such that $D$ is dense in the closed unit ball of $X$ and $U$ is open in $X$ such that $\text{dom}(\varphi) \cap U \neq \emptyset$ and for any $x \in \text{dom}(\varphi) \cap U$ and for any $h \in D$, there exists $\delta > 0$ such that $x - \varepsilon h \in \text{dom}(\varphi) \cap U$ for every $\varepsilon \in (0, \delta)$. Moreover, $\varphi$ is left $D$-Gâteaux subdifferentiable in some non-void subset $G \subset \text{dom}(\varphi) \cap U$, where
\begin{equation}
\{u_s \mid (s, u) \in \Gamma_\alpha(\varphi)\} \cap G \neq \emptyset
\end{equation}
and
\begin{equation}
X_0 := \{u_s \mid (s, u) \in \Gamma_\alpha(\varphi), u_s \in G\}
\end{equation}
is a weakly dense subset of $X$.
\end{hypothesis}

\begin{remark}
Suppose that $\text{dom}(\varphi) = U$ and $\varphi$ is continuous and convex. If $X$ is a weak Asplund space, e.g., if $X$ is reflexive, then according to results from [Phe89], $\varphi$ is both sided Gâteaux differentiable on a dense $G_\delta$-set. In this case, for Hypothesis 3.3 to hold, we need only to verify (3.2).
\end{remark}

\section{Example: The heat flow}

Let us convince ourselves that Hypothesis 3.3 is meaningful. Consider $X = L^2(\Omega, dx)$ for some bounded subdomain $\Omega \subset \mathbb{R}^d$ with sufficiently smooth boundary. We write $H^1_0 = H^1_0(\Omega, dx)$. Define $\varphi : L^2 \to [0, +\infty]$ by
\[ \varphi(u) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx, & \text{if } u \in H^1_0, \\ +\infty, & \text{if } u \in L^2 \setminus H^1_0. \end{cases} \]

By Poincaré inequality, $\|u\|_0 := \sqrt{\varphi(u)}$ is an equivalent norm of $H^1_0$. Let $C > 0$ be the Poincaré constant of $\Omega$, i.e.
\[ C \|u\|_0 \geq \|u\| \quad \forall u \in H^1_0. \]
It is well-known that \( \varphi \) is Fréchet differentiable in \( H^1_0 \) and the differential \( D\varphi \) equals the Riesz isometry of \( H^1_0 \) w.r.t. the inner product coming from \( \| \cdot \|_0 \) which is the Dirichlet Laplacian 
\[
- \Delta : H^1_0 \to H^{-1}.
\]
At each point \( u \in H^1_0 \cap H^2 \), we find \( f := -\Delta u \in L^2 \), such that 
\[
\lim_{t \to 0} \frac{\varphi(u + th) - \varphi(u)}{t} = - \frac{1}{2} \langle D\varphi(u), h \rangle_{H^1_0} = \int_\Omega \langle \nabla u, \nabla h \rangle \, dx = \langle h, f \rangle_{L^2} \quad \forall h \in H^1_0.
\]
Let \( \{ T_t \}_{t \geq 0} \) be the \( L^2 \)-contraction \( C_0 \)-semigroup associated to \(-\Delta\).

We claim that \( \varphi \) satisfies Hypothesis 3.3. Clearly, \( \operatorname{dom}(\varphi) = H^1_0 \). Set \( X = U = L^2 \), \( D = \{ u \in H^1_0 \mid \| u \| \leq 1 \} \) and \( G = H^1_0 \cap H^2 \). Let \( x \in G \). (3.3) verifies the \( D \)-Gâteaux subdifferentiability of \( \varphi \). By the theory of semigroups, \( T_t x \) is differentiable in \( t \) and it holds that 
\[
\frac{d}{dt} T_t x = \Delta T_t x,
\]
see e.g. [Pan93]. Let \( z \in L^2 \). By the subdifferentiability of \( \varphi \) in \( G \) and the differentiability of the \( L^2 \)-norm, we get from 
\[
\frac{1}{2} \left| \frac{d}{dt} \left| T_t x - z \right|^2 \right| = \left( T_t x - z, \Delta T_t x \right)_{L^2} \leq \varphi(z) - \varphi(T_t x)
\]
that (2.2) is satisfied for \( t \mapsto T_t x \), \( t \geq 0 \). Hence \( (t, T_t x) \in \Gamma_0(\varphi) \) for \( t \geq 0 \).

Since \( T_t x \in G \), (3.1) is satisfied. By Poincaré inequality, the quadratic form \( \varphi \) associated to \( -\Delta |_{H^1_0 \cap H^2} \) is bounded and coercive in \( H^1_0 \) (see Remark 3.5 below for the terminology) and therefore \( -\Delta : H^1_0 \to H^{-1} \) is a surjective isometry by the Lax-Milgram theorem. Then \( -\Delta |_{H^1_0 \cap H^2} \) has full range in \( L^2 \), see e.g. [Tan79, p. 27]. As a consequence, taking (3.4) into account, (3.2) is satisfied for \( s = 0 \).

**Remark 3.5.** Let \( X \) be a Hilbert space and let \( V \) be another Hilbert space such that \( V \) is densely and continuously embedded into \( X \). Let \( a : V \times V \to \mathbb{R} \) be a symmetric quadratic form which is bounded, i.e. there exists a constant \( M > 0 \) such that
\[
|a(u, v)| \leq M \| u \|_V \| v \|_V \quad \forall u, v \in V
\]
and coercive, i.e. there exists a constant \( k > 0 \), such that 
\[
a(u, u) \geq k \| u \|_V^2 \quad \forall u \in V.
\]
Let 
\[
\varphi(u) := \begin{cases} 
\frac{1}{2} a(u, u), & \text{if } u \in V, \\
+ \infty, & \text{if } u \in X \setminus V.
\end{cases}
\]
Then Hypothesis 3.3 is satisfied for \( \varphi \).

The proof follows essentially the steps above. We refer to [Tan79, Ch. 2] for the related facts.

**Theorem 3.6.** Suppose that \( \varphi : X \to (-\infty, +\infty] \) is proper and l.s.c. on the Banach space \( X \). Suppose that \( (\varphi, \operatorname{dom}(\varphi)) \) and \( \text{Flow}_\varphi \) satisfy Hypothesis 3.3. Suppose either that \( D = -D \) or that the norm of \( \langle X, \| \cdot \| \rangle \) is both sided Gâteaux differentiable everywhere except in the origin (i.e. \( X \) is smooth).

Then \( X \) is a Hilbert space and \( \partial_{\overline{D}} \varphi \) is single-valued on \( G \) such that for \( (s, u) \in \Gamma_\alpha(\varphi) \) with \( u \in G \) we have that 
\[
\dot{u}_s = -\partial_{\overline{D}} \varphi(u_s).
\]
Moreover, if \( U = X \) and \( G \) is dense in \( \operatorname{dom}(\varphi) \), then \( \varphi \) is \( \alpha \)-convex.
Remark 3.7. Suppose that \( \text{dom}(\varphi) \) is dense in \( X \) and that \( \varphi \) is bounded from below. Suppose for simplicity that \( I = [0, +\infty) \) and that for each \( x \in X \) there exists \( u^x \in \text{Flow}_\alpha(\varphi) \) such that

(i) \( x \mapsto u^x_t \) is continuous for \( x \in X , t \geq 0 \),

(ii) \( u^x_{t+h} = u^{u^x_t}_h \) for \( x \in X , t, h \geq 0 \),

(iii) \( \lim_{t \to 0^+} u^x_t = u^x_0 = x \) for \( x \in X , t \geq 0 \),

(iv) \( u^x_t \in \text{dom}(\varphi) \) for \( t > 0 \).

Then \( \varphi \) is geodesically \( \alpha \)-convex, i.e. for \( x, y \in X \),

\[
\varphi((1-s)x + sy) \leq (1-s)\varphi(x) + s\varphi(y) - \frac{\alpha}{2} s(1-s) ||x - y||^2 \quad \forall s \in [0, 1],
\]

compare with (4.6) below. The proof can be found in [DS08, Theorem 3.2]. In fact, if \( X \) is a Hilbert space, geodesic \( \alpha \)-convexity is equivalent to \( \alpha \)-convexity.

Proof of Theorem 3.6. Let \( (s_0, u) \in \Gamma_\alpha(\varphi) \cap G, \xi \in X_0 \) such that \( \xi = u_{s_0} \), which exist by (3.1). Set \( x := u_{s_0} \).

Let \( h \in D \). According to Hypothesis 3.3, pick \( \delta > 0 \) such that

\[
(3.7) \quad z := x - \varepsilon h \in \text{dom}(\varphi) \cap U \quad \forall \varepsilon \in (0, \delta).
\]

Plugging into (2.6), we get that for \( w \in J(h) \), by homogeneity,

\[
X(\varepsilon w, \xi)_X + \frac{\varepsilon}{2} ||\varepsilon h||^2 \leq \varphi(x - \varepsilon h) - \varphi(x) < +\infty \quad \forall \varepsilon \in (0, \delta).
\]

Furthermore, after multiplying with \( -1 \),

\[
X(-w, \xi)_X \leq \frac{\varepsilon}{2} ||w||^2 \geq \frac{\varphi(x) - \varphi(x - \varepsilon h) - \varphi(x) - \varepsilon}{-\varepsilon} \quad \forall w \in J(h) \forall \varepsilon \in (0, \delta).
\]

For \( y \in \partial_{\Omega} \varphi(x) \), we have that \( (t = -\varepsilon) \)

\[
X(-w, \xi)_X \geq \lim_{t \to 0^-} \frac{\varphi(x + th) - \varphi(x)}{t} \geq X(y, h)_X \quad \forall w \in J(h).
\]

Assume first that \( D = -D \). Then we can replace \( h \) by \( -h \) to get by homogeneity of \( J \) that

\[
X(w, \xi)_X \geq X(y, h)_X \quad \forall h \in D \forall w \in J(h).
\]

Hence

\[
X(w, \xi)_X = X(y, h)_X \quad \forall h \in D \forall w \in J(h).
\]

Since \( D \) is assumed to be dense in the closed unit ball \( \overline{B}_X(0, 1) \) and \( J \) is norm-to-weak* upper semi-continuous (see [Phe89, Proposition 2.5]), we get that

\[
X(-j(h), \xi)_X = X(y, h)_X \quad \forall h \in \overline{B}_X(0, 1),
\]

where \( j(h) \in J(h) \) is some selection (depending on the limit procedure). Alternatively, the above closure could be achieved by the maximal monotonicity of \( J \), weak*-compactness and Minty’s trick.

Hence

\[
(3.8) \quad X(-j(h), \xi)_X = X(y, h)_X \quad \forall h \in S_X(0, 1),
\]

where \( S_X(0, 1) \) denotes the centered unit sphere of \( X \).

Suppose now that \( X \) is smooth but not necessarily that \( D = -D \). Then the duality map \( j = J \) is single-valued by [Phe89, Example 2.26]. As above, we have that

\[
X(-j(h), \xi)_X \geq X(y, h)_X \quad \forall h \in D.
\]
Since $D$ is assumed to be dense in the closed unit ball $\overline{B}_X(0,1)$ and $J$ is norm-to-weak* upper semi-continuous (and hence norm-to-weak* continuous by [Phe89, Proposition 2.8]), we get that
\[ x^* \langle -j(h), \xi \rangle_X \geq x^* \langle y, h \rangle_X \quad \forall h \in \overline{B}_X(0,1), \]
Replacing $h$ by $-h$ yields
\[ x^* \langle j(h), \xi \rangle_X \geq x^* \langle y, -h \rangle_X \quad \forall h \in \overline{B}_X(0,1), \]
hence
\[ x^* \langle -j(h), \xi \rangle_X = x^* \langle y, h \rangle_X \quad \forall h \in S_X(0,1), \]
where $S_X(0,1)$ denotes the centered unit sphere of $X$, which is (3.8) again.

By extending by scaling, we find a homogeneous selection of the duality map $j' \subset J$ such that
\[ x^* \langle j'(h), \xi \rangle_X = x^* \langle -y, h \rangle_X \quad \forall h \in X, \]
i.e.,
\[ j'(u) := \begin{cases} \|u\| \left( \frac{u}{\|u\|} \right), & u \neq 0, \\ 0, & u = 0. \end{cases} \]
Equation (3.9) is true for any $\xi \in X_0$, and the choice of $j'$ in (3.9) does not depend on $\xi$. By weak density of $X_0$, we find that $j'$ is a linear selection of the normalized duality map $J$. By [Ami86, (6.7')] this is equivalent to $\|\cdot\|$ being a Hilbertian norm and $J$ being the Riesz map.

Let us identify $X$ with $X^*$. Setting $h = -\xi$ yields
\[ \|\xi\|^2 = (y, -\xi) \]
and
\[ \|\xi\| \leq \|y\|. \]
Also,
\[ \|y\| \leq \sup_{\|h\| \leq 1} \langle (y, h) \rangle_X = \sup_{\|h\| \leq 1} \|h, -\xi\| \leq \|\xi\|. \]
Hence
\[ y = -\xi \]
and so
\[ \partial_D^\alpha \varphi(x) = -\xi, \]
proving the single-valuedness.

We are left with proving $\alpha$-convexity of $\varphi$, if $U = X$. Since we are in the Hilbert space situation, (2.6) becomes
\[ (x - z, \xi(x))_X + \frac{\alpha}{2} \|x - z\|^2 + \varphi(x) \leq \varphi(z) \quad \forall z \in X, \]
where $x \in G$ and $\xi(x) = \xi \in X_0$ are as above. By the parallelogram law,
\[ (x - z, \xi(x))_X - \alpha(z, x)_X + \varphi(x) + \frac{\alpha}{2} \|x\|^2 \leq \varphi(z) - \frac{\alpha}{2} \|z\|^2 \quad \forall z \in X, \]
Here we use that $G$ is dense in $\text{dom}(\varphi)$ to get that
\[ \sup_{x \in G} \left[ (x - z, \xi(x))_X - \alpha(z, x)_X + \varphi(x) + \frac{\alpha}{2} \|x\|^2 \right] \leq \varphi(z) - \frac{\alpha}{2} \|z\|^2 \quad \forall z \in X, \]
where the supporting point is $x = z$. Hence $\varphi - \frac{\alpha}{2} \|\cdot\|^2$ is convex as a pointwise supremum of affine functions, see e.g. [ABM06, Theorem 9.3.5].
\[ \square \]
4. Consequences

Let $u : I \mapsto X$ be any function. Suppose that the following limit exists for some $s \in I$

$$\lim_{t \to s^+} \frac{y^*}{t-s} \left< y, \frac{u_t - u_s}{t-s} \right> \text{ for every } y \in X^*$$

and is finite. Then $u$ is called weakly right differentiable at $s$ and the above limit defines an element in $X$ which we denote by

$$\frac{d^{+,w}}{ds} u_s$$

and call the weak right derivative of $u$ at $s$.

Remark 4.1. Note that we can weaken Hypothesis 3.3 without breaking the preceding arguments by considering merely points of weak right differentiability of $u_t$, i.e. we reformulate Hypothesis 3.3 with

$$R(s, u_t, \gamma) = (s; u_t) \in \Gamma_\alpha^+ (\varphi),$$

iff

$$u_t \in \text{Flow}_\alpha (\varphi) \text{ and } \frac{d^{+,w}}{ds} u_s \text{ exists}.$$ 

Therefore, let us define:

Hypothesis 4.2. There exist sets $D, U \subset X$ such that $D$ is dense in the closed unit ball of $X$ and $U$ is open in $X$ such that $\text{dom}(\varphi) \cap U \neq \emptyset$ and for any $x \in \text{dom}(\varphi) \cap U$ and for any $h \in D$, there exists $\delta > 0$ such that $x - \varepsilon h \in \text{dom}(\varphi) \cap U$ for every $\varepsilon \in (0, \delta)$. Moreover, $\varphi$ is left $D$-Gâteaux subdifferentiable in some non-void subset $G \subset \text{dom}(\varphi) \cap U$, where

$$X_1 := \{ u_s \mid (s, u) \in \Gamma_\alpha^+ (\varphi) \} \cap G \neq \emptyset.$$ 

and

$$X_0 := \left\{ \varepsilon \frac{d^{+,w}}{ds} u_s \mid (s, u) \in \Gamma_\alpha^+ (\varphi), u_s \in G \right\}$$

is a weakly dense subset of $X$.

The conclusion of Theorem 3.6 remains true under the assumption of Hypothesis 4.2.

Proposition 4.3. Suppose that $X$ is a smooth reflexive Banach space. Let $\varphi$ be everywhere Fréchet differentiable in $X$ such that

$$\nabla \varphi : X \mapsto X^* \text{ is surjective.}$$

Then the following statements are equivalent for any $\alpha \in \mathbb{R}$:

(i) $X_1 := \bigcup_{u \in \text{Flow}_\alpha (\varphi)} \bigcup_{s \in I} \{ u_s \}$ is dense in $X$.

(ii) $X$ is a Hilbert space and $\varphi$ is $\alpha$-convex.

Then, in fact, $X_1 = X$.

Proof. Assume (i). We would like to verify Hypothesis 4.2. Set $D := B_X(0,1)$, $G = U = X$. The first part of the hypothesis is clear by Fréchet differentiability. Since $\varphi$ is continuous and defined everywhere on $X$, the limit in (2.6) always exists and is finite. In other words, $\Gamma_\alpha^+ (\varphi) = I \times \text{Flow}_\alpha (\varphi)$. (4.1) follows. By the proof of Theorem 3.6, we deduce (without employing (4.2)) that

$$x^* \cdot (J(h), -\xi) = x^* \cdot (y, h) \quad \forall h \in X,$$

where $\xi := \frac{d^{+,w}}{ds} u_t$ and $y := \nabla \varphi(u_t)$ and $J : X \mapsto X^*$ is the single-valued normalized duality map of $X$ (due to smoothness).
Setting $h = -\xi$ yields
\[ \|\xi\|^2 = x \cdot \langle y, -\xi\rangle, \]
and
\[ \|\xi\| \leq \|y\|_*. \]
Also,
\[ \|y\|_* = \sup_{\|h\| \leq 1} |x \cdot \langle y, h\rangle| = \sup_{\|h\| \leq 1} |x \cdot \langle J(h), -\xi\rangle| \leq \|\xi\|. \]
Hence
\[ y \in J(-\xi) \]
and therefore
\[ \nabla \varphi(u_t) = J \left( -\frac{d^{+w}}{dt} u_t \right). \]
Then,
\[ J^{-1}(\nabla \varphi(u_t)) = -\frac{d^{+w}}{dt} u_t. \]
Since $J$ is maximal monotone, $J^{-1}$ is maximal monotone as well. By reflexivity of $X$, $J^{-1}$ equals the duality map of $X^*$. We get that
\[ J^{-1}(\nabla \varphi(u_t)) = -\frac{d^{+w}}{dt} u_t. \]
And hence by the preceding discussion, for each $x \in X_1$ there is $\xi \in X_0$ such that
\[ \text{(4.4)} \quad J^{-1}(\nabla \varphi(x)) = -\xi. \]
We claim that the map
\[ \text{(4.5)} \quad F : x \mapsto J^{-1}(\nabla \varphi(x)). \]
is norm-to-weak* continuous and surjective. Indeed, surjectivity follows from coercivity of $J^{-1}$ and the assumption. By Fréchet differentiability, $\nabla \varphi$ is norm-to-norm continuous. $J^{-1}$ is norm-to-weak continuous again by reflexivity, see [Phe89, p. 20].

Noting that surjective continuous maps preserve density of subsets, we combine (4.4) and (4.5) to get that (i) implies the weak density of $X_0$ in $X$. This verifies (4.2) so that (ii) follows from Theorem 3.6 together with Remark 4.1.

The converse implication “(ii) $\Rightarrow$ (i)” follows from [Čle09, Theorem 3.2].

Note that if $\alpha > 0$, we do not need to assume the surjectivity of the gradient in the direction “(ii) $\Rightarrow$ (i)” by [BFV94, Theorem 3.3].

Recall that the local slope $|\partial \varphi|(x)$ of $\varphi$ at $x \in X$ is defined by
\[
|\partial \varphi|(x) := \begin{cases} 
\limsup_{y \to x} \frac{\varphi(x) - \varphi(y)}{|x - y|^\alpha}, & \text{if } x \text{ is not isolated in } \text{dom}(\varphi), \\
\infty, & \text{otherwise}.
\end{cases}
\]
Set $\text{dom}(|\partial \varphi|) := \{x \in \text{dom}(\varphi) \mid |\partial \varphi|(x) < \infty\}$.

The next theorem can be found in [AGS05, Theorem 4.0.4]. For related results we refer to [Čle09, Čle10, CD10a].

**Theorem 4.4.** Let $X$ be a Banach space. Let $\varphi : X \to (-\infty, +\infty]$ be proper, l.s.c.

Suppose that there exists $\alpha \in \mathbb{R}$ such that for every $x, y, z \in \text{dom}(\varphi)$, there exists a map $\gamma : [0, 1] \to \text{dom}(\varphi)$ satisfying $\gamma(0) = x$, $\gamma(1) = y$ for which the following
inequality holds:
\[
\frac{1}{2h} \|z - \gamma(t)\|^2 + \varphi(\gamma(t)) 
\leq (1-t) \left[ \frac{1}{2h} \|z - x\|^2 + \varphi(x) \right] 
+ t \left[ \frac{1}{2h} \|z - y\|^2 + \varphi(y) \right] 
- \left( \frac{1}{h} + 1 \right) \frac{1}{2} t(t-1) \|x - y\|^2, 
\]
for every \( t \in (0, 1) \) and every \( h > 0 \) such that \( 1 + ah > 0 \).

Suppose that there exist \( x_* \in \text{dom}(\varphi) \), \( r_* > 0 \) and \( m_* \in \mathbb{R} \) such that \( \varphi(y) \geq m_* \) for every \( y \in X \) satisfying \( \|x_* - y\| \leq r_* \).

Then \( \text{Flow}_\alpha(\varphi) \) is non-empty and
\[
\bigcup_{u \in \text{Flow}_\alpha(\varphi)} \{ u_0 \} = \text{dom}(\partial \varphi), 
\]
and if \( \alpha = 0 \) in the above condition,
\[
\bigcup_{u \in \text{Flow}_\alpha(\varphi)} \{ u_0 \} = \overline{\text{dom}(\varphi)}. 
\]

We would like to illustrate the connection between the above result and Proposition 4.3. Let \( X \) be a non-Hilbertian Banach space with Fréchet differentiable norm. Set
\[
\varphi(x) := \frac{1}{2} \|x\|^2. 
\]
Obviously, \( \varphi \) is 1-convex and Fréchet differentiable so that
\[
\text{dom}(\partial \varphi) = \text{dom}(\varphi) = X. 
\]
For \( x, y \in X \), let \( \gamma(t) := (1-t)x + ty \). We would like to verify
\[
\frac{1}{2h} \|z - (1-t)x + ty\|^2 + \frac{1}{2} \|(1-t)x + ty\|^2 
\leq (1-t) \left[ \frac{1}{2h} \|z - x\|^2 + \frac{1}{2} \|x\|^2 \right] 
+ t \left[ \frac{1}{2h} \|z - y\|^2 + \frac{1}{2} \|y\|^2 \right] 
- \left( \frac{1}{h} + 1 \right) \frac{1}{2} t(t-1) \|x - y\|^2, 
\]
for all \( t \in (0, 1) \), \( h > 0 \), \( x, y, z \in X \). However, for \( z = 0 \), \( h = 1 \), (4.9) is known to characterize Hilbert spaces, see \[Kas54\]. As a consequence, by means of standard Banach space geodesics \( \gamma(t) = (1-t)x + ty \), we cannot expect to verify (4.7) with the help of Theorem 4.4 for the square of the norm. If we could, however, verify (4.6) for some other curve \( \gamma \) (the lower boundedness condition follows for the square of norm) this would imply that \( X \) is a Hilbert space by Proposition 4.3. We get the following

\textbf{Corollary 4.5.} Suppose \( \varphi \) is a Fréchet differentiable function on a smooth reflexive Banach space \( X \) such that \( \varphi \) satisfies the hypotheses of Theorem 4.4. Then

(i) If \( \nabla \varphi \) is surjective, then \( X \) is a Hilbert space.
(ii) If \( \alpha > 0 \) and \( X \) is a Hilbert space, then \( \varphi \) is \( \alpha \)-convex and \( \nabla \varphi \) is surjective.

Generally speaking, we do not expect to get sufficiently many directions \( u_0 \) in any other than the Hilbert space situation.
4.1. Example: The $p$-Laplacian evolution. Let $1 < p, q < \infty$, $p \neq 2$. Consider $L^2 = L^2(\Omega, dx)$ for some bounded subdomain $\Omega \subset \mathbb{R}^d$. Suppose that the dense and continuous inclusion
\[ W^{1,p}_0(\Omega) \subset L^q(\Omega) \]
holds. Suppose also that we have a Poincaré-type inequality (with constant $C > 0$)
\[ (4.10) \quad C \|\nabla u\|_{L^q} \geq \|u\|_{L^p} \quad \forall u \in W^{1,p}_0(\Omega). \]

Define $\psi : L^2 \to [0, +\infty]$ by
\[ \psi(u) := \begin{cases} \frac{1}{p} \int_\Omega |\nabla u|^p \, dx, & \text{if } u \in W^{1,p}_0(\Omega), \\ +\infty, & \text{if } u \in L^q \setminus W^{1,p}_0(\Omega). \end{cases} \]

It is well-known that $\varphi := \psi|_{W^{1,p}_0}$ is Fréchet differentiable in $W^{1,p}_0$ with differential given by
\[ (\psi(u), h)_{W^{1,p}_0} = \int_\Omega (|\nabla u|^{p-2} \nabla u, \nabla h) \, dx. \]

**Proposition 4.6.** Suppose that in the above situation, \[(4.10)\] holds.

Let $D := \{ u \in W^{1,p}_0 \mid \|u\|_{L^q} \leq 1 \}$, $U := L^q$, $G = \text{dom}(\partial \psi) \subset W^{1,p}_0$. Then conditions (3.1), (3.2) hold if and only if $q = 2$.

**Proof.** We note that in our situation, $D\varphi = \partial \psi$ on $G = \text{dom}(\partial \psi)$. Therefore Hypothesis 3.3 holds for $U, D, G$ as above, $\alpha = 0$. Hence by Theorem 3.6, $q = 2$.

Conversely, if $q = 2$, there is a nonlinear semigroup
\[ t \mapsto S_t \]
such that $S_t$ leaves $\text{dom}\partial \psi$ invariant and
\[ \frac{d}{dt} S_t x = -\partial \psi(S_t x), \]
see [Ze90, Theorem 31.A]. Hence, since $G = \text{dom}\partial \psi$ is dense in $\text{dom} \psi$ by the theory of subdifferentials, we can verify the following. Let $z \in L^2$, $x \in G$. By the subdifferentiability of $\psi$ in $G$ and the differentiability of the $L^2$-norm, we get from
\[ (4.11) \quad \frac{1}{2} \frac{d}{dt} \|S_t x - z\|^2 = \langle z - S_t x, \partial \psi(S_t x) \rangle_{L^2} \leq \psi(z) - \psi(S_t x) \]
that (2.2) is satisfied for $t \mapsto S_t x$, $t \geq 0$. Hence $(t, S_t x) \in \Gamma_0(\psi)$ for $t \geq 0$. By Poincaré inequality (4.10), $\psi$ is coercive and hence $\partial \psi$ has full range in $L^2$. Hence (3.2) is verified. \qed

**Remark 4.7.** Let $V, X$ be reflexive Banach spaces. Suppose that $V \subset X$ densely and continuously such that for $C > 0$
\[ C \|u\|_V \geq \|u\|_X \quad \forall u \in V. \]

Suppose $\varphi : V \to \mathbb{R}$ is everywhere Gâteaux differentiable. Define
\[ \psi(u) := \begin{cases} \varphi(u), & \text{if } u \in V; \\ +\infty, & \text{if } u \in X \setminus V. \end{cases} \]

Suppose that there exists a continuous nondecreasing function $N : [0, \infty) \to [0, \infty)$ with $N(x) = 0$ iff $x = 0$ and $\lim_{t \to -\infty} N(t)/t = +\infty$ such that
\[ \varphi(u) \geq N(\|u\|_V) \quad \forall u \in V. \]

Then $\psi$ is proper and l.s.c. by closed sub-level sets. Also, $\psi$ is coercive. As above, one can prove that (3.1), (3.2) with $U = X$, $D = \{ u \in V \mid \|u\|_X \leq 1 \}$ and $G = \text{dom}(\partial \psi)$ is equivalent to $X$ being a Hilbert space and $\varphi$ being $\alpha$-convex.

Indeed, the $p$-Laplacian is seen to be a special case.
Acknowledgements

The authors are grateful for several helpful remarks by Guiseppe Savaré on a previous version of this work. The authors would like to thank the referee for valuable comments.

References


