Heat Kernel Comparison on Alexandrov Spaces with Curvature Bounded Below

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March 19, 2003

Abstract

In this paper the comparison result for the heat kernel on Riemannian manifolds with lower Ricci curvature bound by Cheeger and Yau [CY81] is extended to locally compact path metric spaces (X, d) with lower curvature bound in the sense of Alexandrov and with sufficiently fast asymptotic decay of the volume of small geodesic balls. As corollaries we recover Varadhan's short time asymptotic formula for the heat kernel [Var67] and Cheng's eigenvalue comparison theorem [Che75]. Finally, we derive an integral inequality for the distance process of a Brownian Motion on (X, d) resembling earlier results in the smooth setting by Debiard, Geavau and Mazet [DGM75].

AMS 2000 Subject Classification: Primary 31C25, 53C22; Secondary 53C21, 58G32.

Keywords: Alexandrov Space, Heat Kernel Estimate, Diffusion Process, Dirichlet Form.

Acknowledgments: This work owes much to my teacher Karl-Theodor Sturm whose advice and support is gratefully acknowledged.

*Supported by the Collaborative Research Center SFB611 of the German Research Council (DFG).

1. Introduction. Metric spaces (X, d) satisfying generalized curvature conditions have attained a lot of interest in recent years. One approach for the definition of such generalized curvature bounds due to Alexandrov is based upon the the observation that, heuristically speaking, in case of smooth Riemannian spaces the convexity of geodesic triangles is an increasing function of curvature. By comparison with Riemannian manifolds of constant curvature this fact can be used for a definition of upper or lower curvature bounds for path metric spaces, which in the latter case are called Alexandrov spaces. In full generality Alexandrov spaces were extensively studied by Burago, Gromov and Perelman [BGP92] who showed that such spaces exhibit a lot of nice geometric and regularity properties. One particularly important feature for further analytical study is the fact that Alexandrov spaces are of integer (topological and) Hausdorff dimension such that in the finite dimensional case every Alexandrov space is in fact a metric measure space (X, d, m) with a natural nontrivial Haussdorff measure m of appropriate dimension. Since then further results on the topology and regularity of Alexandrov spaces have been obtained. For instance, Otsu and Shioya [OS94] showed that an n-dimensional Alexandrov space (X, d) with lower curvature bound is almost Riemannian, i.e. that there is a singular set S_X of Hausdorff dimension $\dim_H(S_X) \leq n-1$ such that $S \setminus S_X$ has a (weak) C^1-Riemannian structure which is compatible with the intrinsic metric on X. Moreover, several definitions of the Laplacian on (X, d) have been given via the construction of a suitable and "natural" Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. One way is to use the weak C^1-Riemannian structure to define the classical energy integral [KMS01], another method for more general metric spaces with doubling measure was proposed by Cheeger who obtained from a local blow-up procedure a finite dimensional Finsler cotangent bundle and an induced quadratic energy form [Che99]. A third construction of the Dirichlet form stems from an idea by Sturm [Stu98] in the context of metric measure spaces with the so called measure contraction property. Here the Dirichlet form is given as the Γ-limit of a sequence of non local forms corresponding to geometric random walks on X. Kuwae and Shioya [KS01] later showed that this method, slightly generalized, applies to Alexandrov spaces as well and yields the same Sobolev spaces as those constructed by means of the weak Riemannian structure [KS98]. All constructions can also be localized to domains $G \subset X$ in which case the coincidence of the resulting energy forms $(\mathcal{E}_G, D(\mathcal{E}_G))$, Sobolev spaces etc. is not obvious and closely related to the regularity properties of the boundary ∂G. We will not discuss the uniqueness question here since we confine ourselves completely to Sturm's framework.

The weak Riemannian structure, however, does not seem to be suitable for a further geometric analysis of semigroups and diffusion processes since it is not defined everywhere and the metric tensor, wherever defined, has only very low smoothness. In particular it is not obvious how to transfer classical analytical proofs involving curvature like the Bochner-Lichnerowicz-Weitzenböck formula to Alexandrov spaces, such that one has to find alternative arguments. As a first step in this direction we present in this paper the extension of well-known comparison properties of the Laplace-Beltrami operator on Riemannian manifolds and a number of corollaries which we prove in a purely intrinsic and hence probably very geometric way.

1.1. Results. We utilize Sturm's scheme to show that if a path metric space (X,d) satisfies the Toponogov comparison property concerning the convexity of geodesic triangles and an additional regularity assumption on the asymptotic decay of the volume of small geodesic balls (see section 2.3), then several other well known comparison results for the heat semigroup on smooth Riemannian manifolds carry over to (X,d). Our results read as follows:

Theorem I (Laplacian Comparison). *Let (X,d) be an n-dimensional Alexandrov space with curvature bounded below by $k \in \mathbb{R}$ which is locally volume $(L^1,1)$-regular with exceptional set $\mathbb{S}_X^{1,1}$ of rough dimension $\leq n-2$ and let Δ^X denote the generator of the canonical Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on (X,d). Then for any $f \in C^3(\mathbb{R})$ with $f' \leq 0$, $0 \leq \zeta \in D(\mathcal{E})$ and $p \in X$ the inequality*

$$\Delta^X(f \circ d_p)(x) \geq S_k^{1-n}(S_k^{n-1} f')' \circ d_p(x) \tag{1}$$

holds in the weak sense, i.e.

$$\mathcal{E}(f \circ d_p, \zeta) \leq \langle -S_k^{1-n}(S_k^{n-1} f')' \circ d_p, \zeta \rangle_{L^2(X,m)} \tag{2}$$

where $d_p(x) := d(p,x)$ and

$$S_k(t) = \begin{cases} 1/\sqrt{k}\sin(\sqrt{k}t) & \text{if } k > 0 \\ t & \text{if } k = 0 \\ 1/\sqrt{(-k)}\sinh(\sqrt{-k}t) & \text{if } k < 0. \end{cases}$$

Note that if X is the simply connected n-dimensional Riemannian manifold $\mathbb{M}_{n,\kappa}$ of constant curvature k then the right hand side of (1) coincides with $\Delta^{\mathbb{M}_{n,k}}(f \circ d_p)(x)$, where $\Delta^{\mathbb{M}_{n,k}}$ is the Laplace-Beltrami Operator on $\mathbb{M}_{n,k}$.

As corollaries to theorem I we obtain the following assertions which extend previous results by Cheeger and Yau [CY81], Cheng [Che75] and Varadhan [Var67] respectively.

Theorem II (Heat Kernel Comparison). *Under the conditions of theorem I let q_t^G be the Dirichlet heat kernel on some domain $G \subset X$ and let $x,\, y \in G$, $r \geq d(x,y)$ such that $B_r(x) \subset G$. Then for $\overline{x}, \overline{y} \in \mathbb{M}_{n,k}$ with $d(x,y) = \overline{d}(\overline{x}, \overline{y})$*

$$q_t^G(x,y) \geq q_t^{k,r}(\overline{x}, \overline{y}) \tag{3}$$

where $\overline{d}$ denotes the distance on $\mathbb{M}_{n,k}$ and $q_t^{k,r}$ is the Dirichlet heat kernel of $B_r(\overline{x}) \subset \mathbb{M}_{n,k}$.

Corollary 1 (Eigenvalue Comparison). *Under the conditions of theorem I for any ball $B_r(x) \subset X$ its first Dirichlet eigenvalue $\lambda_1(B_r(x))$ is bounded from above by*

$$\lambda_1(B_r(x)) \leq \lambda_1^k(r)$$

where $\lambda_1^k(r)$ denote the first Dirichlet eigenvalue for $\overline{B}_r(\overline{0}) \subset \mathbb{M}_{n,k}$.

Corollary 2 (Varadhan's formula). *Under the conditions of theorem I let $G \subset X$ be a domain and $q_t^G(.,.)$ be the Dirichlet heat kernel on G. Then for all $x,\, y \in G$*

$$\lim_{t \to 0} 2t \log q_t^G(x,y) = -d^2(x,y).$$

Our last result gives an upper integral inequality for the distance process $(\rho_p(\Xi_t))_t := (d(p, \Xi_t))_t$ of the canonical Diffusion process $(\Xi_t)_{t \geq 0}$ generated by $(\mathcal{E}, D(\mathcal{E}))$.

Theorem III. *Let Ξ be the diffusion proceess generated by the canonical intrinsic Dirichlet Form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on an n-dimensional locally volume $(L^1, 1)$-regular Alexandrov space with lower curvature bound k. Then for any $p \in X$ the process $(\rho_p(\Xi))_{t \geq 0}$ satisfies the inequality*

$$\rho_p(\Xi_t) - \rho_p(\Xi_0) \leq B_t + (n-1) \int_0^t (\ln S_k)' \circ \rho_p(\Xi_s) ds \quad P_x\text{-a.s.} \tag{4}$$

for all $x \in X$, where B_t is a real-valued standard Brownian Motion.

This statement should be compared to the stochastic differential inequality for $(\rho_p(\Xi_t))_t$ in the smooth Riemannian case with lower Ricci curvature bounds due to Debiard, Gaveau and Mazet [DGM75]: Since the radial process of a Brownian Motion on the Model space $\mathbb{M}_{n,k}$ satisfies (4) with equality sign, theorem III can be considered as a weak formulation of the corresponding comparison principle known from the smooth setting. However, as we do not have an upper bound on the 'drift coefficient' of $(\rho(\Xi_t))_t$ - in fact, it is not even clear whether this process is a semi martingale - inequality (4) is not strong enough to prove a pathwise bound as in [DGM75] of the form $\rho_x(\Xi_{x,t}) \leq \overline{\rho}_{\overline{x}}(\overline{\Xi}_{\overline{x},t})$ where $\overline{\Xi}_{\overline{x}}$ is a Brownian Motion on $\mathbb{M}_{n,k}$ starting in some $\overline{x} \in \mathbb{M}_{n,k}$ and $\overline{\rho}_{\overline{x}}$ is the distance function of $\overline{x}$ on $\mathbb{M}_{n,k}$.

2. Preliminaries. *2.1. Basics on Alexandrov spaces.* Alexandrov's curvature condition for metric spaces (X, d) is based on the comparison of the convexity of geodesic triangles. In a metric space a curve $\gamma_{pq} \subset X$ connecting two points $p,\, q \in X$ is called geodesic (segment) if it is rectifiable and its arclength $L(\gamma_{pq})$ coincides with the distance $d(p, q)$ of its end points. A geodesic triangle $\Delta(pqr) \subset X$ is defined by three vertices p, q and $r \in X$ and three geodesic segments γ_{pq}, γ_{qr} and γ_{rp} which are denoted by pq, qr and rp respectively. - The following definition gives a rigorous formulation how to estimate from above the convexity of geodesic triangles in (X, d) where suitably chosen triangles in the model spaces $\mathbb{M}_{2,k}$ serve as reference objects.

Definition 1 (Toponogov Comparison Property). *For $k \in \mathbb{R}$ let $p, q, r \in X$ be the vertices and $pq, qr, rq \subset X$ the corresponding segments of a geodesic triangle $\Delta(pqr)$ in a metric space (X, d) (with $d(p, q) + d(q, r) + d(r, q) < 2\pi/\sqrt{k}$ if $k > 0$). Then $\Delta(pqr)$ is said to satisfy Toponogov's comparision property with triangles in $\mathbb{M}_{2,k}$ iff for any geodesic triangle $\overline{\Delta}(\overline{pqr})$ in $\mathbb{M}_{2,k}$ with $d(p, q) = \overline{d}(\overline{p}, \overline{q})$, $d(q, r) = \overline{d}(\overline{q}, \overline{r})$ and $d(r, p) = \overline{d}(\overline{r}, \overline{p})$ and for all intermediate points $s \in pq, t \in pr$ the inequality*

$$d(s, t) \geq \overline{d}(\overline{s}, \overline{t}),$$

holds true where the points $\overline{s} \in \overline{pq}$, $\overline{t} \in \overline{pr}$ are chosen such that $d(p, s) = \overline{d}(\overline{p}, \overline{s})$ and $d(p, t) = \overline{d}(\overline{p}, \overline{t})$.

The definition of Alexandrov spaces involves definition 1 locally with a global constant $k \in \mathbb{R}$.

Definition 2 (Alexandrov space). *A complete locally compact path metric space (X, d) is said to have a lower curvature bound $k \in \mathbb{R}$ in the sense of Alexandrov, $Curv(X) \geq k$, iff every point $x \in X$ has a neighborhood $\mathcal{U}_x \subset X$ such that any geodesic trinagle $\Delta(pqr) \subset \mathcal{U}_x$ satisfies the Toponogov comparison property with triangles in $\mathbb{M}_{2,k}$. In this case (X, d) is equivalently called an Alexandrov space with curvature bounded below by k.*

Remember that a metric space (X, d) is called a path metric space iff the distance between any two points in X equals the infimum of the arclengths of curves connecting them. If (X, d) is a locally compact path metric space then it is also a geodesic space, i.e. for each pair of points in X there exists at least one connecting curve whoses arclength actually realizes their distance.

Examples 1. ([BGP92, Shi93])

i) Riemannian manifolds with sectional curvature bounded from below and quotients M/Γ of such Riemannian manifolds by groups Γ acting isometrically (not necessarily free or discrete)

ii) simplicial n-dimensional Riemannian complexes (obtained from gluing together Riemannian simplexes of constant curvature k) which satisfy the 2π-gluing-condition along the faces of codimension 2, cf. [BGP92]

iii) boundaries of convex subsets in Riemannian manifolds with lower sectional curvature bound, and as a special case surfaces of revolution $\subset \mathbb{R}^3$ obtained from graphs of concave functions

iv) spaces obtained by gluing two Alexandrov spaces along their boundaries if the boundaries are intrinsically isometric [Pet97]

v) Hausdorff-limits of Riemannian manifolds with uniform sectional curvature bound. - For this recall the definition of the Hausdorff-distance of two subsets A, $B \subset (X, d)$ of a metric space

$$d_H^X(A, B) = \inf\{\epsilon > 0 \,|\, A \subset U_\epsilon(B),\, B \subset U_\epsilon(A)\}.$$

Now the Hausdorff-distance of two metric spaces (A, d_A), (B, d_B) is defined by

$$d_H(A, B) = \inf d_H^X(f(A), g(B))$$

where the infimum is taken over all metric spaces (X, d) and all isometric embeddings $f : A \to X$ and $g : B \to X$. Then d_H induces a complete metric between compact metric spaces (cf. [Gro99]), i.e. the space $\mathcal{X}_c$ of (equivalence classes of) compact metric spaces together with the function d_H forms a complete (and contractible) not locally compact metric space. By Gromov's compactness theorem for each choice of $n \in \mathbb{N}$, $\kappa \in \mathbb{R}$, $D \in \mathbb{R}_+$ the set $\mathcal{M}(n, \kappa, D)$ of n-dimensional Riemannian manifolds with lower Ricci curvature bound $(n-1)\kappa$ and diameter less than D is precompact in $(\mathcal{X}_c, d_H)$. Moreover, if one assumes also the sectional curvature to be bounded below by $k \in \mathbb{R}$ then any Hausdorff limit of a converging sequence will be an Alexandrov space with curvature bounded below by k (cf. [BBI01]).

Definition 2 has a remarkable amount of consequences, of which we are going to recall and exploit a few. For three points p, q and r in

X let $\overline{\sphericalangle}(qpr)$ denote the angle at $\overline{p}$ of a geodesic comparison triangle $\overline{\Delta}(p,q,r) = \Delta(\overline{p},\overline{q},\overline{r})$ in $\mathbb{M}_{2,k}$ as in definition 1. Then the following property is easily verified, c.f. [BGP92].

Proposition 1 (Alexandrov Convexity, local version). *For a locally compact geodesic metric space the condition $Curv(X) \geq k$ is equivalent to the Alexandrov convexity of geodesic hinges: for each $x \in X$ there is an $\epsilon > 0$ such that for any pair of geodesics γ, η with $\gamma(0) = \eta(0) = x$ the function*

$$\theta(s,t) = \overline{\sphericalangle}(\gamma(s)x\eta(t)) \text{ is non-increasing for } s,\, t \leq \epsilon.$$

Definition 3 (Angles and Tangent Cones). *Let γ, η be two geodesics emanating from $x \in X$ as above. Then*

$$\sphericalangle(\gamma,\eta) := \lim_{s,t\to 0} \theta(s,t) =: d_{\sphericalangle}(\gamma,\eta)$$

defines the angle (or angular distance) between η and γ. The space of directions $(\Sigma_x, d_{\sphericalangle})$ is the (closure of the) set of equivalence classes of all geodesics emanating from x with respect to the angular distance $d_{\sphericalangle}$

$$\Sigma_x = \left(\{\gamma_{xy} | y \in X\}/_{\sim_{d_{\sphericalangle}}}, d_{\sphericalangle}\right)^{\sim}. \tag{5}$$

The tangent cone (K_x, d_x) is the topological cone $(\Sigma_x \times \mathbb{R}_+)/(\Sigma_x \times \{0\})$ over Σ_x equipped with the metric induced on $\mathbb{R}_+ \times \Sigma_x$ by the Euclidean cosine law

$$d_x^2[(\alpha,s),(\beta,t)] = s^2 + t^2 - 2st\cos d_{\sphericalangle}(\alpha,\beta).$$

It is proved in [BGP92] that every Alexandrov space (X,d) has some integer Hausdorff-dimension $n \in \mathbb{N}$ and we will denote by m the corresponding n-dimensional Hausdorff-measure. The singular set $S_X \subset X$ is the collection of points whose tangent cone is not the n-dimensional Euclidean plane or, equivalently, whose space of directions is not isometric to S^{n-1}. Each $x \in X \setminus S_X$ has a neighborhood which is (bi-Lipschitz) homeomorphic to $\mathbb{R}^n$ and for the Hausdorff dimension of S_X one finds $\dim_H(S_X) \leq n-1$ [BGP92, OS94].

2.2. The canonical Dirichlet form and the Laplacian. There are at least two different sequences of non local Dirichlet forms on X leading essentially to the same limiting Dirichlet form, which we will call canonical. For an open subset G in X, $r > 0$ and a measurable function $f : X \mapsto \mathbb{R}$ set

$$\mathcal{E}_G^r(u) = \frac{n}{2} \int_G \int_{B_r^*(x)} \left(\frac{u(x)-u(y)}{d(x,y)}\right)^2 m_r(dy) m_r(dx) \tag{6}$$

$$\mathcal{E}_G^{b,r}(u) = \frac{n}{2} \frac{1}{b_{n,k}(r)} \int_G \int_{B_r^*(x)} \left(\frac{u(x)-u(y)}{d(x,y)}\right)^2 m(dy) m(dx) \tag{7}$$

where $B_r x = \{y \in X \,|\, d(x,y) < r\}$, $m_r(dx) = m(dx)/\sqrt{m(B_r(x))}$ and $b_{n,k}(r)$ denotes the volume of the geodesic r-ball in the constant curvature space form $\mathbb{M}_{n,k}$. In [Stu98] it is shown that if some metric measure space (X,d,m) possesses the so called measure contraction property, which is equivalent to lower Ricci curvature bounds in the smooth Riemannian case, the forms of type (6) converge to a limiting form $\mathcal{E}$ in the Γ-sense, which in particular preserves the Dirichlet form properties of the approximating functionals in the limit. Kuwae and Shioya modified this result for forms of the type (7) defined on Alexandrov spaces with lower curvature bound for which the the notion of generalized measure contraction property was introduced (see [KS01] for details). Also, they proved that on Alexandrov spaces both approximating forms yield the same limit (Corollary 5.1):

Theorem (Existence and uniqueness of the canonical Dirichlet form). *On an n-dimensional Alexandrov space with lower curvature bound k and for some open $G \subset X$ with $m(G) < \infty$ both sequences $\mathcal{E}^r_G$ and $\mathcal{E}^{b,r}_G$ have Γ-limits on $L^2(G)$ which coincide with their common pointwise limit $\mathcal{E}_G$ on Lip(X). The $L^2(G)$-closure $(\mathcal{E}_G, D(\mathcal{E}_G))$ of $\mathcal{E}_G$ on Lip(X) is a strongly local and regular Dirichlet form on $L^2(G)$.*

The generator Δ^G of this form will be called Laplacian, for it coincides with the Neumann Laplace-Beltrami operator if X is smooth. Moreover, since the measure m is doubling and a local Poincaré inequality applies one can show by Moser iteration Hölder continuity for the heat kernel q^G_t of the corresponding semigroup [Stu96, Stu98, KMS01]. In particular the semigroup has the Feller property, which will be used later on. - The same results hold true if one considers the Dirichlet Laplacian instead. Also, we will call the m-symmetric Hunt process on X which is associated with $\mathcal{E}$ intrinsic because it can be understood as the limiting stochastic process for a rescaled sequence of jump processes, where the transition function is just determined by d and m.

2.3. Volume Regularity. On a smooth Riemannian manifold (M^d, g) one obtains from the expansion of the volume density $\det d\exp_x$ for the pull back of the Riemannian volume form on $T_x M$

$$\frac{d\exp_x^* \mathrm{vol}_M}{d\mathcal{L}^n}(z) = \det d\exp_x(z) = 1 - \frac{1}{6}\,\mathrm{Ricc}(z,z) + o(|z|^2). \qquad (8)$$

This formula implies in particular that both tangential and intrinsic mean value operators can be used for the approximation of the Laplace-Beltrami operator on M.

Lemma 1. *Let (M,g) be a smooth (i.e. C^3) Riemannian manifold* *and $f : M \mapsto \mathbb{R}$ some C^3-function. Then for all $x \in M$*

$$\lim_{r\to 0} \frac{1}{r^2}\left[f(x) - \fint_{B_r(x)} f(z) vol_M(dz)\right] = -\frac{1}{2(n+2)}\Delta^M f(x)$$
$$= \lim_{r\to 0} \frac{1}{r^2}\left[f(x) - \fint_{\mathbb{B}_r(0_x)\subset T_xM} f(\exp_x z)dz\right].$$

Proof.

$$\fint_{B_r(x)} f(z)\mathrm{vol}(dz) = \frac{1}{\mathrm{vol}(B_r(x))}\int_{\mathbb{B}_r(0_x)} f(\exp_x(z))\det d_z \exp_x dz$$
$$= \frac{1}{\mathrm{vol}(B_r(x))}\int_{\mathbb{B}_r(0_x)} \Big(f(x) + df_x(z) + \frac{1}{2}\mathrm{Hess}_x f(z,z)$$
$$-\frac{1}{6}f(x)\,\mathrm{Ricc}(z,z) + o(|z|^2)\Big)\, dz$$

and hence, since $\fint_{\mathbb{B}_r(0)} A(z,z)dz = (n+2)^{-1}r^2|\mathbb{B}_r|\mathrm{tr}(A)$

$$= \frac{|\mathbb{B}_r|}{\mathrm{vol}(B_r(x))}\left((1 - \frac{s(x)r^2}{6(n+2)})f(x) + \frac{r^2}{2(n+2)}\Delta f(x) + o(r^2)\right).$$

Thus

$$\frac{1}{r^2}\left[f(x) - \fint_{B_r(x)} f(z)\mathrm{vol}_M(dz)\right] = \frac{-|\mathbb{B}_r|}{\mathrm{vol}(B_r(x))}\frac{1}{2(n+2)}\Delta f(x) + \theta(r)$$
$$+ \frac{|\mathbb{B}_r|}{\mathrm{vol}(B_r(x))}\frac{1}{r^2}\left(\frac{\mathrm{vol}(B_r(x))}{|\mathbb{B}_r|} - \left(1 - \frac{s(x)r^2}{6(n+2)}\right)\right) f(x).$$

By virtue of the expansion of $(r \mapsto \frac{\mathrm{vol}(B_r(x))}{|\mathbb{B}_r|})$ about zero (which follows from integrating (8) over the ball $\mathbb{B}_r(0_x)$) the second term on the right hand side converges to zero and the first equality follows. The second is now also obvious. □

The same assertion is true for the analogous spherical mean value operators. Moreover, by the Kato-Trotter formula from the preceding assertion one deduces for the family of rescaled mean value operators

$$M_t f(x) := \fint_{B_{\sqrt{t/(n+2)}}(x)} f(y)m(dy) \quad \lim_{k\to\infty} M_{t/k}^k f(x) = e^{t\frac{1}{2}\Delta} f(x)$$

on a smooth n-dimensional Riemannian manifold M and for sufficiently smooth f, compare [Blu84].

The proof of lemma 1 relies on the integrated version of (8) only, that is on the asymptotic behaviour of the volume density functions $q_r : X \mapsto \mathbb{R}$

$$q_r(x) = \frac{m(B_r(x))}{b_{n,k}(r)}$$

for r tending to zero. Here $b_{n,k}(r)$ denotes the volume of an r-ball in the model space $\mathbb{M}_{n,k}$. This is the motivation for the following

Definition 4 (Volume Regularity). *For $\alpha > 0$ a point x in an n-dimensional Alexandrov space with curvature bounded below by k is said to be volume α-regular iff*

$$1 - q_r(x) = o(r^\alpha) \text{ for } r \to 0.$$

(X, d) is called locally volume (L^p, α)-regular with exceptional set $\mathbb{S}_X^{p,\alpha} \subset X$, iff

$$r^{-\alpha}(1 - q_r(.)) \to 0 \text{ in } L^p_{loc}(X \setminus \mathbb{S}_X^{p,\alpha}) \text{ for } r \to 0.$$

For the sake of completeness we recall that rough dimension of a subset $V \subset X$ in a metric space is defined as

$$\dim_r V = \inf \left\{ \alpha > 0 \,\middle|\, \lim_{\epsilon \to 0} \epsilon^\alpha \beta_V(\epsilon) = \infty \right\}$$

where $\beta_V(\epsilon)$ is the cardinality of a maximal subset $\{x_i \,|\, i \in I\} \subset V$ such that $d(x_i, x_j) \geq \epsilon$. - In the literature this is also sometimes referred to as Assouad dimension (cf. [Hei01]).

Remarks 1.
a) The existence of the limit of $m(B_r(p))/b_{n,k}(r)$ for r tending to zero follows from the Bishop-Gromov volume comparison on Alexandrov spaces (cf. [Yam96]).
b) In [She93] one finds the statement that if $\lim_{r \to 0} \frac{m(B_r(x))}{b_{n,k}(r)} > \frac{1}{2}$ then p is (topologically) a manifold point. This can be seen as well from the results in [BGP92] and (9).
c) A volume α-regular point is also metrically regular. This is seen as follows: since the tangent cone K_p is the pointed Gromov-Hausdorff-limit of the rescaled metric r-balls centered at p, i.e. for all $R > 0$

$$(B_R^r, d^r) = (B_R(p), \frac{1}{r} d) \xrightarrow{GH} (\mathbb{B}_R(0_p), d_p) \text{ for } r \to 0,$$

by theorem 10.8 in [BGP92] one obtains that the associated Hausdorff measures converge weakly, too. Thus

$$m_{K_p}(\mathbb{B}_1(0_p)) = \lim_{r \to 0} m^r(B_1^r) = \lim_{r \to 0} \frac{m(B_r(p))}{r^n}. \tag{9}$$

The condition of volume α-regularity implies that the limit in (9) equals one, hence $\Sigma_p = \partial\mathbb{B}_1(0_p)$ has the measure ω_n and therefore must be isometric to S^{n-1}. - In general the converse assertion is true only for $\alpha = 0$, see example iii) below.

Examples 2.

i) A smooth Riemannian manifold is trivially volume (L^∞, α)-regular with empty exceptional set $\mathbb{S}_X^{\infty,\alpha}$ for $\alpha < 2$.

ii) Locally finite simplicial Riemannian complexes obtained from Riemannian simplices of constant curvature are locally (L^∞, ∞)-volume-regular with $\mathbb{S}_X^{\infty,\infty} = S_X$.

iii) Take a planar circular cone $C_f := B_1(0) \setminus \Sigma_f$ in the two-dimensional Euclidean plane, i.e. the unit disc minus some sector $\Sigma_f := \{(x_1, x_2) \in \mathbb{R}^2 | \, x_1 > 0, |x_2| < f(x_1)\}$. If f is chosen to be convex, due to the gluing theorem [Pet97], the metric space X_f, which is obtained by gluing C_f (endowed with the induced Euclidean distance) along the graph of $\pm f$, is an Alexandrov space of nonnegative curvature. For linear f, X_f is a flat cone which is a special case of ii). Assume f is not linear and differentiable. Note that if $f'(0) = 0$ then X_f is metrically regular. Let $p = (s, \pm f(s)) \in \mathrm{graph}(f) \subset X_f$ then for sufficiently small r the ball $B_r(p) \subset X_f$ is given by union of the intersection of the Euclidean r-Balls $B_r^+(p)$ (and $B_r^-(p)$) centered at $(s, f(s))$ (and $(s, -f(s))$) with C_f respectively and it is sufficient to compute $|B_r^+(p) \cap C_f|/\pi r^2$, where $|.|$ denotes the two-dimensional Lebesgue measure. If we shift p to the origin and rotate the picture $B_r^+(p) \cap C_f$ is the intersection of the upper half r-ball around 0 with the epigraph of a convex function $\tilde{f}$ such that $\tilde{f}(0) = \tilde{f}'(0) = 0$. If $x_\pm = x_\pm(r)$ denote the x-coordinates of the intersection points of $\partial B_r(0)$ with the graph of $\tilde{f}$ in the first and second quadrant respectively, i.e. $r = x_\pm\sqrt{1 + (\tilde{f}(x\pm)/x_\pm)^2}$, $x_+ > 0$, $x_- < 0$ then the estimates

$$\frac{1}{2}-\frac{\int_{x_-}^{x_+}\tilde f(s)ds}{\pi r^2} \geq \frac{|B_r^+(p)\cap C_f|}{\pi r^2}$$
$$\geq \frac{1}{2}-\frac{\int_{x_-}^{x_+}\tilde f(s)ds + \tilde f(x_-)(r-x_-)+\tilde f(x_+)(r-x_+)}{\pi r^2}$$

hold trivially and since

$$\frac{(r-x_\pm)\tilde f(x_\pm)}{r^{2+\alpha}} \approx \frac{\tilde f(x_\pm)}{|x_\pm|^{1+\alpha}} \text{ and } \frac{\int_0^{x_\pm}\tilde f(s)ds}{|x_\pm|^{2+\alpha}} \approx \frac{f(x_\pm)}{|x_\pm|^{1+\alpha}}$$

for small $r \approx |x_\pm|$ one obtains

$$C_1\frac{\tilde f(x_+)}{x_+^{1+\alpha}} \wedge \frac{\tilde f(x_-)}{|x_-|^{1+\alpha}} \leq \frac{1}{r^\alpha}(1-q_r(p)) \leq C_2\frac{\tilde f(x_+)}{x_+^{1+\alpha}} \vee \frac{\tilde f(x_-)}{|x_-|^{1+\alpha}}. \tag{10}$$

For instance, if the function $\tilde f$ has growth $x^{1+\gamma}$ around 0 then p is an α-volume regular point iff $\gamma > \alpha$. The function $\tilde f$ is given as the image of f under a affine transformation of f depending on the point p. Suppose that the estimate $c_1 r^{1+\gamma} \leq \tilde f(r) \leq c_2 r^{1+\gamma}$ for $r \in (-\epsilon, \epsilon)$ holds true locally uniformly with respect to p. Since the upper estimate in (10) obviously applies to points in a r-neighborhood of graph(f), one obtains that X_f is locally (L^p, α)-volume regular with empty exceptional set $\mathbb{S}_X^{p,\alpha}$, if $\alpha < (1+\gamma p)/p$, in particular for $\mathbb{S}_X^{1,1} = \emptyset$ for any $\gamma > 0$.

This example could also be used as a building block for the construction of a two-dimensional Alexandrov space X with $\mathbb{S}_X^{1,1} \neq \emptyset$ (for a sketch of this procedure cf. [vR02]): Suppose f satisfies the lower estimate of (10) in some $O(r)$-neighborhood of graph($\pm f$). Then the idea is cut off from the left half plane $\mathbb{R}^2_{\{x_1<1\}}$ a fractal subset $F \subset \mathbb{R}^2_{\{x_1 \geq 0\}}$ obtained as a union of (shifted) copies of cusps like Σ_f. By gluing along the adjacent pieces of the boundary $\partial(\mathbb{R}^2_{\{x_1<1\}} \setminus F)$ we obtain a new non-negatively curved Alexandrov surface X with the property that the volume of the r-neighbourhoods of the resulting non-volume regular points in X decays only like r^β with some $\beta < 1$ for r tending to zero. Together with (10) in the integral condition for volume $(L^1, 1)$-regularity this yields $\mathbb{S}_X^{1,1} \neq \emptyset$ if $\gamma + \beta < 1$.

3. Proofs. *3.1. Proof of theorem I.* For the comparison of $(\mathcal{E}, D(\mathcal{E}))$ with the corresponding form in the Riemannian case we introduce a third sequence of operators and associated quadratic forms on (X, d) by

$$A^{E,r}u(x) = n\tfrac{m(B_r(x))}{b_{n,k}(r)}\tfrac{1}{r^2}\left(u(x) - \fint_{B_r(x)} u(y)m(dy)\right)$$

$$\mathcal{E}_G^{E,r}(u) = \tfrac{n}{2b_{n,k}(r)}\int_G\int_{B_r(x)} \tfrac{|u(x)-u(y)|^2}{r^2}m(dx)m(dy).$$

Lemma 2. *On $Lip(G)$ the limit of $\mathcal{E}_G^{E,r}$ for $r \to 0$ exists and coincides with $\frac{n}{n+2}\mathcal{E}_G$.*

Proof. We compare $\mathcal{E}_G^{E,r}$ with $\mathcal{E}_G^{b,r}$. For this purpose define for $\alpha \in (0, 1)$

$$\mathcal{E}_{G,B_\alpha}^{E,r}(u) = \tfrac{n}{2b_{n,k}(r)}\int_G\int_{B_{\alpha r}(x)} \tfrac{|u(x)-u(y)|^2}{r^2}m(dx)m(dy)$$

$$\mathcal{E}_{G,A_\alpha}^{E,r}(u) = \tfrac{n}{2b_{n,k}(r)}\int_G\int_{B_r(x)\setminus B_{\alpha r}(x)} \tfrac{|u(x)-u(y)|^2}{r^2}m(dx)m(dy)$$

and analogoulsly

$$\mathcal{E}_{G,B_\alpha}^{b,r}(u) = \tfrac{n}{2}\tfrac{1}{b_{n,k}(r)}\int_G\int_{B^*_{\alpha r}(x)} \left(\tfrac{u(x)-u(y)}{d(x,y)}\right)^2 m(dy)m(dx)$$

$$\mathcal{E}_{G,A_\alpha}^{b,r}(u) = \tfrac{n}{2}\tfrac{1}{b_{n,k}(r)}\int_G\int_{B_r(x)\setminus B_{\alpha r}(x)} \left(\tfrac{u(x)-u(y)}{d(x,y)}\right)^2 m(dy)m(dx).$$

Then we obtain

$$\begin{aligned}
\mathcal{E}_G^{b,r}(u) &= \mathcal{E}_{G,B_\alpha}^{b,r}(u) + \mathcal{E}_{G,A_\alpha}^{b,r}(u) \\
&\leq \mathcal{E}_{G,B_\alpha}^{b,r}(u) + \frac{1}{\alpha^2}\mathcal{E}_{G,A_\alpha}^{E,r}(u) \\
&= \mathcal{E}_{G,B_\alpha}^{b,r}(u) - \frac{1}{\alpha^2}\mathcal{E}_{G,B_\alpha}^{E,r}(u) + \frac{1}{\alpha^2}\mathcal{E}_G^{E,r}(u) \\
&= \frac{b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{b,\alpha r}(u) - \frac{b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{E,\alpha r}(u) + \frac{1}{\alpha^2}\mathcal{E}_G^{E,r}(u).
\end{aligned}$$

Due to the convergence of $\mathcal{E}_G^{b,r}(u)$ and $\lim_{r\to 0}\frac{b_{n,k}(r\alpha)}{b_{n,k}(r)} = \alpha^n$ in the limit this yields

$$\begin{aligned}
\mathcal{E}_G(u) &\leq \frac{1}{\alpha^2(1-\alpha^n)}\liminf_{r\to 0}\left(\mathcal{E}_G^{E,r}(u) - \alpha^2\frac{b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{E,\alpha r}(u)\right) \\
&\leq \frac{1}{\alpha^2(1-\alpha^n)}\left(\liminf_{r\to 0}\mathcal{E}_G^{E,r}(u) - \alpha^{n+2}\liminf_{r\to 0}\mathcal{E}_G^{E,\alpha r}(u)\right) \\
&= \frac{1-\alpha^{n+2}}{\alpha^2(1-\alpha^n)}\liminf_{r\to 0}\mathcal{E}_G^{E,r}(u).
\end{aligned}$$

As $\lim_{\alpha\to 1}\frac{1-\alpha^{n+2}}{\alpha^2(1-\alpha^n)} = \frac{n+2}{n}$ we see $\mathcal{E}_G(u) \leq \frac{n+2}{n}\liminf_{r\to 0}\mathcal{E}_G^{E,r}(u)$ if we send α to 1. The reverse inequatity can be proved in a similar way. One writes

$$\begin{aligned}\mathcal{E}_G^{E,r}(u) &= \mathcal{E}_{G,B_\alpha}^{E,r}(u) + \mathcal{E}_{G,A_\alpha}^{E,r}(u)\\ &\leq \frac{\alpha^2 b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{E,\alpha r}(u) + \mathcal{E}_G^{b,r}(u) - \frac{b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{b,\alpha r}(u).\end{aligned}$$

Hence

$$\begin{aligned}(1-\alpha^n)\mathcal{E}_G(u) &= \lim_{r\to 4}\left(\mathcal{E}_G^{b,r}(u) - \frac{b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{b,\alpha r}(u)\right)\\ &\geq \limsup_{r\to 0}\left(\mathcal{E}_G^{E,r}(u) - \frac{\alpha^2 b_{n,k}(r\alpha)}{b_{n,k}(r)}\mathcal{E}_G^{E,\alpha r}(u)\right)\\ &\geq (1-\alpha^{n+2})\limsup_{r\to 0}\mathcal{E}_G^{E,r}(u).\end{aligned}$$

Consequently, upon dividing this inequality by $(1-\alpha^n)$ and letting tend α to 1 one obtains $\mathcal{E}_G(u) \geq \frac{n+2}{n}\limsup_{r\to 0}\mathcal{E}_G^{E,r}(u)$ and the claim follows. $\square$

Proof of theorem I. The proof is based upon the choice of a suitable metric on each tangent cone K_x, $x \in X$ and a modification $\Lambda_x : X \mapsto K_x$ of the inverse of the exponential map which allows to apply the Alexandrov convexity of (X,d) - For each $x \in X$ we equip the tangent cone K_x over x with the hyperbolic, spherical or flat metric d_k^x defined by the corresponding cosine law, i.e.

$$\begin{aligned}\cosh(\sqrt{-k}d_k^x[(\alpha,s),(\beta,t)]) &= \cosh(\sqrt{-k}s)\cosh(\sqrt{-k}t)\\ &\quad - \sinh(\sqrt{-k}s)\sinh(\sqrt{-k}t)\cos d_\sphericalangle(\alpha,\beta)\\ \cos(\sqrt{k}d_k^x[(\alpha,s),(\beta,t)]) &= \cos(\sqrt{k}s)\cos(\sqrt{k}t)\\ &\quad - \sin(\sqrt{k}s)\sin(\sqrt{k}t)\cos d_\sphericalangle(\alpha,\beta)\\ (d_k^x[(\alpha,s),(\beta,t)])^2 &= s^2+t^2-2st\cos d_\sphericalangle(\alpha,\beta)\end{aligned}$$

depending on whether $k<0$, $k>0$ or $k=0$ respectively in order to obtain a new curved tangent cone $K_{x,k} = \left((\Sigma_x \times \mathbb{R}_+)/\sim_{d_k^x}, d_k^x\right)^\sim$, which will be denoted $(\mathbb{M}_x, d^x)$. Let $\mathbb{M}_{n,k}(X) = \dot{\bigcup}_{x\in X}\mathbb{M}_x$ denote the corresponding curved tangent cone bundle and note that $(\mathbb{M}_x, d^x) \backsimeq \mathbb{M}_{n,k}$ for each $x \in X \setminus S_X$. Then the map $\Lambda_x : X \mapsto \mathbb{M}_x$ is chosen as the canonical radially isometric extension of (some choice of) the map

$$\lambda_x : X \mapsto \Sigma_x \quad \lambda_x(z) = \gamma'_{xz}(0)$$

which is the projection of z onto one of the directions by which it is seen from x, i.e. $\gamma'_{xz}(0) \in \Sigma_x$ is the image of a unit speed geodesic γ_{xz}

between x and z under the projection map $\lambda_x : X \mapsto \Sigma_x$ from (5). By Kuratowski's measurable selection theorem we may choose λ_x in such a way that the map $\Lambda : X \times X \mapsto \mathbb{M}_{n,k}(X), (x,z) \mapsto \Lambda_x(z) \in \mathbb{M}_{n,k}(x)$ is measurable, where $\mathbb{M}_{n,k}(X)$ is endowed with the product sigma algebra. Now $\mathrm{Curv}(X) \geq k$ implies that for $x \in X \setminus S_X$ the map Λ_x is expanding. In fact, for y, $z \in M$ let $\overline{y}$, $\overline{z}$ be the image points of x, y under Λ_x and denote $0_x = \Lambda_x(x)$, then by construction of Λ_x

$$d^x(0_x, \overline{y}) = d(x,y), \quad d^x(0_x, \overline{z}) = d(x,z)$$
$$\sphericalangle(\gamma_{xy}, \gamma_{xz}) = d_{\sphericalangle}(\gamma'_{xy}, \gamma'_{xz}) = \sphericalangle(\gamma_{0_x\overline{y}}, \gamma_{0_x\overline{z}})$$

where $\gamma_{0_x\overline{y}}$ and $\gamma_{0_x\overline{z}}$ denote the uniquely defined geodesics in $\mathbb{M}_x$ joining 0_x with $\overline{y}$ and $\overline{z}$ respectively. Since $\mathbb{M}_x \backsimeq \mathbb{M}_{n,k}$ one obtains

$$d^x(\Lambda_x(y), \Lambda_x(z)) \geq d(x,y) \tag{11}$$

because the contrary would mean a contradiction to the Alexandrov convexity for geodesic hinges (proposition 1) in the global version.

Let now be f a function as required and $p \in X$. Then $f \circ d_p$ is Lipschitz and thus in $D(\mathcal{E})$. Let us first assume that the nonnegative test function $\zeta \in D(\mathcal{E})$ has compact support. On account of the lemma 2 and the polarization identity we know that

$$\langle A^{E,r}(f \circ d_p), \zeta\rangle_{L^2(X,m)} = \mathcal{E}^{E,r}(f \circ d_p, \zeta) \to \frac{n}{n+2}\mathcal{E}(f \circ d_p, \zeta) \text{ for } r \to 0.$$

If p_x denotes the image point of p under Λ_x the monotonicity property of f together with (11) yields

$$A^{E,r}(f \circ d_p)(x) \leq q_r(x) \frac{n}{r^2} \left(f(d^x(p_x, 0_x)) - \fint_{B_r(x)} f(d^x(p_x, \Lambda_x(y))) m(dy) \right)$$

with $q_r(x) = \frac{m(B_r(x))}{b_{n,k}(r)}$. Also by (11) the image measure of m under Λ_x on $\mathbb{M}_{n,k}(x)$ is absolutely continuous with respect to the volume measure vol_x on $\mathbb{M}_x$

$$\frac{d\left((\Lambda_x)_* m\right)}{d\mathrm{vol}_x} =: \rho_x \leq 1 \quad \mathrm{vol}_x\text{-a.e.}$$

Thus by the general integral transformation formula

$$
\begin{aligned}
&A^{E,r}(f\circ d_p)(x)\\
&\le \frac{q_r(x)n}{r^2}\left(f(d^x(p_x,0_x)) - \frac{1}{q_r(x)}\fint_{\mathbb{B}_r(0_x)} f\circ d^x_{p_x}(z)\rho_x(z)\mathrm{vol}_x(dz)\right)\\
&= q_r(x)\frac{n}{r^2}\left(f(d^x(p_x,0_x)) - \fint_{\mathbb{B}_r(0_x)} f\circ d^x_{p_x}(z)\mathrm{vol}_x(dz)\right)\\
&\qquad + q_r(x)\frac{n}{r^2}\fint_{\mathbb{B}_r(0_x)} f\circ d^x_{p_x}(z)\left[1-\frac{\rho_x(z)}{q_r(x)}\right]\mathrm{vol}_x(dz). \qquad (12)
\end{aligned}
$$

From lemma 3.3 in [KMS01] and the assumption $\dim_r \mathbb{S}^{1,1}_X \le n-2$ it follows that we can find a sequence of cut-off functions in $D(\mathcal{E})$ vanishing on some neighborhood of $\left(\{p\}\cup\mathbb{S}^{1,1}_X\right)\cap \mathrm{supp}(\zeta)$ and converging to the constant function 1 in the Dirichlet space $(D(\mathcal{E}),||.||^{\mathcal{E}}_1)$. So we may assume that ζ is zero on some neighborhood of $\{p\}\cup\mathbb{S}^{1,1}_X$.

The volume regularity of X implies that $q_r(x)\to 1$ for (some subsequence if necessary) $r\to 0$ m-a.e. on $\mathrm{supp}(\xi)$ and thus by lemma 1 and the special structure of the Laplace-Beltrami operator $\Delta^{\mathbb{M}_{n,k}}$ acting on radial functions one obtains for the first term on the right hand side of (12) and $x\in\mathrm{supp}(\xi)$

$$
\begin{aligned}
&\lim_{r\to 0}\frac{q_r(x)n}{r^2}\left(f(d^x(p_x,0_x)) - \fint_{\mathbb{B}_r(0_x)} f\circ d^x_{p_x}(z)\rho_x(z)\mathrm{vol}_x(dz)\right)\\
&= -\frac{n}{n+2}\Delta^{K_x}(f\circ d^x_{p_x})(0_x) = \frac{n}{n+2}S_k^{1-n}(S_k^{n-1}f')'\circ d_{p_x}(0_x)\\
&\qquad\qquad = \frac{n}{n+2}S_k^{1-n}(S_k^{n-1}f')'(d(p,x)).
\end{aligned}
$$

For the second term on the right hand side of (12) note first that we may assume without loss of generality that $f(0)=0$. Then by the regularity of f

$$
\begin{aligned}
&\frac{n}{r^2}\fint_{\mathbb{B}_r(0_x)} f\circ d^x_{p_x}(z)\left[1-\frac{\rho(z)}{q_r(x)}\right]\mathrm{vol}(dz)\\
&= \frac{n}{r^2}\frac{f'(d_p(x))}{d_p(x)}\fint_{\mathbb{B}_r(0_x)}\langle p_x,z\rangle\left[1-\frac{\rho(z)}{q_r(x)}\right]\mathrm{vol}(dz)\\
&\qquad + \frac{1}{2}\frac{n}{r^2}\fint_{\mathbb{B}_r(0_x)}\mathrm{Hess}_{0_x}[f\circ d^x_{p_x}](z,z) + o_f(|z|^2)\left[1-\frac{\rho(z)}{q_r(x)}\right]\mathrm{vol}(dz).
\end{aligned}
$$

Using the relation $1-\frac{\rho(z)}{q_r(x)} = (1-q_r^{-1}(x))(1-\frac{1-\rho(z)}{1-q_r(x)})$ and rescaling the integrals we see that the second term in the right hand side of (12) equals

$$n\frac{q_r(x)-1}{r}\frac{f'(d_p(x))}{d_p(x)}\fint_{\mathbb{B}_1(0_x)}\langle p_x, z\rangle\frac{1-\rho(rz)}{1-q_r(x)}\mathrm{vol}(dz)+$$

$$\frac{n}{2}(1-q_r(x))\fint_{\mathbb{B}_1(0_x)}\mathrm{Hess}_{0_x}[f\circ d^x_{p_x}](z,z)\left[1-\frac{1-\rho(zr)}{1-q_r(x)}\right]\mathrm{vol}(dz)+\vartheta_f(r)$$

$$\leq nf'(d_p(x))\frac{q_r(x)-1}{r}+n(1-q_r(x))C(f'',\mathrm{supp}(\zeta))+\vartheta_f(r)$$

with a function $\vartheta_f:\mathbb{R}\to\mathbb{R}$ depending on f such that $\vartheta_f(r)\to 0$ for $r\to 0$. Thus we get the desired inequality for compactly supported ζ from the local $(L^1,1)$-volume regularity if we multiply (12) by ζ, integrate over X and let r tend to zero.

For general ζ take a sequence of smooth nonnegative functions with compact support $\eta_k:\mathbb{R}_+\mapsto[0,1]$ such that $\eta(t)=1$ for $t\in[0,k]$ and set $\zeta_k=\zeta\cdot\eta_k\circ d_p$ in order to obtain (2) for ζ_k. The Dirichlet form $\mathcal{E}$ is strongly local and hence the corresponding energy measure $\mu_{\langle .,.\rangle}$, which is a measure valued symmetric bilinear form on $D(\mathcal{E})$ defined by

$$\int_X\phi(x)\mu_{\langle u,u\rangle}(dx)=2\mathcal{E}(u,\phi u)-\mathcal{E}(u^2,\phi)\quad\forall\phi\in C_0(X)$$

has the derivation property

$$d\mu_{\langle u\cdot v,w\rangle}=ud\mu_{\langle v,w\rangle}+vd\mu_{\langle u,w\rangle}$$

for all u, v and $w\in\mathcal{D}(\mathcal{E})$ (cf. [FOT94], section 3.3.2). Also from the construction of $\mathcal{E}$ it is obvious that $\mu_{\langle u,v\rangle}\ll m$ for u, $v\in D(\mathcal{E})$ and thus $\mathcal{E}$ admits the Carré du Champ operator Γ which is defined via the corresponding density, i.e.

$$\Gamma(u,v):=\frac{d\mu_{\langle u,v\rangle}}{dm}\in L^1(X,m)\quad\forall\,u,\,v\in D(\mathcal{E})$$

which yields the representation $\mathcal{E}(u,v)=\int_X\Gamma(u,v)dm$ and for u, $v\in D(\mathcal{E})$. Hence a twofold application of Lebesgue's theorem together with $\lim_{k\to\infty}\Gamma(f\circ d_p,\eta_k\circ d_p)=0$ m-a.e. yields

$$\begin{aligned}\mathcal{E}(f\circ d_p,\zeta) &= \int_X \Gamma(f\circ d_p,\zeta)dm \\ &= \lim_{k\to\infty}\int_X \eta_k\circ d_p\Gamma(f\circ d_p,\zeta)dm + \lim_{k\to\infty}\int_X \zeta\Gamma(f\circ d_p,\eta_k\circ d_p)dm \\ &= \lim_{k\to\infty}\int_X \Gamma(f\circ d_p,\eta_k\circ d_p\cdot\zeta)dm = \lim_{k\to\infty}\mathcal{E}(f\circ d_p,\zeta_k) \\ &\leq -\lim_{k\to\infty}\langle S_k^{1-n}(S_k^{n-1}f')'\circ d_p,\zeta_k\rangle_{L^2(X,m)} \\ &= -\langle S_k^{1-n}(S_k^{n-1}f')'\circ d_p,\zeta\rangle_m. \end{aligned}$$

□

One might think about a different approach to proving a comparison theorem for the Laplacian on Alexandrov spaces via some sort of second variation formula for the arclength functional and generalized Jacobi fields, cf. [Ots98]. However, except resulting in very delicate technicalities this idea would probably be also much harder to pursue in more general situations than the proof given above.

Under the weaker assumption that $1-q_r(x)\leq O(r)$ locally uniformly on $X\setminus\mathbb{S}_X$ one obtains an additional drift term in the Laplacian comparison principle which then takes the form

$$\begin{aligned}\mathcal{E}(f\circ d_p,\zeta) &\leq (-S_k^{1-n}(S_k^{n-1}f')'\circ d_p,\zeta)_{L^2(X,m)} \\ &\quad -(n+2)\sup_\nu \int_{\mathbb{M}_{n,k}(X)} \frac{f'(d_p(x))}{d_p(x)}\langle p_x,z\rangle\zeta(x)\nu(dx,dz) \end{aligned}\tag{13}$$

where supremum with respect to ν is taken over all weak accumulation points of the weakly precompact sequence of measures on $\mathbb{M}_{n,k}(X)$

$$\nu_r(dz,dx)=\frac{1-q_r(x)}{r}\frac{1-\rho_x(rz)}{1-q_r(x)}\mathbb{1}_{\mathbb{B}_1(0_x)}(z)\mathrm{vol}_x(dz)m(dx).$$

If $z\mapsto\rho_x(z)$ is differentiable in 0_x we find

$$\nu_r(dx,dz)\to\nu(dx,dz)=d\rho_x(z)\mathbb{1}_{\mathbb{B}_1(0_x)}(z)\mathrm{vol}_x(dz)m(dx)$$

and the drift part in (13) becomes

$$\int_X \frac{f'(d_p(x))}{d_p(x)}d\rho_x(p_x)\zeta(x)m(dx).$$

However, the drift term in (13) can be interpreted as a measure for the local approximation of X by its tangent spaces.

3.2. Proof of theorem II. The proof of theorem II is an application of the weak Laplacian comparison theorem I to the heat kernel on the model space together with following simple version of a parabolic maximum principle for Δ^X:

Lemma 3. *Let $f : \Omega \times (0,T) \to \mathbb{R}$ with $f \in L^2([0,T], D_c(\mathcal{E}_\Omega)) \cap C([0,T]), L^2(\Omega))$. If $f_0 := f(0,.) \leq 0$ m-a. e. and $Lf \geq 0$ with $L = \Delta^\Omega - \partial_t$ in the following weak sense*

$$\int_\sigma^\tau \mathcal{E}(f(t,.),\xi)dt \leq -\left.\langle f(.),\xi\rangle_m\right|_\sigma^\tau \tag{14}$$

for all $\sigma,\tau \in (0,T)$ and $0 \leq \xi \in D_c(\mathcal{E}_\Omega)$, then $f(t,x) \leq 0$ for m-a.e. $x \in \Omega$ and $t \in [0,T]$.

Proof. For $\epsilon > 0$ consider $f_\epsilon(t,x) = \frac{1}{\epsilon}\int_t^{t+\epsilon} f(s,x)ds$. Then f_ϵ is a subsolution to the heat equation in the following sense: for all nonnegative $\xi \in L^2([0,T], D_c(\mathcal{E}_\Omega))$ and $\sigma,\tau \in (0,T-\epsilon)$:

$$\begin{aligned}\int_\sigma^\tau \mathcal{E}(f_\epsilon(t,.),\xi(t))dt &= \int_\sigma^\tau \frac{1}{\epsilon}\int_t^{t+\epsilon} \mathcal{E}(f(s,.),\xi(t))dsdt \\ &\leq -\int_\sigma^\tau \frac{1}{\epsilon}\langle f(t+\epsilon,.) - f(t,.),\xi(t,.)\rangle_m dt \\ &= -\int_\sigma^\tau \langle \partial_t f_\epsilon(t,.),\xi(t,.)\rangle_m dt\end{aligned}$$

Now we take ξ to be $j(f_\epsilon)$ where $j \in C^\infty(\mathbb{R})$ with $j' \geq 0$, $\|j'\|_\infty < \infty$ and $j = 0$ on $(-\infty,\delta]$, $j > 0$ on $(\delta,+\infty)$ for some $\delta > 0$. From the definition of $D_c(\mathcal{E}_\Omega)$ it follows that this choice of ξ is admissible. Inserting this into the last inequality yields

$$\begin{aligned}0 &\leq \int_\sigma^\tau\int_\Omega j'(f_\epsilon(t,x))\mu_{\langle f_\epsilon(t)\rangle}(dx)dt = \int_\sigma^\tau\int_\Omega \mathcal{E}(f_\epsilon(t,.))dt \\ &\leq \int_\sigma^\tau \langle \partial_t f_\epsilon(t), j(f_\epsilon)\rangle_m dt = \langle J(f_\epsilon(\sigma))\rangle_m - \langle J(f_\epsilon(\tau))\rangle_m\end{aligned}$$

with $J(t) = \int_0^t j(s)ds$. Using $f_\epsilon(\sigma) \to f_\epsilon(0)$ and $f_\epsilon(0) \to f_0$ in $L^2(\Omega)$ as well as $J(f_0) = 0$ m-a.s., by sending first $\sigma \to 0$ and then $\epsilon \to 0$ we find that $\langle J(f(\sigma))\rangle_m \leq 0$. From the definition of j and J resp. this implies that $f(t,x) \leq \delta$ for m a.e. $x \in \Omega$. Sending $\delta \to 0$ yields the claim. □

Proof of theorem II. Let us first assume that $\overline{B_r(x)} \subset X$. If $q_t^{k,r}(.,.)$ denotes the Dirichlet heat kernel on $B_r(\overline{x}) \subset \mathbb{M}_{n,k}$ then there is a uniquely defined real valued function $(t,s) \mapsto h_t(s) = h_t^{k,r}(s)$ satisfying the differential equation $\partial_t h_t(s) = -S_k^{1-n}\partial_s(S_k^{n-1}\partial_s h_t(s))$ for $(t,s) \in \mathbb{R}_+ \times (0,r)$ and such that $q_t^{k,r}(\overline{x},\overline{y}) = h_t(\overline{d}(\overline{x},\overline{y}))$ (see [Cha84]). Furthermore, since $s \mapsto h_t(s)$ is non-increasing (ibid., lemma 2.3), the continuation of h_t (still denoted by $h_t(s)$)

$$\mathbb{R}_+ \times \mathbb{R}_+ \ni (t,s) \to h_t^{k,r}(s) = \begin{cases} h_t(s) & \text{for } s \in [0,r] \\ 0 & \text{for } s > r \end{cases}$$

is locally Lipschitz in both variables, non-increasing in s and satisfies

$$\partial_t h_t(s) \geq -S_k^{1-n}\partial_s(S_k^{n-1}\partial_s h_t(s)) \tag{15}$$

in the distributional sense. In order to prove (15) it is sufficient to note that for the Laplacian $\Delta^{\mathbb{M}_{n,k}}$ of the function $\overline{q}_t^{k,r,\overline{x}} : \mathbb{R}_+ \times \mathbb{M}_{n,k} \to \mathbb{R}_+$, $(t,\overline{y}) \to h_t(d(\overline{x},\overline{y}))$ one finds

$$-\Delta^{\mathbb{M}_{n,k}}\overline{q}_t^{k,r,\overline{x}} = -\Delta_y^{\mathbb{M}_{n,k}} q_t^{k,r}(\overline{x},.)1\!\!1_{B_r(\overline{x})} + \frac{\partial q_t^{k,r}(\overline{x},.)}{\partial \nu_{|\partial B_r(\overline{x})}} d\mathcal{H}^{n-1}_{|\partial B_r(\overline{x})}$$

in the distributional sense, where the density in front of the Hausdorff-measure is obviously non-positive. Testing this inequality with radially symmetric test functions and using the special form of $\Delta^{\mathbb{M}_{n,k}}$ we obtain (15). Hence, if we mollify $(t,s) \to h_t(s)$ with respect to s by a non-negative smooth kernel we obtain a family $(t,s) \to h_t^\rho(s)$ of smooth functions, non-increasing in s, satisfying (15) in a pointwise sense and which converge to h locally uniformly on $\mathbb{R}_+ \times \mathbb{R}_+$ for $\rho \to 0$, which in particular implies that for all $T > 0$ and ρ sufficiently small (depending on T) the function

$$G \ni y \to \psi_t^\rho(y) = h_t^\rho(d(x,y))$$

belongs to $D_c(\mathcal{E}(G))$ for all $t \in (0,T)$. Using the weak Laplacian comparison inequality (2) and (15) one deduces that $(t,y) \to \psi_t^\rho(y)$ satisfies (14) for $0 < \sigma \leq \tau$ for sufficiently small ρ and all $0 \leq \xi \in D_c(\mathcal{E}_G)$. If we assume also that $\xi \in D(\Delta^G)$ then we may integrate by parts on the left hand side of (15), pass to the limit for $\rho \to 0$ and integrate by parts again which yields for the function $\psi_t(y) = h_t(d(x,y))$

$$\int_\sigma^\tau \mathcal{E}(\psi_t,\xi)dt \leq -\langle\psi_\tau,\xi\rangle_m + \langle\psi_\sigma,\xi\rangle_m \quad \forall\, 0 \leq \xi \in D(\Delta^G). \tag{16}$$

Standard arguments of Dirichlet form theory now show that for $\xi \in D(\mathcal{E})$ the sequence $\xi_\lambda = \lambda R_\lambda \xi \in D(\Delta^G)$ converges to ξ in the Dirichlet space $(D(\mathcal{E}), \|.\|_1)$ for λ tending to infinity, where R_λ is the $\lambda-$resolvent associated to $(\mathcal{E}, D(\mathcal{E}))$. From the representation $R_\lambda = \lambda \int_0^\infty e^{-\lambda s} P_s ds$ and the fact that the heat semigroup is positivity preserving $\xi \geq 0$ implies $\xi_\lambda \geq 0$. Hence we obtain (16) for arbitrary $0 \leq \xi \in D_c(\mathcal{E}_G)$ by approximation.

For $\delta > 0$ let $\psi_t^\delta(y) = (P_\rho^G \psi_t)(y) = \int_G q_\rho^G(y,z)\psi_t(z)m(dz)$ and $q_t^{G,\delta}(y) = q_{t+\delta}^G(x,y)$. Then ψ_t^δ and $q_t^{G,\delta}$ belong to $D_c(\mathcal{E}_G)$, where $q_t^{G,\delta}$ obviously satisfies (14) with equality sign, whereas for $\psi_t^\delta(y)$ we find

$$\begin{aligned}
\int_\sigma^\tau \mathcal{E}_G(\psi_t^\delta, \xi)dt &= \int_\sigma^\tau \mathcal{E}_G(\psi_t, P_\delta^G \xi)dt \\
&\leq -\langle \psi_\tau, P_\delta^G \xi\rangle_m + \langle \psi_\sigma, P_\delta^G \xi\rangle_m \\
&= -\langle \psi_\tau^\delta, \xi\rangle_m + \langle \psi_\sigma^\delta, \xi\rangle_m \quad \forall\, 0 \leq \xi \in D_c(\mathcal{E}(G)).
\end{aligned}$$

Hence the function $(t,y) \to f_t(y) = \psi^\delta(y) - q_t^{G,\delta}(y)$ satisfies (14). As for the initial boundary value $f_0(.)$ we study the behaviour of $\psi_t(.)$ when t tends to zero: as before the Alexandrov convexity implies that there is a radially isometric (i.e. $d(x,y) = \overline{d}(\Lambda_x(x), \Lambda_x(y))$ for all $y \in X$) and non-expanding map $\Lambda_x : (X,d) \mapsto (\mathbb{M}_{n,k}, \overline{d})$. Thus

$$\begin{aligned}
\lim_{t\to 0} \int_X h_t(d(x,y))m(dy) &= \lim_{t\to 0} \int_{B_r(\Lambda_x(x))} h_t(\overline{d}(\Lambda_x(x), \overline{y}))\Lambda_{x*}m(d\overline{y}) \\
&= \frac{d(\Lambda_{x*}m)}{d\,vol_{\mathbb{M}_{n,k}}}(\Lambda_x(x)) = \lim_{r\to 0} \frac{m(B_r(x))}{b_{n,k}(r)} =: q_0(x)
\end{aligned}$$

(with $q_0(x) \leq 1$ and $m(\{q_0 < 1\}) = 0$). From this and the standard Gaussian estimates for the heat kernel on smooth manifolds it follows that for any $\zeta \in C_c^0(X)$

$$\lim_{t\to 0} \int_X h_t(d(x,y))\zeta(y)m(dy) = C_x \zeta(x)$$

with $C_x = 1/q_0(x)$, i.e. the function $y \mapsto h_t(d(x,y))$ converges weakly to $C_x\delta_x$ for t tending to zero. From the local $(L^1, 1)$-volume regularity of (X,d) it follows that $C_x = 1$ for m-almost every X. Hence let us assume that $C_x = 1$ for the previously fixed x. Then it follows that in fact $f_t = \psi_t^\delta - q_t^{G,\delta}$ converges to $f_0(y) = q_\delta^G(x,y) - q_\delta^G(x,y) = 0$ in $L^2(G, dm)$ for $t \to 0$ and we may apply lemma 3 which yields $f_t(.) \leq 0$ m-almost everywhere in G. Since $\delta > 0$ was arbitrary and $P_\delta^G \to 1$ in $L^2(G)$ for $\rho \to 0$ this implies also $\psi_t(y) \leq q_t^G(y)$ for m-almost every $y \in G$. Hence from the continuity of $\psi_t(.) = h_t(d(x,.))$ and $q_t^G(.) = q_t(x,.)$ we may conclude

$$q_t^{k,r}(\overline{x},\overline{y}) = h_t(d(x,y)) \leq q_t^G(x,y) \quad \forall\, y \in B_r(x). \tag{17}$$

Since the set $R = \{x \in G | C_x = 1\}$ is dense in G for general $x \in G$ with $\overline{B_r(x)} \subset G$ and $y \in B_r(x)$ we may find an approximating sequence (x_l, y_l) with $d(x_l, y_l) = d(x,y) = \overline{d}(\overline{x},\overline{y})$ such that $y_l \in B_r(x_l) \Subset G$ and $C_{x_l} = 1$, which by the continuity of $q_t^G(.,.)$ establishes (17) also in the case $C_x > 1$. Finally, we can offset the assumption $B_r(x) \Subset G$ by considering $B_{r-\epsilon}(x)$ first which gives $(17)_{r'}$ for $r' = r - \epsilon$ and fixed $y \in B_{r-\epsilon}(x)$. Using the continuity of the Dirichlet heat kernel $q_t^r(\overline{x},\overline{y})$ on $B_r(\overline{x}) \subset \mathbb{M}_{n,k}$ with respect to r (which follows from the parabolic maximum principle on $\mathbb{M}_{n,k}$) we may pass to the limit for $\epsilon \to 0$ in the left hand side of (17), obtaining

$$q_t^{k,r}(\overline{x},\overline{y}) = h_t(d(x,y)) \leq q_t^G(x,y) \quad \forall\, y \in B_{r-\epsilon}(x)$$

where $\epsilon > 0$ is arbitrary. Hence, for general $y \in B_r(x)$ the claim follows from the continuity of $q_t^G(x,.)$ by an approximation with $B_{r-1/l}(x) \ni y_l \to y$ for $l \to \infty$. □

Example 3. Consider the heat kernel $q_t^{C_f}$ on C_f (where C_f is defined as in example iii) of section) satisfying the boundary conditions

$$f = 0 \text{ on } S^1 \cap \partial C_f, \quad \frac{\partial u}{\partial \nu}(x, f(x)) = -\frac{\partial u}{\partial \nu}(x, -f(x)) \text{ on } \partial\Sigma_f$$

where $\frac{\partial u}{\partial \nu}$ is the exterior normal derivative of u. The conditions on $\partial\Sigma_f$ are chosen in such a way that solving this boundary value problem is consistent with gluing the two half-sectors together along the graph of $\pm f$. By the heat kernel comparison theorem we now get a lower bound for the flat heat kernel $q_t^{C_f}$ on C_f of the form $q_t^{C_f}(x,y) \geq q_t(x,y)$, where q_t is the Euclidean Dirichlet heat kernel on B_1.

Proof of corollary 1. This follows directly from the heat kernel comparison theorem and the eigenfunction expansion of the heat kernel on X [KMS01] and on $\mathbb{M}_{n,k}$, c.f. [SY94]. □

Remark 2. Let (X,d) be as above and for $r > 0$ $p \in X$ such that $X_0 := B_r(p) \subset X$ let $\lambda_j(X_0)$ denote the j-th (counted with multiplicity) Dirichlet eigenvalue of $B_j(p)$ with $0 = \lambda_0(X_0) < \lambda_1(X_0) \leq \lambda_2(X_0) \leq \cdots$. Then

$$\lambda_j(X_0) \leq \lambda_1^k(\operatorname{diam}(X_0)/2j).$$

This is a consequence of the max-min-principle for the higher eigenvalues of the Laplace operator. For further standard results in this direction see [Cha84], chapter III.

Proof of corollary 2. This follows from the Davies' sharp upper Gaussian estimate which persists on local Dirichlet spaces with Poincaré inequality and doubling base measure (cf. [Stu95]). The other inequality follows from the lower bound of theorem II for the heat kernel and Varadhan's formula on manifolds with lower Ricci bound (cf. [Dav89]) applied to $q_t^{k,r}$ on $\mathbb{M}_{n,k}$. □

In [Nor96] Varadhan's formula is extended to the case of Lipschitz Riemannian manifolds with measurable and uniformly elliptic metric tensor (g_{ij}), which rules out the most general Alexandrov spaces. However, since no curvature condition on the resulting metric space (X, d) are imposed this result is not a special case of ours.

3.3. Proof of theorem III. In the classical situation where $\Xi \in M$ is some semi-martingale on a smooth Riemannian manifold (M^d, g) and $f \in C^2(M;\mathbb{R})$ the geometric Ito-formula yields for the composite process $f(\Xi)$ the representation

$$d(f\circ\Xi) = \sum_{i=1}^{d} df(\Xi)(Ue_i)dZ^i + \frac{1}{2}\sum_{i,j=1}^{d}(\nabla df)(\Xi)(Ue_i, Ue_j)d[Z^i, Z^j]$$

where $U \in \mathcal{O}(M)$ is the horizontal lift of Ξ onto the orthonormal frame bundle of (M, g) and $Z \in R^d$ is the stochastic anti-development of Ξ (see, for instance, [HT94]). In particular if Ξ is a Brownian Motion on M this formula reduces to

$$d(f\circ\Xi) = df(Ue_i)dW^i + \frac{1}{2}\Delta f(\Xi)dt$$

with $W \in \mathbb{R}^d$ being a Brownian Motion on $\mathbb{R}^d$ and Δ the Laplace-Beltrami Operator on (M, g). In this section a decomposition of the same type will be established for the process $\rho_p(\Xi)$, where Ξ is the Hunt process generated by the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on the Alexandrov space (X, d). We start with two general observations concerning the martingale and zero energy part in the Fukushima decomposition of the Dirichlet process $df(\Xi)$:

Lemma 4. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a strongly local symmetric Dirichlet form on some Hilbert space $L^2(X, \sigma, m)$ and $f \in D(\mathcal{E})$. Then for arbitrary g, $h \in L^2 \cap L^\infty(X, m)$ such that $gf^2 \in D(\mathcal{E})$*

$$\mathbb{E}_{h\cdot m}\left(\tilde{g}(\Xi_S)\langle M^{[f]}\rangle_S\right) = \int_0^S\int_X P_t h \cdot P_{S-t} g \mu_{\langle f\rangle} dt$$

where $\langle M^{[f]}\rangle$ is the quadratic variation process of the martingale additive functional part of $df(\Xi)$ and $\mu_{\langle f\rangle}$ is the energy measure associated to f.

Proof. This fact essentially follows from [FOT94], theorem 5.2.3. and lemma 5.1.10. However, we present here an almost self contained proof which requires only a certain familiarity with the concept of energy measures. For $\Delta \in \mathbb{R}$ small let S, T be some fixed positive numbers such that $S > T + \Delta$. For general sufficiently regular g, h the Markov property of Ξ yields

$$\begin{aligned}
&\mathbb{E}_{h\cdot m}(g(\Xi_S)[f(\Xi_{T+\Delta}) - f(\Xi_T)]^2) \\
&= \mathbb{E}_{h\cdot m}(\mathbb{E}_{\Xi_{T+\Delta}}(g(\Xi_{S-(T+\Delta)})[f(\Xi_{T+\Delta}) - f(\Xi_T))]^2) \\
&= \mathbb{E}_{h\cdot m}((P_{S-(T+\Delta)}g)(\Xi_{T+\Delta})[f(\Xi_{T+\Delta}) - f(\Xi_T))]^2) \\
&= \mathbb{E}_{P_T h\cdot m}((P_{S-(T+\Delta)}g)(\Xi_{\Delta})[f(\Xi_{\Delta}) - f(\Xi_0))]^2). \qquad (18)
\end{aligned}$$

We want to divide in (18) by Δ and pass to the limit for Δ tending to zero. Before doing so one notices that in general for $f, g \in D(\mathcal{E})$

$$\lim_{\Delta\to 0} \frac{1}{\Delta}\mathbb{E}_{h\cdot m}(g(\Xi_\Delta)[f(\Xi_\Delta) - f(\Xi_0)]^2) = \int_X h \cdot g\mu_{\langle f\rangle}. \qquad (19)$$

If in (19) the term $g(\Xi_\Delta)$ was replaced by $g(\Xi_0)$ this would just be the well known coincidence of the energy measure $\mu_{\langle f\rangle}$ and the Revuz-measure $\mu_{\langle M^{[f]}\rangle}$ ([FOT94], lemma 5.3.3.), but in the given form (19) can be verified as a consequence of the chain rule for the energy measure, which holds true by the strong locality of $\mathcal{E}$. In a second step one has to verify that whenever hf^2, $g \in D(\mathcal{E})$

$$\begin{aligned}
&\lim_{\Delta\to 0} \frac{1}{\Delta}\mathbb{E}_{h\cdot m}(P_\Delta g(\Xi_\Delta)[f(\Xi_{T+\Delta}) - f(\Xi_T)]^2) \\
&\qquad - \lim_{\Delta\to 0} \frac{1}{\Delta}\mathbb{E}_{h\cdot m}(P_\Delta g(\Xi_\Delta)[f(\Xi_{T+\Delta}) - f(\Xi_T)]^2) \\
&= \lim_{\Delta\to 0} \frac{1}{\Delta}\langle P_\Delta h, [P_\Delta g - g]f^2\rangle_m \\
&\quad - \lim_{\Delta\to 0} 2\langle P_\Delta(fh), [P_\Delta g - g]f\rangle_m + \lim_{\Delta\to 0}\langle P_\Delta(f^2h), [P_\Delta g - g]\rangle_m \\
&= \mathcal{E}(hf^2, g) - 2\mathcal{E}(hf^2, g) + \mathcal{E}(hf^2, g) = 0.
\end{aligned}$$

Due to these two assertions taking the limit in (18) gives

$$\lim_{\Delta\to 0} \frac{1}{\Delta}\mathbb{E}_{h\cdot m}(g(\Xi_S)[f(\Xi_{T+\Delta}) - f(\Xi_T)]^2) = \int_X P_T h \cdot P_{S-T} g\mu_{\langle f\rangle}.$$

Now it is easy to compute the quadratic variation of $M^{[f]}$ because we know that by general Ito theory

$$\mathbb{E}_{h\cdot m}(g(\Xi_S)\langle M^{[f]}\rangle_S) = \lim_{\Delta\to 0}\mathbb{E}_{h\cdot m}(g(\Xi_S)\langle M^{[f]}\rangle_{S-\Delta})$$
$$= \lim_{\Delta\to 0}\mathbb{E}_{h\cdot m}\Big(g(\Xi_S)\sum_{i=0}^{\lfloor\frac{S}{\Delta}\rfloor-2}[f(\Xi_{(i+1)\Delta} - f(\Xi_{i\Delta})]^2\Big)$$
$$= \lim_{\Delta\to 0}\sum_{i=0}^{\lfloor\frac{S}{\Delta}\rfloor-2}\big(\Delta\int_X P_{i\Delta}h\cdot P_{S-i\Delta}g\mu_{\langle f\rangle} + O(\Delta)\big)$$
$$= \int_0^S\int_X P_t h\cdot P_{S-t}g\mu_{\langle f\rangle}dt. \qquad \square$$

Corollary 3. *Let $(\mathcal{E}, D(\mathcal{E}))$ be a strongly local Dirichlet form defined on $L^2(X,\sigma,m)$ such that the associated semigroup has the Feller property. Then for $f\in D(\mathcal{E})$ the following implication holds:*

$$\big\{\mu_{\langle f\rangle} = m\big\} \Longrightarrow M^{\langle f\rangle} \textit{ is a real } P_x\textit{-Brownian Motion for all } x\in X.$$

Proof. For $\mu_{\langle f\rangle} = m$ the previous lemma gives

$$\mathbb{E}_{h\cdot m}\left(\tilde{g}(\Xi_S)\langle M^{[f]}\rangle_S\right) = \int_0^S \langle P_t h, P_{S-t}g\rangle_m dt$$
$$= S\langle h, P_S g\rangle_m = S\mathbb{E}_{h\cdot m}(g(\Xi_S)).$$

By a monotone class argument this implies $\langle M^{[f]}\rangle_S = S$ $P_{h\cdot m}$-almost surely and thus for all $x\in X$ also P_x-almost surely, since we can let tend $h\cdot m$ tend to δ_x when utilizing the Feller property of P_t. Levy's characterization of Brownian Motion then yields the claim. $\square$

Lemma 5. *Let the $(\mathcal{E}, D(\mathcal{E}))$ and Ξ be as in lemma 4 and f in $D(\mathcal{E})$. Then for the CAF of zero energy $A^{[f]}$ belonging to $(f(\Xi_t) - f(\Xi_0))_{t\geq 0}$ one has*

$$\mathbb{E}_{h\cdot m}(\tilde{g}(\Xi_S)A_S^{[f]}) = -\int_0^S \mathcal{E}(P_t h P_{S-t}g, f)dt \quad \forall\, h,g\in L^\infty(X,m)\cap D(\mathcal{E}),$$

where P_t is the semigroup generated by $(\mathcal{E}, D(\mathcal{E}))$ and $\tilde{\zeta}$ is a quasi-continuous version of ζ.

Proof. We proceed as in the proof of lemma 4 by using the Markov property and the additivity of $A^{[f]}$ to obtain for $S > T+\Delta$

$$\mathbb{E}_{h\cdot m}(\tilde{g}(\Xi_S)(A_{T+\Delta}^{[f]} - A_T^{[f]}) = \mathbb{E}_{P_T h\cdot m}((P_{S-(T+\Delta)}\tilde{g})(\Xi_\Delta)A_\Delta^{[f]})).$$

For $f = R_1\eta$ with $\eta \in L^2$ the right hand side equals

$$\langle P_T h, \mathbb{E}_{\cdot}\Big(P_{S-(T+\Delta)}\tilde{g}(X_\Delta)\int_0^\Delta f(\Xi_u) + \eta(\Xi_u)du\Big)\rangle_m$$

and from this representation and the continuity of $\tilde{g}(\Xi)$ it is clear that

$$\begin{aligned}\lim_{\Delta\to 0}\frac{1}{\Delta}\mathbb{E}_{h\cdot m}(\tilde{g}(\Xi_S)(A^{[f]}_{T+\Delta} - A^{[f]}_T) &= \langle P_T h \cdot P_{S-T} g, f + \eta\rangle_m \\ &= -\mathcal{E}(P_T h \cdot P_{S-T} g, f).\end{aligned}$$

For general $f \in D(\mathcal{E})$ one establishes this result by the usual approximation argument (compare [FOT94], thm. 5.2.4.). As before we now can compute

$$\begin{aligned}\mathbb{E}_{h\cdot m}(\tilde{g}(\Xi_S)A^{[f]}_S) &= \lim_{\Delta\to 0}\sum_{i=0}^{\lfloor\frac{S}{\Delta}\rfloor - 1}\Delta\,\frac{1}{\Delta}\mathbb{E}_{h\cdot m}(\tilde{g}(\Xi_S)(A^{[f]}_{(i+1)\Delta} - A^{[f]}_{i\Delta})) \\ &= -\lim_{\Delta\to 0}\Big(\sum_{i=0}^{\lfloor\frac{S}{\Delta}\rfloor - 1}\Delta\ \mathcal{E}(P_{i\Delta}h \cdot P_{S-i\Delta}g, f) + O(\Delta)S\Big) \\ &= -\int_0^S \mathcal{E}(P_t h \cdot P_{S-t} g, f)dt.\end{aligned}$$

□

Lemma 6. *Let (X, d) be an n-dimensional locally $(L^1, 1)$-volume regular Alexandrov space with lower curvature bound k and $(\mathcal{E}, D(\mathcal{E}))$ the canonical intrinsic Dirichlet form on $L^2(X, \mathcal{B}(X), m)$, where m is the n-dimensional Hausdorff-measure. Then for each $p \in X$ the distance function $\rho_p(.) = d(p, .) : X \mapsto \mathbb{R}$ has the energy measure $\mu_{\langle\rho_p\rangle} = m$.*

Proof. From the fact that $\mathcal{E}^r \to \mathcal{E}$ for $r \to 0$ pointwise on the set of Lipschitz functions on (X, d), which serves as a common core for the forms $\mathcal{E}^r$ and $\mathcal{E}$, it follows that

$$\mu^r_{\langle f\rangle} \to \mu_{\langle f\rangle} \text{ for } r \to 0 \text{ weakly in the sense of Radon measures.}$$

Now obviously

$$\mu^r_{\langle f\rangle}(dx) = \int_{B^*_r(x)}\left(\frac{f(x) - f(y)}{d(x, y)}\right)^2 m_r(dy)m_r(dx)$$

and in the special case $f = d_p$ the first variation formula for the distance function on Alexandrov spaces ([OS94], thm 3.5) says that for fixed p, $x \in X$ any choice of segments γ_{xy} for $y \in X$ the formula

$$\rho_p(x) - \rho_p(y) = d(x, y)\cos\inf_{\gamma_{px}}\sphericalangle pxy + o_x(d(x, y))$$

obtains, where the infimum is taken over all possible choices of segments γ_{px} connecting p with x. Furthermore, for fixed p the cut locus C_p has measure zero (ibid., prop. 3.1), which implies that ρ_p is differentiable in m-a.e. $x \in X$. Moreover, using $\Sigma_x \simeq R^d$ for m-a.e. $x \in X$, the volume regularity and the weak Riemannian structure of (X, d) we find that

$$\lim_{r\to 0} \int_{B_r^*(x)} \left(\frac{u(x)-u(y)}{d(x,y)}\right)^2 m_r(dy)\frac{1}{\sqrt{m(B_r(x))}} = \fint_{B_1(0_x)} \langle \gamma'_{xp}, z\rangle^2_{g_x} dz = 1 \text{ for } m\text{-a.e. } x \in X,$$

which yields the claim by the dominated convergence theorem. □

Proof of theorem III. Due to corollary 3 and lemma 6 we know that the MCAF-part in the Ito decomposition of $\rho_p(\Xi)$ is a real-valued Brownian motion. As for the CAF part $A^{[\rho_p]}$ of zero energy we apply lemma 5 to $f = \rho_p$ for arbitrary nonnegative $h, \zeta \in L^\infty(X, m) \cap D(\mathcal{E})$ and the weak Laplacian comparison theorem I, which gives

$$\begin{aligned}
\mathbb{E}_{h\cdot m}(\tilde{\zeta}(\Xi_s)A_s^{[f]}) &= -\int_0^s \mathcal{E}(P_t h \cdot P_{s-t}\zeta, \rho_p)dt \\
&\leq (n-1)\int_0^s \langle P_t h \cdot P_{s-t}\zeta, (\ln S_k)' \circ \rho_p\rangle_m dt \\
&= (n-1)\mathbb{E}_{h\cdot m}\left[\tilde{\zeta}(\Xi_s)\int_0^s (\ln S_k)' \circ \rho_p(\Xi_t)dt\right].
\end{aligned}$$

Letting tend $h \cdot m$ to δ_x and using the monotone class theorem this means that $A_s^{[f]} \leq (n-1)\int_0^s (\ln S_k)' \circ \rho_p(\Xi_t)dt$ P_x-a.s. and thus the proof is complete. □

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