On Local Poincaré via Transportation

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Abstract

It is shown that curvature-dimension bounds $CD(N, K)$ for a metric measure space (X, d, m) in the sense of Sturm imply a weak L^1-Poincaré-inequality provided (X, d) has m-almost surely no branching points.

1. Introduction. From analysis on manifolds and metric measure spaces (X, d, m) the fundamental importance of Poincaré-type inequalities for the regularity of harmonic, Lipschitz or Sobolev functions is known (cf. [Stu96, Che99, HK00] and [Hei01, SC02, AT04]). In this note we show that metric measure spaces (X, d, m) with upper dimension-lower Ricci curvature $CD(N, K)$ bounds in the generalized sense of Sturm [Stu05b] (cf. [LV05] for $CD(N, 0)$) support a weak local L^1-Poincaré inequality provided (X, d) supports m-almost no branching points.

In fact we show a slightly stronger result, namely that the *segment inequality* of Cheeger-Colding [CC96], which in particular implies the Poincaré inequality in the sense of upper gradients, follows from the (N, K)-measure contraction property $(MCP(N, K))$ of [Oht05], assuming an additional symmetry condition on the choice of the geodesics involved. Furthermore, we show that the symmetry condition is trivially satisfied if the set of branching points in (X, d) is m-negligible, in which case $MCP(N, K)$ is also a known consequence of $CD(N, K)$, [Stu05b].

The assumption on the m-almost sure absence of branching points of (X, d) may be quite restrictive. However, as a first step in understanding the full meaning of curvature-dimension bounds for metric measure spaces it may be a useful task to study the regularity of admissible spaces (X, d, m) without branching points and study their relation with Alexandrov spaces, for instance.

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2. Preliminary study: Proof of the segment inequality on smooth manifolds by mass transportation.

For illustration let us derive the segment inequality in the smooth case using the language of mass transportation. This approach proved recently very useful for the generalization of certain concepts in smooth Riemannian analysis to general metric measure spaces [vRS04, Stu05a, Stu05b, LV05].

We review some standard terminology first. Throughout this note we call a curve $\gamma : [0,1] \to (X,d)$ in a metric space a *geodesic* (segment) if $d(\gamma(s),\gamma(t)) = d(\gamma(0),\gamma(1))|s-t|$ for all $s,t \in [0,1]$. For $A, B \subset X$ we define the set $\Gamma(A,B)$ as the collection of all geodesics γ with $\gamma(0) \in A$ and $\gamma(1) \in B$. For $x, y \in X$ any $\gamma \in \Gamma(x,y)$ will be denoted γ_{xy} which may not be unique. If $X = M$ is a Riemannian manifold (M^n, g) then d is the intrinsic metric induced by g.

Proposition. ([CC96]) *Let (M^n, g) be a smooth Riemannian manifold with Ricci curvature $\mathrm{Ricc}_M \geq (n-1)k$, $k \in \mathbb{R}$. Let $A_1, A_2 \subset B_R$ be measurable subsets contained in a geodesic R-Ball and let $g : B_{2R} \to \mathbb{R}$ be a nonnegative measurable function, then*

$$\int_{A_1}\int_{A_2}\int_0^1 g(\gamma_{xy}(t))dtdxdy \leq C_k(n,D)(|A_1|+|A_2|)\int_{B_{2R}} g(z)dz,$$

where

$$C_k(n,D) = \sup\{\left(\frac{s_k(s)}{s_k(ts)}\right)^{n-1} \mid s \in [0,D], t \in [\frac{1}{2},1]\},$$

$D = D(A_1,A_2) = \sup_{(x,y)\in A_1\times A_2} d(x,y)$ *and* $|A_i| = \mathrm{vol}_g(A_i), i = 1,2.$

Here and in the sequel s_k denotes the usual Sturm-Liouville function

$$s_k(t) = \begin{cases} \frac{\sin(\sqrt{k}t)}{\sqrt{k}} & \text{if } k > 0 \\ t & \text{if } k = 0 \\ \frac{\sinh(\sqrt{-k}t)}{\sqrt{-k}} & \text{if } k < 0. \end{cases}$$

Let us recall the necessary basic facts from optimal mass transportation theory we need (cf. [CEMS01] and [Vil03] as general reference).

Let μ_0 and μ_1 be two probability measures on a Riemannian manifold M^n with $\mathrm{Ricc}(M) \geq (n-1)k$ and let $\tau_t : M \to M$, $t \in [0,1]$ the optimal transportation map associated to the L^2-Wasserstein metric with the squared distance as cost function. General theory says that for $\mu_0 \ll d\mathrm{vol}_g$ this map is of the form

$$\tau_t(x) = \exp_x(-t\nabla\psi(x))$$

where $\psi : M \to \mathbb{R}$ is a $d^2/2$-concave function (i.e ψ it is the inf-convolution of another function ϕ with respect to the potential $d^2/2$).

A central element in the theory of optimal transportation on Riemannian manifolds is the following nonlinear concavity of the n-th root of the Jacobian of τ_t with respect to $t \in [0,1]$. Namely, if $J_t(x) := \det d\tau_t(x)$ denotes the Jacobian determinant of the map τ_t in x then

$$J_t^{1/n}(x) \geq \sigma_k^{(1-t)}(x) J_0^{1/n}(x) + \sigma_k^{(t)}(x) J_1^{1/n}(x)$$

with

$$\sigma_k^{(t)}(x) := t^{1/n} \left(\frac{s_k(td(x,\tau_1(x)))}{s_k(d(x,\tau_1(x)))} \right)^{1-1/n}.$$

Here $d(x,\tau_1(x))$ is the distance between a point x and its target point $\tau_1(x)$. This estimate is obtained in the fundamental paper [CEMS01] (cf. corollary 2.2. and lemma 6.1.), using the structure of τ_t, from Jacobi-field estimates and the arithmetic-geometric mean inequality on nonnegative matrices.

Proof of the Riemannian segment inequality. Let be A_1 and A_2 be two sets which are both embedded in a larger ball B_R and let $y \in A_2$ be a point. Let $\mu_t = \tau_{t*}\mu_0$ the Wasserstein geodesic connecting $\mu_0 = \frac{1}{|A_1|} dx_{|A_1}$ on A_1 with $\mu_1 = \delta_y$, the Dirac measure in y. From the structure of τ_t it is clear that each $x \in A_1$ travels along a geodesic connecting y with x (i.e. $\psi(x) = d_y^2(x)/2$ in τ_t). Since the cut locus of y has measure zero we can assume below that there is only one such geodesic for each pair $(x,y) \in M \times M$. Let $g : B_{2R} \to \mathbb{R}$ be nonnegative, then

$$\frac{1}{|A_1|} \int_{A_1} \int_0^{\frac{1}{2}} g(\gamma_{xy}(t)) dt dx = \frac{1}{|A_1|} \int_{A_1} \int_0^{\frac{1}{2}} g(\tau_t(x)) dt dx$$

which by the general integral transformation formula equals

$$= \int_0^{\frac{1}{2}} \int_{A_{1_t}} g(z) \mu_t(z) dz dt = \int_0^{\frac{1}{2}} \int_{A_{1_t}} g(z) \frac{m_0(x(t,z))}{\det d\tau_t(x(t,z))} dz dt$$

with $A_{1_t} := \{\gamma_{yx}(t) \,|\, y \in A_1\}$. Here we used the Jacobi identity for $m_t = \frac{d\mu_t}{dx}$

$$m_t(\tau_t(.)) \det d\tau_t(.) = m_0(.)$$

with $x(t,z) = \tau_t^{-1}(z)$ being the origin of the transport ray which hits z at time t. By the Jacobian concavity

$$\frac{1}{(\det d\tau_t(x(t,z))} \leq \left(\sigma_\kappa^{(1-t)}(x(t,z)) + \sigma_\kappa^{(t)}(x(t,z)) \det d\tau_1^{1/n}(x(t,z)) \right)^{-n}$$

where in the present case $\det d\tau_1(x(t,z)) = 0$ since $\mu_1 = \delta_y$. (To see this approximate $\mu_1 = \delta_y$ by the family $\mu_1^\epsilon = \frac{1}{|B_\epsilon(y)|} dx_{|B_\epsilon(y)}$ and use Jacobian identity for the density m_1^ϵ of μ_1^ϵ

$$m_1^\epsilon(z) = \frac{1}{|B_\epsilon(y)|} \mathbb{1}_{B_\epsilon(y)}(z) = \frac{m_0(x(z))}{\det d\tau_1^\epsilon(x(z))} = \frac{1/|A_1| \mathbb{1}_{A_1}(x(z))}{\det d\tau_1^\epsilon(x(z))}$$

which implies $\det d\tau_1^\epsilon(x(z)) \equiv \frac{|B_\epsilon(y)|}{|A_1|}$ for all $z \in B_\epsilon(y)$, equivalently $\det d\tau_1^\epsilon(x) \equiv \frac{|B_\epsilon(y)|}{|A_1|}$ for all $x \in A_1$ or $\det d\tau_1^\epsilon(x(t,z)) \equiv \frac{|B_\epsilon(y)|}{|A_1|}$ for all $z \in A_{1_t}$, and let $\epsilon \to 0$.) Since $m_0(x(t,z)) = \mathbb{1}_{A_{1_t}}(z)/|A_1|$ we arrive at the estimate

$$\begin{aligned}\frac{1}{|A_1|}\int_{A_1}\int_0^{\frac{1}{2}} g(\gamma_{xy}(t))dtdx \\ &\leq \int_0^{\frac{1}{2}}\int_{A_{1_t}} g(z)m_0(x(t,z))\left(\sigma_k^{(1-t)}(x(t,z))\right)^{-n} dzdt \\ &\leq \frac{1}{2|A_1|}\sup_{\substack{t\in[\frac{1}{2},1]\\ x\in A_1}}\left(\sigma_k^{(t)}(x)\right)^{-n}\int_{B_{2R}} g(z)dz,\end{aligned}$$

where we used $A_1, A_2 \subset B_R$. Integration respect to $y \in A_2$ yields

$$\int_{A_1}\int_{A_2}\int_0^{\frac{1}{2}} g(\gamma_{xy}(t))dtdxdy \leq C_k(n,D)|A_2|\int_{B_{2R}} g(z)dz$$

where $C_k(n,D)$ is the constant as defined in the statement of the proposition. Using the symmetry of the integral estimate we can bound the expression

$$\int_{A_1}\int_{A_2}\int_{\frac{1}{2}}^1 g(\gamma_{xy}(t))dtdxdy \leq C_k(n,D)|A_2|\int_{B_{2R}} g(z)dz$$

by repeating the previous arguments to the corresponding integral over the time interval $[0,\frac{1}{2}]$ when A_1 and A_2 are interchanged (see also section 3.) which by adding the two estimates concludes the proof. □

In the estimate above we defined the geodesic to be parameterized on the unit interval. Using unit speed parameterization it reads

$$\int_{A_1}\int_{A_2}\int_0^{d(x,y)} \frac{g(\gamma_{xy}(t))}{d(x,y)}dtdxdy \leq C_k(n,D)(|A_1|+|A_2|)\int_{B_{2R}} g(z)dz.$$

3. Segment Inequality on metric measure spaces with transportation lower Ricci bounds.

3.1. Measure Contraction Property.

In the sequel we shall basically adopt the framework from [Oht05] which builds on the concept of the measure contraction property defined in [Stu98]. For this let (X,d) be a complete length metric space, i.e. d coincides with its induced length metric, and let m be a locally finite fully supported measure on the Borel sigma algebra $\mathcal{B}(X)$ of X.

For $N > 1$, $K \in \mathbb{R}$, $r \in [0, \pi\sqrt{(N-1)/\max(K,0)}[$ and $t \in [0,1]$ define

$$\tau_{K,N}^{(t)}(r) = t^{1/N}\left(\frac{s_{K/(N-1)}(tr)}{s_{K/(N-1)}(r)}\right)^{1-1/N}.$$

Definition. (**[Oht05]**) *For $N > 1$ and $K \in \mathbb{R}$ the metric measure space (X, d, m) is having the (N, K) measure contraction property ($MCP(N, K)$, for short) if for each pair $(x, M) \in X \times \mathcal{B}(X)$ with $m(M) > 0$ there exists a probability measure Π on the set of geodesics $\Gamma(x, M) = \{\gamma_{xy} | y \in M\}$ with $e_{0*}\Pi = m_M := \frac{1}{m(M)} m_{|M}$ and $e_{1*}\Pi = \delta_x$ such that*

$$dm \geq m(M) \cdot e_{t*} \left(\tau_{K,N}^{(1-t)}(\ell(\gamma))\Pi(d\gamma) \right),$$

where $l : \Gamma \to \mathbb{R}$ is the length functional and $e_t : \Gamma \to X$ with $e_t\gamma := \gamma(t)$, $t \in [0, 1]$, is the evaluation map.

For simplicity the case $N = 1$ is omitted here, which is interesting only for $K \leq 1$, cf. [Oht05], and can be treated as below. Note also that for $K > 0$, in the definition above a bound on $\operatorname{diam}(A \cup \{x\})$ is redundant a posteriori because $MCP(N, K)$ implies the corresponding Bonnet-Myers diameter bound on (X, d) (ibid. theorem 4.3.). Finally, from the Bishop-Gromov volume comparison implied by $MCP(N, K)$ (theorem 5.1.) it follows that (X, d) is locally compact.

For the interpretation of the $MCP(N, K)$ we may disintegrate the measure Π with respect to the evaluation map e_0 and use the condition that $e_{0*}\Pi = m_M$. This yields the mixing representation $\Pi(d\gamma) = \lambda_{yx}(d\gamma)m_M(y)$ where the measures λ_{yx} are supported on $\Gamma(y, x)$. Moreover, the measures λ_{yx} are determined uniquely by Π for m-almost y and vice versa.

Let now $M_t := e_t\Gamma(M, x)$ be the set hit by all geodesics from M to x then the statement above is equivalent to

$$m(Z) \geq (1-t) \int_M \int_{\Gamma(y,x)} 1\!\!1_Z(\gamma_t) \left\{ \frac{s_{K/(N-1)}((1-t)d(x, y))}{s_{K/(N-1)}(d(x, y))} \right\}^{N-1} \lambda_{yx}(d\gamma)m(dy)$$

for all measurable Z with w.l.o.g $Z \subset M_t$, since for $Z \subset X \setminus M_t$ the right hand side is zero. - Written in this form the MCP-condition gives a lower bound for the concentration of m under the generalized homothetic map defined by the Markov kernels $\Lambda_t^y(dx) = e_{t*}(\lambda_y)(dx)$ (cf. next section). It may also be seen as a requirement on the minimal 'mean spreading' of all geodesics to x, where the mean is taken with respect to the collection of weights (λ_{yx}) and m.

From the classical Bishop-Gromov volume comparison it follows that a smooth Riemannian manifold (M^n, g) with $\operatorname{Ricc}(M^n, g) \geq (n-1)k$ satisfies the $CD(n, K)$ condition for any $K \leq (n-1)k$. The relevance of the $MCP(N, K)$ condition comes from its robustness with respect to the measured Gromov-Hausdorff convergence (cf. [Oht05]). Moreover, it is essentially implied by the generalized dimension-curvature bounds defined recently by Sturm cf. section 3.4

3.2. Segment inequality for (X, d, m).

For the precise formulation of the subsequent results we need a little more notation. Let $B \subset X$ be a set and $t \in [0, 1]$. Then we define the following set valued geodesic contraction map in direction B

$$\Gamma_t(., B) : 2^X \to 2^X; \quad \Gamma_t(A, B) := e_t(\Gamma(A, B)).$$

By abuse of notation for $t \in [0, 1]$ we define also $\Gamma_{-t}(A, B) = \{z \in X \,|\, \exists\, b \in B, \gamma_{zb} : \gamma_{zb}(t) \in A\}$ the inverse A with respect to geodesic contraction in direction B. When $B = \{b\}$ we write $A_t(b) := \Gamma_t(A, \{b\})$ and $A_t^{-1}(b) := \Gamma_{-t}(A, \{b\})$. For later reference note that compactness of A implies the same for $A_t(b)$, due to the local compactness of (X, d) and the Arzela-Ascoli theorem.

In the (N, K)-MCP statement above the 'transference plan' Π and thus the measures $(\lambda_{yx})_{y\in M} = (\lambda^M_{yx})_{y\in M}$ may depend on M. Let us say that the family of measures $(\lambda^M_{xy})_{x,y\in X}$ is *symmetric* in (x, y) if

$$\lambda^M_{xy}(d\gamma) = \lambda^M_{yx}(d\overline{\gamma}),$$

where $\gamma \to \overline{\gamma}$ is the inversion map $\overline{\gamma}(t) = \gamma(1 - t)$, $t \in [0, 1]$.

Proposition. *Let (X, d, m) satisfy the (N, K)-measure contraction property and assume that the map $(x, y) \to \lambda^M_{xy} \in \mathcal{P}(\Gamma_{xy})$ can be chosen to be symmetric for $m \times m$ almost every pair (x, y) and for some set $M \subset X$. Then for two measurable subsets $A_1, A_2 \subset B_R$ contained in a geodesic ball $B_R \subset M$ and $g : B_{2R} \to \mathbb{R}$ nonnegative*

$$\int_{A_1}\int_{A_2}\int_{\Gamma(x,y)}\int_0^1 g(\gamma_{xy}(t))dt\lambda^M_{xy}(d\gamma)m(dx)m(dy)$$

$$\leq C_{K/(N-1)}(N, D)(m(A_1) + m(A_2)) \int_{B_{2R}} g(z)m(dz),$$

where $D = D(A_1, A_2)$ and $C_{K/(N-1)}(N, D)$ are defined as above.

Proof. We write the (N, K)-measure contraction inequality relative to the ambient set M yet in another form, namely for all $(x, t) \in X \times [0, 1]$

$$m \geq (\tau^x_t \cdot \Lambda^x_t) * m,$$

where the sign $*$ means convolution with the transition kernel

$$(\tau^x_t \cdot \Lambda^x_t)(y, dz) = \tau_t(x, y)\Lambda^x_t(y, dz)$$

with the symmetric function $\tau_t(x, y) := \tau^{(1-t)}_{K,N}(d(x, y)) =: \tau_t(d(x, y))$ and the Markov kernel

$$(y, Z) \to \Lambda^x_t(y, Z) = \int_{\Gamma(y,x)} 1\!\!1_Z(\gamma_t)\lambda_{yx}(d\gamma), \quad y \in X, Z \subset X.$$

Here we omitted the upper index for $\lambda_{yx} = \lambda^M_{yx}$ as we shall in the rest of the proof because M is fixed.

With this notation the (N,K)-measure contraction inequality is written

$$\int_Z g(z)m(dz) \geq \int_X \int_Z g(y)\tau_t(x,z)\Lambda_t^x(z,dy)m(dz) \tag{MCP}$$

for all measurable $Z \subset X$ and nonnegative measurable $f : X \to \mathbb{R}$. Note that it suffices to take the outer integral on the set $Z_t^{-1}(x)$. Since $d(z,x) = d(\gamma_{zx}(t),x)/(1-t)$ for all $z \in X$ we have

$$\inf_{z\in Z_t^{-1}(x)} [\tau_t(x,z)] = \inf_{z\in Z} \left[\tau_t\big(d(x,z)/(1-t)\big)\right]$$

such that the estimate

$$\int_Z g(z)m(dz) \geq \inf_{z\in Z} [\tau_t(d(x,z)/(1-t))] \int_X \int_Z g(y)\Lambda_t^x(z,dy)m(dz).$$

is obtained. For compact $A \subset X$ we apply this inequality to the set $Z = A_t(x)$, which by the previous remark is compact and hence also measurable. Assuming $A \subset B_R, x \in B_R$ it follows $A_t(x) \subset B_{2R}$, thus

$$\int_{B_{2R}} g(z)m(dz) \geq \inf_{z\in A} [\tau_t(z,x)] \int_X \int_{A_t(x)} g(y)\Lambda_t^x(z,dy)m(dz).$$

From

$$\begin{aligned}\int\limits_X \int\limits_{A_t(x)} g(y)\Lambda_t^x(z,dy)m(dz) &= \int\limits_{\Gamma_{-t}(A_t(x),x)} \int\limits_{A_t(x)} g(y)\Lambda_t^x(z,dy)m(dz)\\ &\geq \int_A \int_{\Gamma(X,X)} g(\gamma(t))\lambda_{zx}(d\gamma)m(dy)\end{aligned}$$

and integration with respect to time one obtains

$$\int_A \int_0^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t))\lambda_{zx}(d\gamma)dtm(dz) \leq \frac{1}{2} \sup_{\substack{t\in[0,\frac{1}{2}]\\ z\in A}} [\tau_t^{-1}(x,z)] \int_A g(z)m(dz).$$

Now, in order to prove the claim of the proposition we may assume that $A_1, A_2 \subset B_R$ are closed, because the general statement will follow from this via approximation of A_1, A_2 by compacts from inside. (Here we may use the inner regularity of m, which is a consequence of the σ-compactness of (X,d).) Hence, putting $A = A_1$ and integrating the last inequality with respect to $x \in A_2$ leads to

$$\begin{aligned}&\int_{A_2} \int_{A_1} \int_0^{\frac{1}{2}} \int_{\Gamma(X,X)} g(\gamma(t))\lambda_{zx}(d\gamma)dtm(dz)m(dx)\\ &\qquad \leq \frac{1}{2} m(A_2) \cdot \sup_{\substack{t\in[0,\frac{1}{2}]\\ (z,x)\in A_1\times A_2}} [\tau_t^{-1}(x,z)] \int_{A_1} g(z)m(dz)\\ &\qquad \leq m(A_2) \cdot C_{K/(N-1)}(N,D) \int_{B_{2R}} g(z)m(dz).\end{aligned}$$

Interchanging the roles of A_1 and A_2 and using the symmetry of the measures (λ_{xy}) yields

$$\begin{aligned}
m(A_1)\cdot C_{K/(N-1)}(N,D)\int_{B_{2R}} g(z)m(dz)& \\
&\geq \int_{A_1}\int_{A_2}\int_0^{\frac{1}{2}}\int_{\Gamma(X,X)} g(\gamma(t))\lambda_{zx}(d\gamma)dtm(dz)m(dx)\\
&= \int_{A_1}\int_{A_2}\int_0^{\frac{1}{2}}\int_{\Gamma(X,X)} g(\gamma(t))\lambda_{xz}(d\overline{\gamma})dtm(dx)m(dz)\\
&= \int_{A_1}\int_{A_2}\int_0^{\frac{1}{2}}\int_{\Gamma(X,X)} g(\overline{\gamma}(1-t))\lambda_{xz}(d\overline{\gamma})dtm(dx)m(dz)\\
&= \int_{A_1}\int_{A_2}\int_{\frac{1}{2}}^{1}\int_{\Gamma(X,X)} g(\overline{\gamma}(t))\lambda_{xz}(d\overline{\gamma})dtm(dx)m(dz)\\
&= \int_{A_1}\int_{A_2}\int_{\frac{1}{2}}^{1}\int_{\Gamma(X,X)} g(\gamma(t))\lambda_{xz}(d\gamma)dtm(dx)m(dz)\\
&= \int_{A_2}\int_{A_1}\int_{\frac{1}{2}}^{1}\int_{\Gamma(X,X)} g(\gamma(t))\lambda_{zx}(d\gamma)dtm(dz)m(dx).
\end{aligned}$$

Adding this inequality to the first one the claim is established. □

Corollary. *The assertion of the proposition above is true in particular when the cut-locus* $C_x := \{y \in X \mid \#\Gamma(x,y) \geq 2\}$ *satisfies* $m(C_x) = 0$ *for* m*-a.e.* $x \in X$*.*

Proof. If $x \notin C_y$ then $\lambda_{xy} = \delta_{\gamma_{xy}}$ for the unique $\gamma_{xy} \in \Gamma(x,y)$. This forces λ_{xy} to be symmetric $m \times m$-almost surely. □

For the following version of the Poincaré inequality recall that for $f : X \to \mathbb{R}$ the function $g : X \to \mathbb{R}_+$ is called an *upper gradient* if

$$|f(x) - f(x)| \leq \int_0^{d(x,y)} h(\gamma_s)ds$$

for any unit speed geodesic connecting x and y.

Corollary. **(L^1-Poincaré-inequality)** *Under the conditions above let* h *be an upper gradient of* f *then*

$$\int_{B_R}\int_{B_R}\frac{|f(x)-f(y)|}{d(x,y)}m(dx)m(dy) \leq |B_R|C_{n,k}(D)\int_{B_{2R}} h(x)m(dx).$$

Proof. In order to prove this inequality for each pair (x,y) let λ_{xy} be the associated measure from proposition above, then the assertion follows from

$$\frac{|f(x)-f(y)|}{d(x,y)} = \int_{\Gamma_{xy}} |f(\gamma(0))-f(\gamma(1))|\lambda_{xy}(d\gamma) \leq \int_{\Gamma_{xy}}\int_0^1 h(\gamma_s)ds\lambda_{xy}(d\gamma)$$

which can be inserted into the segment inequality. □

Examples. Consider the Banach space $(X,d) = (\mathbb{R}^n, \|.\|_p)$, $p \in [1,\infty]$, equipped with $m = \lambda^n$ the n-dimensional Lebesgue measure, where $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$. For $p \in]1,\infty[$ the geodesics are straight Euclidean line segments $\gamma_{xy} = x + t(x-y)$, hence $C_x = \emptyset$ for all $x \in X$. Obviously $(\mathbb{R}^n, \|.\|_p, \lambda^n)$ satisfies $MCP(0,n)$. In the cases $p = 1$ or $p = \infty$ one finds $m(X \setminus C_x) = 0$ for all $x \in X$. However, choosing $\lambda_{xy}(d\gamma) = \delta_{(x+t(x-y))}(d\gamma)$ for all $x, y \in X$ the $MCP(0,n)$ property remains true. Since this choice of λ is symmetric, the segment and Poincaré inequalities hold.

3.3. Extendable geodesics and branching points.

Definition. *Let (X,d) be a metric set and $x \in X$. Define $I_p := \{\gamma_{xp}(t) | t \in (0,1], x \in X\}$ as all points x which are connected to p by at least one extendable geodesic segment and let $T_p = X \setminus I_p$.*

Proposition. *Let (X,d,m) satisfy an (N,K)-measure contraction property then $m(T_p) = 0$ for all $p \in X$.*

Proof. Adapting the idea from proposition 3.1. in [OS94], let $X^l \subset X$, $l \in \mathbb{N}$, be an exhaustion of X by compact subsets. Then $I_p = \bigcup_{l\in\mathbb{N}} I_p^l$ with $I_p^l := \bigcup_{t\in(0,1]} X_t^l(p)$ and where the sets $X_t^l(p)$ are monotone decreasing for $t \in [0,1]$, i.e. $X_t^l(p) \subset X_s^l(p) \subset$ for $s \le t$. Let $A \subset X$ be an open bounded set and choose the weight functions $\lambda_{xy} = \lambda_{xy}^M$ for M large enough such that $A \subset M$. Then (MCP) with $x = p$ and $Z = A \cap X_t^l(p)$ yields

$$\begin{aligned} m(A \cap X_t^l(p)) &\ge \int_X \tau_t(d(p,y))\Lambda_t^p(y, A \cap X_t^l(p))m(dy) \\ &= \int_X \tau_t(d(p,y))\Lambda_t^p(y,A)m(dy) \end{aligned}$$

because $\Lambda_t^p(y, A \setminus X_t^l(p)) = 0$ for all $y \in X$. Hence for $s \le t$ we obtain from the monotonicity of $\{A \cap X_t^l(p), s \in [0,1]\}$

$$m(A \cap X_s^l(p)) \ge \int_X \tau_t(d(p,y))\Lambda_t^p(y,A)m(dy)$$

Hence, for $s \to 0$, by monotone convergence

$$m(A \cap I_p^l) \ge \int_X \tau_t(d(p,y))\Lambda_t^p(y,A)m(dy).$$

Sending $t \to 0$ and using $\Lambda_t^p(y,A) \to \mathbb{1}_A(y)$ dominated convergence yields

$$m(A \cap I_p^l) \ge m(A),$$

such that the claim follows, from $l \to \infty$, by the arbitrariness of A. □

Recall that $p \in X$ by definition is a branching point of (X,d) if p is a common end point of at least three nontrivial and disjoint segments.

Corollary. *If (X,d) admits no branching points m-almost surely and (X,d,m) satisfies an $MCP(N,K)$-property then $m(C_x) = 0$ for $x \in$*

X. Also, the segment and Poincaré inequalities hold for (X,d,m) in this case.

Proof. Since for m-almost all $y \in C_x$ at least one $\gamma_{xy} \in \Gamma(x,y)$ can be extended beyond y as segment, y must be a branching point. By assumption branching points are m-negligible. □

Remark. The examples $(\mathbb{R}^n, \|.\|_p, \lambda^n)$, $p \in \{1,\infty\}$, show that the $MCP(N,K)$-property is not strong enough to prevent a 'large' (with respect to m) amount of branching points, even if branching points indicate infinite negative sectional curvature in Alexandrov sense. It is natural to ask which additional assumptions on (X,d,m) inhibit a set of branching points with positive m-mass. For example, (X,d) admits no branching if it is a limit of Riemannian manifolds with uniform local lower sectional curvature bounds.

3.4. Upper Dimension-Lower Curvature Bounds CD(N,K).
The following definition is introduced in [Stu05b] as a stable (w.r.t. measured Gromov-Hausdorff convergence) notion of sharp upper dimension-lower Ricci curvature bounds for metric measure spaces. For this let as above (X,d,m) be a complete length metric measure space, where m is fully supported locally finite Borel, and let $(P_2(X,d), d_W)$ denote the associated space of probability measures on X equipped with the the quadratic Wasserstein distance d_W.

Definition. *A metric measure space (X,d,m) satisfies the curvature dimension condition $CD(N,K)$ if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M,d,m)$ with $d\rho_i/dm = \rho_i \in L^1(X,m)$, $i = 1,2$, there exists an optimal d^2-coupling $q \in \mathcal{P}(X \times X)$ and a geodesic $\Gamma : [0,1] \to P_2(X,d)$ connecting ν_0 and ν_1 such that for all $t \in [0,1]$, $N' \geq N$,*

$$S_{N'}(\Gamma(t)|m) \leq -\int_{X\times X} \Big[\tau^{(1-t)}_{K,N'}(d(x_0,x_1)) \cdot \rho_0^{-1/N'}(x_0) + \tau^{(t)}_{K,N'}(d(x_0,x_1)) \cdot \rho_1^{-1/N'}(x_1)\Big]\, dq(x_0,x_1)$$

where $S_{N'}(\mu|m) = -\int_X \rho^{-1/N'} dm$ for $\mu = \rho dm \in \mathcal{P}(X)$.
Furthermore, it is shown (ibid., theorem 5.1.) that $CD(N,K)$ implies $MCP(N,K)$ provided the geodesic γ_{xy} in (X,d) is unique for $m \times m$-almost all $(x,y) \in X \times X$. By the same argument as in ibid., lemma 4.1., the latter will hold if (X,d) admits m-almost surely no branching. Hence the following conclusion is obtained.

Proposition. *$CD(N,K)$ implies the segment and Poincare inequalities for a metric measure space (X,d,m) provided (X,d) has m-almost no branching points.*

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