EVOLUTION BY MEAN CURVATURE FLOW IN SUB-RIEMANNIAN GEOMETRIES: A STOCHASTIC APPROACH.

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Abstract. We study evolution by horizontal mean curvature flow in sub-Riemannian geometries by using stochastic approach to prove the existence of a generalized evolution in these spaces. In particular we show that the value function of suitable family of stochastic control problems solves in the viscosity sense the level set equation for the evolution by horizontal mean curvature flow.

1. Introduction.

In Euclidean spaces, the motion by mean curvature flow of a hypersurface is a geometrical evolution such that the normal velocity at each point of the hypersurface is equal to the mean curvature at that point. Unfortunately, even smooth surfaces can develop singularities in finite time, so a weak notion of evolution is necessary. The notion that we are going to use here follows a nonlinear PDE-approach, based on Chen-Giga-Goto ([CGG]) and Evans-Spruck ([ES]). Roughly speaking, the idea consists in associating a PDE to a smooth hypersurface evolving such that the function which solves this PDE has level sets which evolve by mean curvature flow. Then one can define the solutions of the "generalized evolution by mean curvature flow" as the zero-level sets of the viscosity solution of this PDE. In this paper we study the corresponding evolution in sub-Riemannian geometries with the help of stochastic control methods.

Sub-Riemannian geometries are degenerate Riemannian spaces where the Riemannian inner product is defined just on a sub-bundle of the tangent bundle. To be more precise, we will consider $X_1, \dots, X_m$ smooth vector fields on $\mathbb{R}^n$ and a Riemannian inner product defined on the distribution $\mathcal{H}$ generated by such vector fields. Then it is possible to define intrinsic derivatives of any order by taking the derivatives along the vector fields $X_1, \dots, X_m$. That allows us to write differential operators like Laplacian, infinite-Laplacian etc, using intrinsic derivatives. In particular one can define a notion of horizontal mean curvature and horizontal mean curvature flow.

While there are many results for evolution by mean curvature flow in the Euclidean setting, only little is known in these degenerate spaces. This evolution in a sub-Riemannian manifold is very different from the corresponding Euclidean motion, in particular because of the existence of the so-called characteristic points, which are points where the Euclidean normal is perpendicular to the horizontal space and so not admissible. In such points the horizontal gradient of the level-set function vanishes and so they correspond to singularities for the associated level set equation.

The structure of the set of these points is far more complicated than in the Euclidean case because the set of Euclidean gradients for which the associated horizontal gradient vanishes is space-dependent and, at each point, of nonzero dimension. The different nature of these degeneracies creates serious difficulties in applying most of those techniques which are known to work for the Euclidean setting. To avoid the problems created from the presence of these singularities, we will use a stochastic approach for showing existence of solutions and for defining a generalized evolution.

A connection between a certain stochastic control problem and a large class of geometric evolution equations, including the (Euclidean) evolution by mean curvature flow, has been found by Buckdahn, Cardaliaguet and Quincampoix (in [BCQ]) and Soner and Touzi (in [ST2, ST3]). The control, loosely speaking, constrains the increments of the stochastic process to a lower dimensional subspace of $\mathbb{R}^n$, while the cost functional consists only of the terminal cost but involves an essential supremum over the probability space. It turns out that the value function solves the level set equation associated with the geometric evolution. Moreover, one can show that the set of points from which the initial hypersurface can be reached almost surely in a given time by choosing an appropriate control coincides with the set evolving by mean curvature flow. This stochastic approach generalizes very naturally to sub-Riemannian geometries by using an intrinsic Brownian motion associated with the sub-Riemannian geometry. This allows us to obtain certain existence results in general sub-Riemannian manifolds which were previously unknown. In particular, the value function may be used for defining a generalized flow.
More precisely, the value function $v(t,x)$ associated to this stochastic control problem is defined as the infimum, over the admissible controls, of the essential supremum of the final cost g (at some fixed terminal time $T>t$), for the controlled path ξ^ν starting from x at the time t. We can show that $u(t,x):=v(T-t,x)$ is a viscosity solution of the level set equation of the evolution by horizontal mean curvature flow. So $\Gamma(t)=\{x\in\mathbb{R}^n \mid u(t,x)=0\}$ is a generalized evolution by horizontal mean curvature flow in general sub-Riemannian manifolds.
We would like to mention that there is a recent work by Capogna and Citti ([CC]) where the authors show existence of solutions of the level set equation for horizontal mean curvature flow in Carnot groups, using a different approach and a slightly different formulation of the level set equation in the visosity sense.

The organization of the paper is the following.
In Section 2 we introduce sub-Riemannian geometries, the horizontal mean curvature and the definition of the characteristic points.
In Section 3 we give a notion of generalized evolution by mean curvature flow, following the level set formulation introduced by Chen-Giga-Goto in [CGG] for the corresponding Euclidean evolution.
In Section 4 we define and study a stochastic control problem, whose associated value function solves in the viscosity sense the level set equation for the evolution by horizontal mean curvature flow. We introduce a family of (Stratonovich) stochastic ODEs driven by a "horizontally constrained Brownian motion" and we will show that the associated generator is exactly the horizontal Laplacian. Moreover we study some properties of value function with stochastic methods.

In Section 5 we show that the value function is a bounded and lower semicontinuous viscosity solution of the level set equation for the evolution by horizontal mean curvature flow in the sub-Riemannian case. We first sketch how to derive the PDE solved by the value function, assuming more regularity for the solution. This explains heuristically why the optimal control is, at any point, the projection on the horizontal tangent space of the level set. Moreover we show that if there exist comparison principles for the degenerate parabolic PDE introduced in Section 3, the value function is continuous in any sub-Riemannian geometry.

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2. Mean curvature in sub-Riemannian geometries.

2.1. **Sub-Riemannian geometries.**

We recall briefly what sub-Riemannian geometries are (e.g. see [Be, M]).
Let $X_1(x), \dots, X_m(x)$ be a family of smooth vector fields on $\mathbb{R}^n$ and

$$\mathcal{H}_x := \mathrm{Span}(X_1(x), \dots, X_m(x)) \quad \text{and} \quad \mathcal{H} := \{(x, v) \,|\, x \in \mathbb{R}^n, v \in \mathcal{H}_x\}.$$

Definition 2.1. A *sub-Riemannian metric* in $\mathbb{R}^n$ is a triple $(\mathbb{R}^n, \mathcal{H}, \langle\cdot,\cdot\rangle_g)$, where $\langle\cdot,\cdot\rangle_g$ is a Riemannian metric defined on $\mathcal{H}$.

An absolutely continuous curve $\gamma : [0, T] \to \mathbb{R}^n$ is called *horizontal* if $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$, a.e. $t \in [0, T]$, i.e. $\exists\, \alpha(t) = (\alpha_1(t), \dots, \alpha_m(t))$ measurable function such that

$$\dot\gamma(t) = \sum_{i=1}^{m} \alpha_i(t) X_i(\gamma(t)), \quad \text{a.e. } t \in [0, T]. \tag{1}$$

We set $|\dot\gamma(t)|_g = \langle\dot\gamma(t), \dot\gamma(t)\rangle_g^{\frac{1}{2}}$ and define the length-functional

$$l(\gamma) = \int_0^T |\dot\gamma(t)|_g dt = \int_0^T \sqrt{\alpha_1^2(t) + \cdots + \alpha_m^2(t)}\, dt,$$

choosing $\langle\cdot,\cdot\rangle_g$ such that the vector fields $X_1, \dots, X_m$ are orthonormal.
Once defined the length-functional, we can introduce the following distance

$$d(x, y) := \inf\{l(\gamma) \,|\, \gamma \text{ horizontal curve joining } x \text{ to } y\}. \tag{2}$$

Whenever the Hörmander condition (i.e. the Lie algebra associated to $\mathcal{H}$ generates at any point the whole of $\mathbb{R}^n$) is satisfied the above defined distance is finite, continuous with respect to the Euclidean topology, and minimizing geodesics exist (but they are in general not even locally unique).
Carnot groups are particular sub-Riemannian geometries, where a structure of Lie group is defined. For more details on this particular class of sub-Riemannian geometries, we refer to [CDPT, DGN].

2.2. **Horizontal mean curvature.**
We introduce the notion of horizontal mean curvature in sub-Riemannian manifolds (see also [CDPT, DGN, HP]).
Given $X_1, \dots, X_m$ smooth vector fields on $\mathbb{R}^n$, satisfying the Hörmander condition, the *horizontal gradient* of a (smooth) function $u : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\Upsilon u(x) = (X_1 u) X_1(x) + \dots + (X_m u) X_m(x) \in \mathbb{R}^n.$$

From now on, we will often omit the dependency on the point x and use the coordinate-vector field of Υu w.r.t. $X_1, \dots, X_m$, that is

$$\mathcal{X} u = (X_1 u, .., X_m u)^T \in \mathbb{R}^m.$$

Note that $|\Upsilon u|_g^2 = \sum_{i=1}^m \big(X_i u\big)^2 = |\mathcal{X} u|^2$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$.
Fix a point $x \in \mathbb{R}^n$. We call *horizontal space* the tangent space of the sub-Riemannian manifold, denoted by $\mathcal{H}_x \mathbb{R}^n$ while for a hypersurface $\Sigma \subset \mathbb{R}^n$ the *tangent space* and *horizontal tangent space* are, respectively, the Euclidean tangent space of Σ and the intersection of the Euclidean tangent space with the horizontal space. We indicate the latter two objects by $T_x\Sigma$ and $\mathcal{H}T_x\Sigma$.

Definition 2.2. Let Σ be a hypersurface in $\mathbb{R}^n$, we call *horizontal normal of* Σ the renormalized projection of the Euclidean normal on the horizontal space, which, if $\Sigma = \{x \in \mathbb{R}^n \,|\, u(x) = 0\}$ smooth (with $|\nabla u(x)| \neq 0$ on Σ), is

$$n_0(x) := \frac{\Upsilon u}{|\Upsilon u|_g} = \frac{\sum_{i=1}^m (X_i u) X_i(x)}{|\mathcal{X} u|}. \tag{3}$$

The *horizontal mean curvature* is defined as the horizontal divergence of the horizontal normal, i.e.

$$k_0(x) := \sum_{i=1}^m X_i \left(\frac{X_i u}{|\mathcal{X} u|} \right). \tag{4}$$

Unlike in the Euclidean case, the horizontal normal to a smooth hypersurface is not always well defined. In fact, whenever the Euclidean normal is “vertical”, which means that its projection on the horizontal space vanishes, then n_0 and hence k_0 are not defined.

Definition 2.3. Let Σ be a hypersurface in $\mathbb{R}^n$, the *set of the characteristic points of* Σ is

$$char(\Sigma) = \{x \in \mathbb{R}^n \,|\, \mathcal{H}T_x\Sigma = \mathcal{H}_x\mathbb{R}^n\}. \tag{5}$$

We remark that the existence of characteristic points makes the evolution by horizontal mean curvature flow much different from the corresponding Euclidean or Riemannian evolution. Note that, if $\Sigma = \{u = 0\}$, then $|\mathcal{X} u| = 0$, at any characteristic points (while the reverse is in general not true).
The aim of the paper is to introduce and study a PDE associated to this evolution, so we need to introduce some intrinsic differential operators.
The symmetrized matrix of second-order derivatives is the $m \times m$ matrix given by

$$(\mathcal{X}^2 u)^*_{i,j} = \frac{X_i(X_j u) + X_j(X_i u)}{2}.$$

We call *horizontal Laplacian* and *horizontal infinite-Laplacian*, respectively, the following second-order operators:

$$\Delta_0 u = \sum_{i=1}^{m} X_i(X_i u), \quad \Delta_{0,\infty} u = \left\langle (\mathcal{X}^2 u)^* \frac{\mathcal{X}u}{|\mathcal{X}u|}, \frac{\mathcal{X}u}{|\mathcal{X}u|} \right\rangle. \tag{6}$$

As in the Euclidean case, if $u : \mathbb{R}^n \to \mathbb{R}$ is smooth and $\Sigma = \{x \in \mathbb{R}^n : \ u(x) = 0\}$, with $|\nabla u(x)| \neq 0$ on Σ, at any non-characteristic point, (4) can be written as

$$k_0(x) = |\mathcal{X}u|^{-1}\big(\Delta_0 u - \Delta_{0,\infty} u\big). \tag{7}$$

For later use, we need to express the previous intrinsic differential operators in terms of the matrix associated to the sub-Riemannian geometry and the corresponding Euclidean objects. So we introduce the matrix $\sigma(x) := [X_1(x), \dots, X_m(x)]^T$, then

$$\mathcal{X}u(x) = \sigma(x) Du(x),$$

and

$$\big(\mathcal{X}^2 u\big)^* = \sigma(x)(D^2 u)\sigma^T(x) + A(X_1, \dots, X_m, Du), \tag{8}$$

where the matrix A is a symmetric $m \times m$ matrix defined as

$$A_{i,j}(X_1, \dots, X_m, Du) = \frac{1}{2} \left\langle \nabla_{X_i} X_j + \nabla_{X_j} X_i, Du \right\rangle, \quad \text{for } i, j = 1, \dots, m, \tag{9}$$

and $\nabla_{X_i} X_j = DX_j X_i$ is the derivative of the vector field X_j w.r.t. the vector field X_i. Note that $\big(\mathcal{X}^2 u\big)^*$ does not depend on just the second-order derivatives but also on the first-order derivatives (due to the derivation of the vector fields). Moreover

$$\begin{aligned} \Delta_0 u &= \mathrm{Tr}\big(\sigma(x)(D^2 u)\sigma^T(x)\big) + \sum_{i=1}^{m} \left\langle \nabla_{X_i} X_i, Du \right\rangle, \\ \Delta_{0,\infty} u &= \left\langle \big(\sigma(x)(D^2 u)\sigma^T(x)\big) \frac{\sigma(x)Du}{|\sigma(x)Du|}, \frac{\sigma(x)Du}{|\sigma(x)Du|} \right\rangle \\ &\quad + \left\langle A(X_1, \dots, X_m, Du) \frac{\sigma(x)Du}{|\sigma(x)Du|}, \frac{\sigma(x)Du}{|\sigma(x)Du|} \right\rangle. \end{aligned} \tag{10}$$

This paves the way for studying the horizontal mean curvature flow by the techniques from stochastic control theory which we explain later. Note that the situation is much easier when there are no characteristic points, because the previous operators are not degenerate.

Definition 2.4. We call *regular hypersurface* any C^1 hypersurface such that all the points are not characteristic.

Example 2.1. In the particular case of the Heisenberg group (see e.g. [CDPT] for a definition and several details), an easy calculation shows that $\nabla_{X_i} X_j + \nabla_{X_j} X_i = 0$, for any $i, j = 1, 2$, hence the first-order part in (8) and (10) vanishes. That makes it easier to study several explicit examples ([CDPT]). In the Heisenberg case, examples of regular surfaces are any vertical plane $ax + by = d$, any cylinder around the z-axis, any torus around the z-axis. Nevertheless, there are very few regular surfaces compared to the Euclidean or the Riemannian case. As remarked by Roberto Monti, one easily sees by the "hairy ball theorem" that any C^1 compact surface $\Sigma \subset \mathbb{H}^1$, topologically equivalent to the sphere, is not regular: In fact, using the complex interpretation of the Heisenberg group, one can consider as vector field the horizontal normal vector. Assuming that the surface is regular, such a vector field is different from zero at any point, so also the rotated vector by $\frac{\pi}{2}$ is not vanishing and it is tangent to the surface, which contradicts the "hairy ball theorem". Hence non-regular surfaces are far more interesting, because all sphere-type surfaces are not regular. Moreover, results for short time existence of classical solutions starting from regular surfaces, which were very important in the euclidean context, would apply only to very few surfaces in our context.

3. Generalized evolution by horizontal mean curvature.

In Euclidean spaces, the motion by mean curvature flow of a manifold of codimension 1 is the geometrical evolution defined by requiring that the normal velocity at each point of the manifold equals the negative of the mean curvature at that point. Only few results are known for mean curvature flow in sub-Riemannian manifolds, i.e. for the evolution obtained by replacing all the geometrical objects by the corresponding horizontal quantities. In these degenerate spaces, such a kind of evolution is very different from the corresponding Euclidean motion, especially because of the existence of characteristic points.
Let us define rigorously this evolution. We give first a notion assuming that the hypersurface is *regular* (i.e. smooth without characteristic points) and then we give a weak formulation.

Definition 3.1. For $t > 0$, let $\Gamma(t)$ be a family of regular hypersurfaces in a sub-Riemannian geometry $(\mathbb{R}^n, \mathcal{H}, \langle \cdot, \cdot \rangle_g)$. We say that $\Gamma(t)$ is an evolution by horizontal mean curvature flow of the hypersurface Γ_0 if $\Gamma(0) = \Gamma_0$ and, for any smooth horizontal curve $x(t) : [0, T] \to \mathbb{R}^n$ such that $x(t) \in \Gamma(t)$ for all $t \in [0, T]$, the velocity in the horizontal normal direction is equal to minus the horizontal mean curvature, that means

$$v_0(x(t)) := \langle \dot{x}(t), n_0(x(t)) \rangle_g = -k_0(x(t)), \tag{11}$$

for any $x(t) \in \Gamma(t)$ and where n_0 and k_0 are defined in (3) and (4).

Nevertheless the previous definition is not sufficient to describe the evolution since, like in the Euclidean case, it is not defined whenever the hypersurface develops singularities (which can happen in the Euclidean case starting from a smooth hypersurface) and it is not defined at the characteristic points, which are a specific feature of the sub-Riemannian mean curvature flow.
We introduce a weak notion of evolution by mean curvature flow, using the level set approach. Such a definition was given first by Chen, Giga and Goto [CGG] and, independently, by Evans and Spruck [ES]. It is based on the idea of defining the evolution of a function $u(t, x)$ by a degenerate parabolic PDE in such a way that each level set $\{x \in \mathbb{R}^n : \ u(t, x) = c\}$ evolves by mean curvature as long as it is a smooth manifold. Exploiting the fact that this PDE is degenerate parabolic, one can define a generalized solution, called viscosity solution.
The associated PDE can be derived for regular hypersurfaces in a way similar to the Euclidean case. Considering a smooth horizontal curve $x(t) \in \Gamma(t) = \{u(t, x) = c\}$ and taking the derivative in time of the expression $u(t, x(t)) = c$, one can get from (11)

$$u_t = \operatorname{Tr}\left((\mathcal{X}^2 u)^*\right) - \left\langle (\mathcal{X}^2 u)^* \frac{\mathcal{X}u}{|\mathcal{X}u|}, \frac{\mathcal{X}u}{|\mathcal{X}u|} \right\rangle = \Delta_0 u - \Delta_{0,\infty} u. \tag{12}$$

We want to point out that equation (12) is parabolic degenerate whenever $\mathcal{X}u = \sigma(x)Du = 0$. We call the points where the horizontal gradient vanishes *singularities*. In the Euclidean case it is known that singularities can lead to the so-called fattening of level sets. We say fattening occurs when the level set has no-empty interior, that means in particular that the gradient vanishes in an open subset, i.e. the co-dimension of the level set is locally zero (see [AAG, Gi], for more information). In the sub-Riemannian geometry, singularities are more difficult to study and they can occur in different situations. In particular characteristic points lead always to singularities for equation (12). Note that the co-dimension in the horizontal tangent space is zero at a characteristic point.

In order to introduce a generalized motion by horizontal mean curvature, we follow the definition introduced by Chen, Giga and Goto in [CGG] for the Euclidean evolution and by Giga in [Gi] for generic degenerate parabolic equations.
Let $A(x,p)$ be defined in (9) and, for sake of semplicity, $\widetilde{S} = \sigma(x)S\sigma^T(x) + A(x,p)$, then equation (12) can be written as

$$u_t + H(x, Du, D^2u) = 0, \tag{13}$$

where

$$H(x,p,S) = -\mathrm{Tr}\big(\widetilde{S}\big) + \left\langle \widetilde{S}\frac{\sigma(x)p}{|\sigma(x)p|}, \frac{\sigma(x)p}{|\sigma(x)p|} \right\rangle.$$

One can easily calculate that the upper and lower semicontinuous envelopes of the function H are, respectively,

$$H^*(x,p,S) = \begin{cases} -\mathrm{Tr}\big(\widetilde{S}\big) + \left\langle \widetilde{S}\frac{\sigma(x)p}{|\sigma(x)p|}, \frac{\sigma(x)p}{|\sigma(x)p|} \right\rangle, & |\sigma(x)p| \neq 0, \\ -\mathrm{Tr}\big(\widetilde{S}\big) + \lambda_{max}\big(\widetilde{S}\big), & |\sigma(x)p| = 0, \end{cases} \tag{14}$$

and

$$H_*(x,p,S) = \begin{cases} -\mathrm{Tr}\big(\widetilde{S}\big) + \left\langle \widetilde{S}\frac{\sigma(x)p}{|\sigma(x)p|}, \frac{\sigma(x)p}{|\sigma(x)p|} \right\rangle, & |\sigma(x)p| \neq 0, \\ -\mathrm{Tr}\big(\widetilde{S}\big) + \lambda_{min}\big(\widetilde{S}\big), & |\sigma(x)p| = 0, \end{cases} \tag{15}$$

where $\lambda_{\max}(\widetilde{S})$ and $\lambda_{\min}(\widetilde{S})$ are the maximal and minimal eigenvalues of $\widetilde{S}$.
Taking $\widetilde{S} = (\mathcal{X}^2u)^*$ and $\sigma(x)p = \mathcal{X}u$, we can give the following definition for the generalized motion by horizontal mean curvature flow.

Definition 3.2. Let $\Gamma_0 = \{x \in \mathbb{R}^n | u_0(x) = 0\}$ be a hypersurface in $\mathbb{R}^n$.
We say that $\Gamma(t) = \{x \in \mathbb{R}^n | u(t,x) = 0\}$ is a *generalized evolution by horizontal mean curvature flow* if u is a continuous function satisfying $u(0,x) = u_0(x)$ and

(1) for any $\varphi \in \mathrm{C}^2(\mathbb{R}^n \times (0,+\infty))$ such that $u - \varphi$ has a local minimum at (t_0, x_0), then

$$\begin{cases} \varphi_t - \Delta_0\varphi + \Delta_{0,\infty}\varphi \geq 0, & \text{at } (t_0,x_0), \text{ if } \mathcal{X}\varphi(t_0,x_0) \neq 0, \\ \varphi_t - \Delta_0\varphi + \lambda_{\max}((\mathcal{X}^2\varphi)^*) \geq 0, & \text{at } (t_0,x_0), \text{ if } \mathcal{X}\varphi(t_0,x_0) = 0. \end{cases} \tag{16}$$

(2) for any $\varphi \in \mathrm{C}^2(\mathbb{R}^n \times (0,+\infty))$ such that $u - \varphi$ has a local maximum at (t_0, x_0), then

$$\begin{cases} \varphi_t - \Delta_0\varphi + \Delta_{0,\infty}\varphi \leq 0, & \text{at } (t_0,x_0), \text{ if } \mathcal{X}\varphi(t_0,x_0) \neq 0, \\ \varphi_t - \Delta_0\varphi + \lambda_{\min}((\mathcal{X}^2\varphi)^*) \leq 0, & \text{at } (t_0,x_0), \text{ if } \mathcal{X}\varphi(t_0,x_0) = 0. \end{cases} \tag{17}$$

If u is locally bounded but not continous, one can give the same definition but requiring the viscosity supersolution condition (17) for the lower semicontinuous envelope of u and the viscosity subsolution condition (16) for the upper semicontinuous envelope of u, which are defined, respectively, as

$$u_*(t,x) := \sup\{v(t,x) | v \text{ cont. and } v \leq u\} = \liminf_{r\to 0^+}\{u(s,y) |\ |y-x| \leq r |t-s| \leq r\},$$

and

$$u^*(t,x) := \inf\{v(t,x) | v \text{ cont. and } v \geq u\} = \limsup_{r\to 0^+}\{u(s,y) |\ |y-x| \leq r, |t-s| \leq r\}.$$

(see [Ba] for more information on discontinuous viscosity solutions).

Definition 3.2 means that u is a viscosity solution of the equation (12). We will prove the existence of such a solution but not the uniqueness. Very little is known about comparison principles (and hence uniqueness) for the viscosity solutions of (12). In [CC], the authors prove some comparison principles in Carnot groups for a special class of initial data (e.g. spheres, tori and any compact surfaces are not covered by such a result). Since (12) is a strong geometric equation, once comparisons are known, one can prove (exaclty as in Theorem 4.2.8, [Gi]) that $\Gamma(t)$, defined in Definition 3.2, does not depend on the chosen parametrization u_0 but just on the level set Γ_0. Nevertheless, at the present, except for the class of initial hypersurfaces covered by [CC], the level set approach does not give a well-posed notion of evolution. Let us also point out that the definition introduced in [CC] looks slightly different from ours, since there the authors used the definition introduced by Evans and Spruck [ES], for the Euclidean case. The two definitions are indeed equivalent in the Euclidean case (see [Gi]), while this equivalence is not yet clear in the sub-Riemannian case.
We would like to remark that the results proved for general nonlinear degenerate parabolic equations in [Gi] (like equivalence of the definitions, comparison principles, existence, etc.) rely on techniques which are not applicable in our case. The main difference between the usual degenerate parabolic equations and the level set equation for the evolution by horizontal mean curvature flow is that equation (12) is discontinuous at the points $(p,x) \in \mathbb{R}^n \times \mathbb{R}^n$ such that $\sigma(x)p = 0$, which is a space-variable-depending set which has *non-zero dimension* in p.

4. Controlled diffusion processes.

Let us first recall some elementary facts from stochastic analysis for continuous semi-martingales which can be found in any standard textbook such as e.g. [KS]. Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}t\}_{t\geq 0}$ let $\xi(t)$ be continuous and adapted (i.e. $\xi(t)$ is $\mathcal{F}_t$-measurable), and let $B(t)$ be a Brownian motion adapted to the filtration. Then the Itô integral $\xi dB(t)$is defined as the following limit (as the step size of the partition decreases) in $L^2(\Omega)$:

$$\int_0^t \xi(s)dB(s) \ \stackrel{L^2}{:=} \lim_{N\to+\infty} \sum_{i=1}^{N} \xi(t_i)\big(B(t_{i+1}) - B(t_i)\big).$$

Note that this holds actually in a far more general setting: The convergence holds in the space of continuous square integrable martingales, the deterministic partition may be replaced by one constructed via stopping times, the integrand ξ need not be continuous, but merely previsible, and the Brownian motion as integrator can

be replaced by any square-integrable continuous (semi-) martingale $\eta(t)$. In latter, more general case, we write $(\xi d\eta)(t)$ for the Itô-integral.
The Stratonovich integral $\xi \circ d\eta$ is defined as

$$\int_0^t \xi(s) \circ d\eta(s) \overset{L^2}{:=} \lim_{N\to+\infty} \sum_{i=1}^{N} \frac{\xi(t_i)+\xi(t_{i+1})}{2} \big(\eta(t_{i+1}) - \eta(t_i)\big),$$

both integrals are related by the formula

$$\xi \circ d\eta = \xi d\eta + \frac{1}{2} d\langle \xi, \eta \rangle,$$

where $\langle \xi, \eta \rangle$ denotes the quadratic covariation of the processes ξ and η which is defined as

$$\int_0^t d\langle \xi, \eta \rangle(s) \overset{L^2}{=} \lim_{N\to+\infty} \sum_{i=1}^{N} \big(\xi(t_{i+1}) - \xi(t_i)\big)\big(\eta(t_{i+1}) - \eta(t_i)\big).$$

The chain rule looks classical if we use the Stratonovich integral. In fact, for any smooth f, the process $f(\xi(t))$ satisfies

$$d\big[f(\xi(t))\big] = f'(\xi(t)) \circ d\xi,$$

which can be re-written as

$$d\big[f(\xi(t))\big] = f'(\xi(t)) d\xi + \frac{1}{2} f''(\xi(t)) d\langle \xi, \xi \rangle.$$

Note that, whenever $\xi = B$ is a Brownian motion, we have $d\langle \xi, \xi \rangle = d\langle B, B \rangle = dt$ and the formula above is the well known Itô formula. This establishes the basic connection between second-order PDE and stochastic processes which yields an extension of the classical method of characteristics to the case of second-order equations.
We would like to point out that we will use the Stratonovich calculus for defining our controlled stochastic processes since, because the chain rule is the classical one, it does not depend on the chosen parametrization and so it is intrinsic in Riemannian and sub-Riemannian geometries (see e.g. [H]). Nevertheless the Itô calculus will be very useful for proofs and computations (see Sec. 5).

4.1. **The stochastic control problem.**

It is well known that viscosity solutions of certain second-order equations are closely related to the value function of stochastic control problems, see e.g. [FS]. The relation between solutions of degenerate equations like in Definition 3.2 and stochastic control problems is more complicated. Nevertheless, Soner and Touzi (in [ST2, ST3]) and, using another approach, Buckdahn, Cardaliaguet and Quincampoix (in [BCQ]) derived a stochastic representation for a set evolving by mean curvature flow (in the Euclidean case).
Our construction of controlled paths yields an analogue to the processes considered for the Euclidean case in [BCQ, ST2, ST3], which could be called *locally codimension one constrained Brownian motion*.
In the Euclidean case, any control $\nu(s)$, taking values in the space of co-rank-one orthogonal projections induces a *locally codimension one constrained Brownian motion* B^ν as solution of the following Itô SDE $dB^\nu = \nu(s) dB$. In the present sub-Riemannian case, in order to define the locally codimension one constrained

or unconstrained Brownian motion, some extra care has to be taken due to the geometry.
We define an "horizontal Brownian motion" as the stochastic process whose generator is the horizontal Laplacian operator. The construction of the associated *unconstrained horizontal Brownian motion* by means of the following Stratonovich SDE, is natural: $d\xi(s) = \sum_{i=1}^m X_i(\xi(s)) \circ dB^i(s)$, where $B = (B^1, \dots, B^m)$ is a standard Brownian motion in $\mathbb{R}^m$.
The use of the Stratonovich instead of the Itô formulation reflects the fact that we do not work in an Euclidean space.
Replacing in the previous Stratonovich SDE the unconstrained Brownian motion B by a locally codimension one constrained Euclidean Brownian motion B^ν in $\mathbb{R}^m$, we get the *locally constrained codimension one horizontal Brownian motion* $\xi^\nu(s)$ associated to Δ_0 and $\nu(s)$, which constitutes a controlled path for our problem.

Now we will make these ideas precise. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space and B is a m-dimensional Browinian motion adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Let S_m the set of all $m \times m$ symmetric matrices, we define the set of admissible controls by

$$\mathcal{V} = \{(\nu(s))_{s\geq 0} \text{ predictable} \mid \nu(s) \in S_m, \nu \geq 0, I_m - \nu^2 \geq 0, \mathrm{Tr}(I_m - \nu^2) = 1\}.$$

Under suitable assumptions, each $\nu(s)$ determines a (unique)control path $\xi^{t,x,\nu(\cdot)}$ as a solution to the SDE

$$\begin{cases} d\xi^{t,x,\nu(\cdot)}(s) = \sqrt{2}\sigma^T(\xi^{t,x,\nu(\cdot)}(s)) \circ dB^\nu(s), & s \in (t,T], \\ dB^\nu(s) = \nu(s) dB(s), & s \in (t,T], \\ \xi^{t,x,\nu(\cdot)}(t) = x, \end{cases} \tag{18}$$

where $\circ dB^\nu$ denotes the integral w.r.t to B^ν in the sense of Stratonovich.
Using the relation $\xi \circ d\eta = \xi d\eta + \frac{1}{2}\langle \xi, \eta \rangle$ between the Stratonovich and the Itô formulation, we get the following equivalent Itô formulation for SDE (18)

$$\begin{cases} d\xi^{t,x,\nu(\cdot)}(s) = \sqrt{2}\sigma^T(\xi^{t,x,\nu(\cdot)}(s))\nu(s)dB(s) \\ \qquad\qquad + \displaystyle\sum_{i,j=1}^m (\nu^2(s))_{ij} \nabla_{X_i} X_j(\xi^{t,x,\nu(\cdot)}(s))ds, \quad s \in (t,T], \\ \xi^{t,x,\nu(\cdot)}(t) = x, \end{cases} \tag{19}$$

where $\nabla_{X_i} X_j$ is the derivative of the vector field X_j in the direction X_i, already introduced. A straightforward application of Itô's formula gives for smooth bounded $u : \mathbb{R}^n \to \mathbb{R}$ that

$$\begin{aligned} du(\xi^{t,x,\nu}(s)) &= \sqrt{2}\sum_{i=1}^m X_i(u)(\xi^{t,x,\nu}(s))\nu(s)dB(s) \\ &\quad + \sum_{i,j=1}^m (\nu^2(s))_{ij} \left[\sum_{k,l=1}^n u_{kl} X_i^k X_j^l + \sum_{k=1}^n u_k \nabla_{X_i} X_j^k \right] \left(\xi^{t,x,\nu}(s)\right) ds, \end{aligned}$$

where we used the notation $X_i = (X_i^1, \dots, X_i^n) \in \mathbb{R}^n$, $u_k = \frac{\partial u}{\partial x_k}$ and $u_{kl} = \frac{\partial^2 u}{\partial x_k \partial x_l}$, so that the previous identity can be written as

$$du(\xi^{t,x,\nu}(s)) = \sqrt{2} \sum_{i=1}^{m} X_i(u)(\xi^{t,x,\nu}(s))\nu(s)dB(s) + tr\left[(\nu(s))^2(\mathcal{X}^2 u)\right](\xi^{t,x,\nu}(s))ds.$$

From now on, we assume that the matrix $\sigma(x)$ as well as the drift

$$\mu(x) := \sum_{i,j=1}^{m} \nabla_{X_i} X_j(x)$$

are Lipschitz in x. Under the Lipschitz condition, classical results for stochastic ODEs give that for any fixed control ν, (18) has a unique strong solution (see e.g. [YZ], Chapter 1, Corollary 6.1). Recall that the main difference between the notions of strong and weak solutions is that, in the first case, the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ and the Brownian motion B are fixed while a weak solutions mean that there exists a process on *some* filtered probability space equipped with an adapted Brownian motion which satisfies the equation, for more details see Definitions 6.2 and 6.5, [YZ]. This difference becomes very important for the stochastic control problem, i.e. when considering

$$\inf_{\nu} \mathbb{E}[f\big(\xi^{t,x,\nu}(T)\big)], \tag{20}$$

where usually f is a suitably regular teminal cost function and $\xi^{t,x,\nu}(\cdot)$ are solutions of a controlled Itô SDE as e.g. (19). It is clear that the properties of the previous minimum problem depend on the set where we take the infimum, and we will need the additional freedom that comes with varying the filtered probability space.

Hence we define the set $\mathcal{A}$ of all the weak-admissible controlled pairs ([YZ], Definition 4.2) which are, roughly speaking, 6-tuple

$$\pi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, B(\cdot), \nu(\cdot))$$

such that B is a Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$, ν is previsible and $(\xi^{t,x,\nu}(\cdot), (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}))$ is a solution of the controlled SDE (18), w.r.t. the control ν and the Brownian motion B in $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$. Under certain structural assumptions for the control set and assuming sufficient regularity of the coefficients, the existence of an optimal control for a large class of problems as in (20) is known, (see for example Theorem 5.3 in [YZ]). For these results it is crucial to use the weak formulation.
Following [ST2, ST3, BCQ], for a given bounded and uniformly continuous function $g : \mathbb{R}^n \to \mathbb{R}$, we define the value function associated to the stochastic control probems (18), as

$$V(t,x) = \inf_{\nu \in \mathcal{A}} \operatorname*{ess\,sup}_{\omega \in \Omega} g(\xi^{t,x,\nu(\cdot)}(T)(\omega)), \tag{21}$$

where the set $\mathcal{A}$ is the set of the weak-admissible controlled pairs, defined above.
From now on we will often omit the dependency on ω.
In the Euclidean case (i.e. $\sigma(x) = Id$) the value function (21) is the solution of the level set equation for the evolution by mean curvature flow (backward) in the viscosity sense (cf. [BCQ], Theorem 1.1).

Our goal is to show that this result is still true in the general sub-Riemannian case, that means $V(t,x)$ is a viscosity solution of

$$\begin{cases} -V_t - \Delta_0 V + \Delta_{\infty,0} V = 0, & x \times \mathbb{R}^n,\ t \in [0,T), \\ V(T,x) = g(x), & x \in \mathbb{R}^n. \end{cases} \tag{22}$$

Remark 4.1 (Non-Lipschitz coefficients)**.** If the coefficients of the matrix $\sigma(x)$ and of the drift part $\mu(x)$ are smooth but not globally Lipschitz, the solutions of the SDE could explode in finite time. There are results on non-explosion for some classes of non-Lipschitz coefficients, but we will not investigate this issue further, but instead assume global in time existence of solutions for the controlled SDEs. In many important examples, e.g. in the Heisenberg group, the Lipschitz condition is satisfied and so the non-explosion follows. In particular, in the latter case the drift part is zero, so the Stratonovich and the Itô formulations coincide.

4.2. **Properties of the value function.**

In this section we study the main properties of the value function (21).

Lemma 4.1 (Comparison Principle)**.** Let g_1, g_2 be bounded and uniformly continuous functions with $g_1 \le g_2$ on $[0,T] \times \mathbb{R}^n$, and let V_i, $i \in \{1,2\}$, be defined as in (21) with g_i as terminal cost. Then

$$V_1(x,t) \le V_2(x,t), \quad \text{on } [0,T] \times \mathbb{R}^n.$$

The proof is obvious and therefore omitted.

Lemma 4.2 (Value function is geometric)**.** Let g be bounded and uniformly continuous, and let V_g be defined as in (21) with g as terminal cost. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous and strictly increasing. Then

$$\varphi(V_g(t,x)) = V_{\varphi(g)}(t,x).$$

Proof. As φ is increasing and continuous, $\varphi(\inf A) = \inf \varphi(A)$ for any bounded set $A \subseteq \mathbb{R}$. Hence, for any measurable function $f : \Omega \to \mathbb{R}$, it is trivial to note:

$$\varphi(\operatorname{ess\,sup} f) = \operatorname{ess\,sup}(\varphi(f)),$$

and so we can easily conclude the proof. □

Remark 4.2. Lemmas 4.1 and 4.2 allow us to conclude, reasoning as in [Gi], that the sublevel set $\{V(t,x) \le 0\}$ depends only on the set $\{g(x) \le 0\}$, and not on the specific form of g. Hence the levels sets of the value function exist and depend only on the level sets of g, so these level sets could be considered as generalized evolution by horizontal mean curvature flow.

Lemma 4.3 (Boundedness)**.** Assume that g is bounded then the value function $V(t,x)$ defined in (21) is bounded.

Proof. The property follows immediately once we know that the infimum is taken over a non-empty set, since we can always consider constant controls. □

In order to investigate the continuity of the value function, we have to restrict our attention to the case of Carnot groups (see [DGN] for the main definitions).

Lemma 4.4 (Continuity in space)**.** *Let $\mathbb{G} = (\mathbb{R}^n, \cdot)$ be a Carnot group, and suppose $g : \mathbb{G} \to \mathbb{R}$ is bounded and uniformly continuous on the one-point-compactification of $\mathbb{G}$, i.e. it is uniformly continuous on $\mathbb{G}$ and there exists $\lim_{|x|\to\infty} g(x)$.*
Then $V(t,x)$ defined in (21) is continuous in space.

Proof. Denote by $L_a(\cdot)$ the left translation in the Carnot group by the element $a \in \mathbb{G}$. . As $\mathbb{G}$ is a Lie group, we may assume that the vector fields $X_1, \dots, X_m$ are left-invariant, i.e.

$$X_i(a \cdot x) = X_i(L_a(x)) = (DL_a)(X_i(x)), \tag{23}$$

for $i = 1, \dots, m$, where DL_a is the derivative of the left translation (see e.g. [W] for more details on Lie groups). Let $\xi^{t,x,\nu(\cdot)}$ be a constrained codimension one horizontal Brownian motion, with $d\xi^{t,x,\nu(\cdot)} = \sqrt{2}\sigma^T(\xi^{t,x,\nu(\cdot)}(s)) \circ dB^\nu(s)$, then, by the chain rule for Stratonovich integrals, it holds

$$\begin{aligned} d\left(L_a\left(\xi^{t,x,\nu(\cdot)}\right)\right) &= (DL_a) \circ \left(\sqrt{2}\sigma^T\left(\xi^{t,x,\nu(\cdot)}(s)\right) \circ dB^\nu(s)\right) \\ &= \sqrt{2}(DL_a)\left(\sigma^T\left(\xi^{t,x,\nu(\cdot)}(s)\right)\right) \circ dB^\nu(s) = \sqrt{2}\sigma^T\left(L_a\left(\xi^{t,x,\nu(\cdot)}(s)\right)\right) \circ dB^\nu(s), \end{aligned} \tag{24}$$

where we have used (23) for the last equality. Hence the left translation of a codimension 1 horizontal Brownian motion yields another one. Now fix a point x, $\epsilon > 0$ and choose a control ν_x such that

$$V(t,x) + \epsilon \geq \operatorname{ess\,sup} g(\xi^{t,x,\nu_x(\cdot)}(T)).$$

Let $a = y \cdot x^{-1}$. By (24), the path $\eta^{t,y,\nu_x(\cdot)}$ starting at the time t in y, is equal to $L_a(\xi^{t,x,\nu_x(\cdot)})$. (Note that the control ν_x is the same for both points x and y.) Therefore

$$\begin{aligned} V(t,y) &\leq \operatorname{ess\,sup} g(\eta^{t,x,\nu_x(\cdot)}(T)) = \operatorname{ess\,sup} g(L_a(\xi^{t,x,\nu_x(\cdot)}(T))) \\ &= \operatorname{ess\,sup}\left(g(\xi^{t,x,\nu_x(\cdot)}(T)) + \left(g(L_a(\xi^{t,x,\nu_x(\cdot)}(T))) - g(\xi^{t,x,\nu_x(\cdot)}(T))\right)\right) \\ &\leq V(t,x) + \epsilon + \operatorname{ess\,sup}\left|g\left(L_a(\xi^{t,x,\nu_x(\cdot)}(T))\right) - g\left(\xi^{t,x,\nu_x(\cdot)}(T)\right)\right|. \end{aligned}$$

Choose a large number $R > 0$ then

$$\begin{aligned} &\operatorname*{ess\,sup}_{\omega\in\Omega}\left|g\left(\xi^{t,x,\nu_x(\cdot)}(T)(\omega)\right) - g\left(L_a(\xi^{t,x,\nu_x(\cdot)}(T)(\omega))\right)\right| \\ &\leq \sup_{\{z\in\mathbb{R}^n:\ |z|<R\}} |g(z) - g(a\cdot z)| + \sup_{\{z\in\mathbb{R}^n:\ |z|\geq R\}} |g(z) - g(a\cdot z)| =: A + B, \end{aligned}$$

where we set $z = \xi^{t,x,\nu_x(\cdot)}(T)(\omega)$ and so $a \cdot z = L_a(\xi^{t,x,\nu_x(\cdot)}(T)(\omega))$.
Note that $|a \cdot x| \to \infty$ if $|x| \to \infty$. Therefore we can use the continuity of g at ∞ to find a sufficiently large R such that $B < \epsilon$. As, by continuity of the group operation, $|a \cdot x - x| \to 0$ (uniformly on compact sets) as $|a| = |y \cdot x^{-1}| \to 0$, we can use the uniform continuity of g to find $\delta > 0$ such that $V(t,y) \leq V(t,x) + 3\epsilon$ for $|x - y| < \delta$. Reversing the role of x and y yields the continuity. □

5. Existence of a generalized evolution by horizontal mean curvature flow in sub-Riemannian manifolds.

Using the value function for the stochastic control problem introduced in the previous section as representation for the viscosity solution of equation (12), we get an existence result for the generalized evolution by horizontal mean curvature flow as given in Definition 3.2. By classical results (see e.g. [FS], [T]), it is known how to find the equation solved by value functions of the form $\inf_{\nu\in\mathcal{A}} \mathbb{E}\left[g(\xi^{t,x,\nu(\cdot)}(T)\right]$. Unfortunately, the value function $V(t,x)$ defined in (21) looks different, because of the essential supremum instead of the expectation. Hence the idea (already used in [BCQ]) is to approximate formula (21) with functions that look like the infimum of an expectation and then to pass to the limit, essentially using the fact that the L^q-norm of a fixed nonnegative function converges to the essential supremum as $q \to \infty$. The main result of this paper is the following existence theorem.

Theorem 5.1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and Hölder continuous, $T > 0$ and $\sigma(x) = [X_1(x), \dots, X_m(x)]^T$ a $m \times n$-Hörmander matrix with $m \leq n$ and smooth coefficients. Assume that $\sigma(x)$ and $\mu(x) = \sum_{i=1}^m \nabla_{X_i} X_j(x)$ are Lipschitz, the value function $V(t,x)$ defined by (21) is a bounded lower semicontinuous viscosity solution of the level set equation for the evolution by horizontal mean curvature flow (22), with terminal condition $V(T,x) = g(x)$.

Remark 5.1. The Lipschitz assumptions on $\sigma(x)$ and $\mu(x)$ is in order to have non-explosion for the solution of the SDE. These are satisfied in many sub-Riemannian geometries (e.g. the Heisenberg group, the Grušin plane and the roto-translation geometry).

As in [BCQ], let us introduce the following regularization

$$V_q(t,x) = \inf_{\nu \in \mathcal{A}} \Big(\mathbb{E}\big[g^q(\xi^{t,x,\nu(\cdot)}(T)) \big] \Big)^{1/q}, \quad \text{for any } 1 < q < +\infty, \tag{25}$$

where $\mathcal{A}$ is the set of all the admissible control defined in Section 4.1.
The idea is to derive the PDE solved by the value functions (25) and then to show that V is their limit as $q \to +\infty$ and solves a limit-equation (22). In fact, we have

Lemma 5.1. Under the assumptions of Theorem 5.1, we have

$$V(t,x) = \lim_{q \to +\infty} V_q(t,x) \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^N \ (\textit{pointwise convergence}), \tag{26}$$

with $V(t,x)$ as in (21).

Proof. As the L^q norms are increasing and bounded by the essential supremum, for each fixed control, it is clear that $V(t,x) \geq \lim V_q(t,x)$.
In order to show equality, we can argue as in [BCQ]. For any $q \geq 1$ we find a control ν_q such that

$$\Big(\mathbb{E}\big[g^q(\xi^{t,x,\nu_q(\cdot)}(T)) \big] \Big)^{1/q} \leq V_q(t,x) + q^{-1}.$$

The controlled SDE (19) has a drift part, which depends on the control only through ν^2 and our control set is, as the one in [BCQ], convex in ν^2. So standard arguments (see e.g. [YZ] Theorem 5.3), yield the existence of a $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, B(\cdot), \nu(\cdot))$ such that for a subsequence q_k the processes $\xi^{t,x,\nu_{q_k}(\cdot)}$ converge weakly to $\xi^{t,x,\nu(\cdot)}$ and so for any fixed $\bar{q} \geq 1$

$$\lim_{k \to \infty} \Big(\mathbb{E}\big[g^{\bar{q}}(\xi^{t,x,\nu_{q_k}(\cdot)}(T)) \big] \Big)^{1/\bar{q}} = \Big(\mathbb{E}\big[g^{\bar{q}}(\xi^{t,x,\nu(\cdot)}(T)) \big] \Big)^{1/\bar{q}}.$$

Since the L^q-norm is non-decreasing in q,

$$\Big(\mathbb{E}\big[g^{\bar{q}}(\xi^{t,x,\nu(\cdot)}(T)) \big] \Big)^{1/\bar{q}} \leq \lim_{q \to \infty} V_q(t,x).$$

Now $V(x,t) \leq \lim_{q \to \infty} V_q(x,t)$ follows from the convergence of the L^q-norms to the L^∞-norm and the definition of V as infimum.

□

Before giving a rigorous proof of Theorem 5.1, let us show a proof which works only under stonger regularity assumptions, which are in general not satisfied. Nevertheless explains how to derive the limit equation solved by V much clearer than the technical viscosity proof, and it makes clear how optimal controls (if they exist) should look like.

Heuristic proof of Theorem 5.1. We first look at

$$U_q(t,x)=V_q^q(t,x)=\inf_{v\in\mathcal{A}}\mathbb{E}\big[g^q(\xi^{t,v,x}(T))\big]. \tag{27}$$

It is known (see e.g. [FS], [T]) that $U_q(t,x)$ is a viscosity solution of

$$\begin{cases} -(U_q)_t+H(x,DU_q,D^2U_q)=0, & x\in\mathbb{R}^n,\ t\in[0,T),\\ \qquad U_q(T,x)=g^q(x), & x\in\mathbb{R}^n, \end{cases} \tag{28}$$

where H, for any $x,p\in\mathbb{R}^n$ and any symmetric $n\times n$ matrix S, is given by

$$H(x,p,S)=\sup_{\nu\in\mathcal{A}}\left[-\mathrm{Tr}\big(\sigma(x)S\sigma^T(x)\nu^2(s)\big)+\sum_{i,j=1}^m(\nu^2(s))_{i,j}\big\langle\nabla_{X_i}X_j(x),p\big\rangle\right]. \tag{29}$$

Assuming g locally Lipschitz (or locally Hölder), it is possible to show that U_q is a continuous viscosity solution of (28) (see [T]).
Note that we can replace g by $ag+b$ for real numbers a and b and therefore, as g is bounded, assume $C\geq g(x)\geq 1$, (for any $C>0$) and so $C\geq V_q(t,x)\geq 1>0$. Hence, we can divide by qV_q^{q-1}. Assuming that all functions involved are smooth, a trivial calculation tells that V_q solves

$$\begin{cases} -(V_q)_t+H(x,DV_q,(q-1)V_q^{-1}DV_q(DV_q)^T+D^2V_q)=0, & x\in\mathbb{R}^n,\ t\in[0,T),\\ \qquad V_q(T,x)=g(x), & x\in\mathbb{R}^n. \end{cases} \tag{30}$$

Whenever V_q is just continuous, one can show that V_q solves equation (30) in the viscosity sense, by applying the previous calculation to the (smooth) test functions. Moreover, the continuity for V_q follows from the continuity for U_q.
Let now assume that V_q and V are C^2 and

$$\big(V_q\big)_t\to V_t,\quad DV_q\to DV,\quad D^2V_q\to D^2V\quad \text{as } q\to+\infty.$$

Let us first look at the case $\mathcal{X}V(t,x)=\sigma(x)DV(t,x)\neq 0$ (which implies $\mathcal{X}V_q(t,x)\neq 0$, at least, for large q). We can rewrite explicitely the Hamiltonian in (30) as

$$\begin{aligned} &H(x,DV_q,(q-1)V_q^{-1}DV_q(DV_q)^T+D^2V_q)=\\ &\sup_{\nu\in\mathcal{A}}\Big[-(q-1)\mathrm{Tr}\big[V_q^{-1}\nu\nu^T(\sigma(x)DV_q)(\sigma(x)DV_q)^T\big]+\mathrm{Tr}\big[\nu\nu^T\sigma(x)D^2V_q\sigma^T(x)\big]\\ &\qquad\qquad+\sum_{i,j=1}^m(\nu^2(s))_{i,j}\big\langle\nabla_{X_i}X_j(x),DV_q\big\rangle\Big] \end{aligned} \tag{31}$$

Recalling that $(\mathcal{X}^2V_q)^*=\sigma(x)D^2V_q\sigma^T(x)+A(x,DV_q)$ where $A(x,p)$ is defined by (9), we observe that $\mathrm{Tr}(\mathcal{X}^2V_q)^*=\mathrm{Tr}\big(\sigma(x)D^2V_q\sigma^T(x)\big)+\mathrm{Tr}\big(A(x,DV_q)\big)$, with

$\mathrm{Tr}A(x, DV_q) = \sum_{i=1}^{m} \langle \nabla_{X_i} X_i(x), DV_q \rangle$. Then (31) can be also written as

$$H(x, DV_q, (q-1)V_q^{-1}DV_q(DV_q)^T + D^2V_q) = \sup_{\nu \in \mathcal{A}} \Big[-(q-1)V_q^{-1}\mathrm{Tr}\big[\nu\nu^T(\mathcal{X}V_q)(\mathcal{X}V_q)^T\big] + \mathrm{Tr}\big[\nu\nu^T(\mathcal{X}^2V_q)^*\big]\Big] \tag{32}$$

Note that

$$-(q-1)V_q^{-1}\mathrm{Tr}\big[\nu\nu^T(\mathcal{X}V_q)(\mathcal{X}V_q)^T\big] = -(q-1)V_q^{-1}\mathrm{Tr}\big[\big(\nu^T\mathcal{X}V_q\big)\big(\nu^T\mathcal{X}V_q\big)^T\big] \leq 0,$$

and so it goes to $-\infty$ as $q \to +\infty$. Hence, in order to attain the supremum, we need (at least for large q) that $\nu^T\mathcal{X}V_q = 0$. Since the horizontal gradient is in the direction of the horizontal normal, the optimal control $\overline{\nu}$ has to coincide with the projection on the tangent space, that means $\overline{\nu} = I_m - n_0 \otimes n_0$.
To get the level set equation, we have to write Hamiltonian (31) in a bit different way. Let I_m be the $m \times m$ identity-matrix, we can replace ν^2 by $I_m - a \otimes a$ with $a \in \mathbb{R}^m$, then, for any $n \times n$ matrix S, it holds

$$\sup_{\nu \in \mathcal{A}} \big[-\mathrm{Tr}\big(\nu^2 S\big)\big] = \sup_{\nu \in \mathcal{A}} \big[-\mathrm{Tr}\big((I_m - a \otimes a)S\big)\big] = -\mathrm{Tr}[S] + \max_{|a|=1} \langle Sa, a\rangle.$$

Using the optimal control $\overline{\nu}$ and recalling that $n_0 = \mathcal{X}V/|\mathcal{X}V|$ and $S = (\mathcal{X}^2V)^*$, we can conclude that the limit Hamiltonian, as $q \to \infty$, is

$$H(x, DV, D^2V) = -\mathrm{Tr}\big[(\mathcal{X}^2V)^*\big] + \Big\langle (\mathcal{X}^2V)^*\frac{\mathcal{X}V}{|\mathcal{X}V|}, \frac{\mathcal{X}V}{|\mathcal{X}V|}\Big\rangle = -\Delta_0 V + \Delta_{0,\infty}V.$$

So the limit equation of (30), as $q \to +\infty$, is exactly (22).
It remains to consider the case $\mathcal{X}V(t,x) = 0$. In this case, passing to the limit in (32), the first-order disappears whatever the control ν looks like. Then, for any control $\nu = I_m - a \otimes a$ and $S = (\mathcal{X}^2V)^*$,

$$\begin{aligned} 0 = -V_t + \sup_{\nu \in \mathcal{A}} \Big[-\mathrm{Tr}\big[\nu\nu^T(\mathcal{X}^2V)^*\big]\Big] &= -V_t - \mathrm{Tr}(\mathcal{X}^2V)^* + \max_{|a|=1}\mathrm{Tr}\big(aa^T(\mathcal{X}^2V)^*\big) \\ &= -V_t - \Delta_0 V + \lambda_{\max}(\mathcal{X}^2V)^*. \end{aligned}$$

So we find, as expected, the upper semicontinous regularization of the equation.

□

Note that, in general $V(t,x)$ is bounded since the datum g is, and, since V_q is a non-decreasing sequence of continuous functions, then

$$V(t,x) = \lim_{q \to +\infty} V_q(t,x) = \sup_{q>1} V_q(t,x), \tag{33}$$

Hence $V(t,x)$ is, a priori, just lower semicontinuous. Therefore we need to consider the upper and lower envelopes of V.

Proof of Theorem 5.1. We have to show that $V(T-t,x)$ satisfies Definition 3.2.
First we recall that, since $V(t,x)$ is lower semicontinuous, $V_*(t,x) = V(t,x)$.
Let us introduce the half-relaxed upper-limit of $V_q(t,x)$ which are defined in (25).

$$V^\sharp(t,x) := \limsup_{\substack{(s,y) \to (t,x) \\ q \to +\infty}} V_q(s,y).$$

Note that $V^\sharp \geq V$ and $V^\sharp$ is upper semicontinuous. Since the upper semicontinuous envelope V^* is the smallest upper semicontinuous function above V, we have

$V^*(t,x) \le V^\sharp(t,x)$. Moreover since $V_q(t,x)$ is non-decreasing, $V_q(t,x) \le V(t,x)$, for any x, t and $q>1$. Taking the limsup in x,t,q, we get also the reverse inequality $V^\sharp(t,x) \le V^*(t,x)$, hence $V^* = V^\sharp$.
Therefore, to verify Definition 3.2, we have to show that $V(t,x)$ is a viscosity supersolution of (22)while $V^\sharp(t,x)$ is a viscosity subsolution.
First we show that $V(t,x)$ is a supersolution. Let $\varphi \in C^1([0,T];C^2(\mathbb{R}^n))$ be such that $V-\varphi$ has a local minimum at (t,x).
Two different cases occur. If $\sigma(x)D\varphi(t,x) \neq 0$, we have to verify that

$$-\varphi_t(t,x) - \Delta_0\varphi(t,x) + \Delta_{0,\infty}\varphi(t,x) \ge 0, \tag{34}$$

while, if $\sigma(x)D\varphi(t,x)=0$, we need to check

$$-\varphi_t(t,x) - \Delta_0\varphi(t,x) + \lambda_{\max}\big((\mathcal{X}^2\varphi)^*(t,x)\big) \ge 0. \tag{35}$$

First note that, for any $q>1$, there exists (t_q,x_q) such that $V_q-\varphi$ has a local minimum at (t_q,x_q) and $(t_q,x_q)\to(t,x)$.
In fact, we can always assume that (t,x) is a strict minimum in some $B_R(t,x)$. Set $K=\overline{B_{\frac{R}{2}}(t,x)}$, the sequence of minimum points (t_q,x_q) converge to some $(\overline{t},\overline{x})\in K$. As V is the limit of the V_q and lower semicontinuous, therefore a standard argument yields that $(\overline{x},\overline{t})$ is a minimum, hence it equals (x,t).
Since V_q is a solution of (30), we have for H as in (29),

$$-\varphi_t(t_q,x_q) + H(x_q,(q-1)V_q^{-1}D\varphi(D\varphi)^T + D^2\varphi)(t_q,x_q) \ge 0.$$

Now consider the case $\sigma(x)D\varphi(t,x)\neq 0$. Let us write H more explicitly. Set

$$\begin{aligned} S_1 &= (q-1)V_q^{-1}\mathcal{X}\varphi(t_q,x_q)(\mathcal{X}\varphi(t_q,x_q))^T,\\ S_2 &= (\mathcal{X}^2\varphi)^*(t_q,x_q), \end{aligned}$$

Since the trace operator is linear and $\mathrm{Tr}((\mathcal{X}\varphi(x_q))(\mathcal{X}\varphi)^T(x_q)) = |\mathcal{X}\varphi(x_q)|^2$,

$$\begin{aligned} H(x_q,S_1,S_2) &= -\mathrm{Tr}(S_1+S_2)+\lambda_{\max}(S_1+S_2) = -\mathrm{Tr}(S_1)-\mathrm{Tr}(S_2)+\lambda_{\max}(S_1+S_2)\\ &= -(q-1)V_q^{-1}(t_q,x_q)|\mathcal{X}\varphi(t_q,x_q)|^2 - \Delta_0\varphi(t_q,x_q) + \lambda_{\max}(S_1+S_2). \end{aligned} \tag{36}$$

Now we need to apply the following result to the matrix $S_q = \big(V_q^{-1}\mathcal{X}\varphi(\mathcal{X}\varphi)^T\big)(t_q,x_q)$.

Lemma 5.2 ([BCQ], Lemma 1.2)**.** Let S be a symmetric $m\times m$-matrix such that the space of the eigenvectors associated to the maximum eigevalue is of dimension one. Then, $S\to\lambda_{\max}(S)$ is C^1 in a neighborhood of S. Moreover, $D\lambda_{\max}(S)(H) = \big\langle Ha,a\big\rangle$, for any $a\in\mathbb{R}^m$ eigenvector associated to $\lambda_{\max}(S)$ and $|a|=1$.

Expanding the Hamiltonian (36) around S_q and then, passing to the limit as $q\to+\infty$, we get exactly (34).

For the remaining case, $\sigma(x)D\varphi(t,x)=0$, we use the subadditivity of the function $S\to\lambda_{\max}(S)$ and remark that, since V_q is supersolution

$$\begin{aligned} 0 &\le -\varphi_t + H(x_q, DV_q, (q-1)V_q^{-1}D\varphi(D\varphi)^T + D^2\varphi)\\ &\le -\varphi_t - (q-1)V_q^{-1}|\mathcal{X}\varphi|^2 - \mathrm{Tr}\big((\mathcal{X}^2\varphi)^*\big) + \lambda_{\max}\big((q-1)V_q^{-1}\mathcal{X}\varphi(\mathcal{X}\varphi)^T + (\mathcal{X}^2\varphi)^*\big)\\ &\le -\varphi_t - (q-1)V_q^{-1}|\mathcal{X}\varphi|^2 - \Delta_0\varphi + (q-1)V_q^{-1}|\mathcal{X}\varphi|^2 + \lambda_{\max}((\mathcal{X}^2\varphi)^*), \end{aligned}$$

at the point (t_q,x_q). So, passing to the limit as $q\to+\infty$, we find (35).
To verify the subsolution property for $V^*=V^\sharp$, let $\varphi\in C^1([0,T];C^2(\mathbb{R}^n))$ be such

that $V^\sharp - \varphi$ has a maximum at (t,x) and we may assume that such a maximum is strict. Let (t_q, x_q) be a sequence of maximum points of $V_q - \varphi$, then we can again find a subsequence converging to (t,x). Hence, since V_q are solutions of (30), we have at (t_q, x_q)

$$0 = -\varphi_t + H(x, (q-1)\varphi^{-1} D\varphi (D\varphi)^T + D^2\varphi). \tag{37}$$

We define, for any $z > 0$, $x, d \in \mathbb{R}^n$ and any $n \times n$ symmetric matrix S

$$\begin{aligned} H_q(x,z,p,S) = &-\frac{(q-1)}{z} |\sigma(x)p + A(x,p)|^2 - \mathrm{Tr}(\sigma^T(x) S \sigma(x) + A(x,p)) \\ &+ \lambda_{\max}\bigg(\frac{(q-1)}{z} (\sigma(x)p)(\sigma(x)p)^T + \sigma^T(x) S \sigma(x) + A(x,p) \bigg). \end{aligned} \tag{38}$$

It is clear that for H^* and H_* as in (14) and (15) respectively,

$$H^*(x,p,S) \geq H_*(x,p,S).$$

Moreover we can observe that $H_q(x,z,p,S) \geq H^*(x,p,S)$, for any z. This is trivial for $|\sigma(x)p| = 0$ (by (38)) while for $|\sigma(x)p| \neq 0$, it follows by taking $a = \frac{\sigma(x)p}{|\sigma(x)p|}$ in the variational characterization of the maximum eigenvalue as

$$\lambda_{\max}(\widetilde{S}) = \max_{|a|=1} \langle \widetilde{S}a, a \rangle.$$

Set $z = \varphi^{-1}(t_q, x_q) > 0$, $p = D\varphi(t_q, x_q)$, $S = D^2\varphi(t_q, x_q)$, then by taking the limsup as $q \to +\infty$ in (37), we can deduce that

$$0 \geq -\varphi_t + H_*(x, D\varphi, D^2\varphi)$$

at (t,x). □

In Lemma 4.4, we have shown that the function (21) is continuous in space (in Carnot groups). We would like to be able to get the continuity in both time and space in the general sub-Riemannian setting, in order to conclude the existence of a continuous evolution by horizontal mean curvature flow.

Unfortunately, the strategy introduced in [BCQ] needs comparison principles for viscosity subsolutions and supersolutions for the PDE, which, as we have already remarked, are known just in the particular setting covered in [CC]. However, whenever comparisons hold, we can generalize the strategy used in [BCQ] and obtain a much stronger result. But first let us show the following Lemma, which does not rely on comparison arguments.

Lemma 5.3. For any $x \in \mathbb{R}^n$, $V^\sharp(T,x) \leq g(x)$.

Proof. Assuming that it is not true, there exists a point x_0 such that $V^\sharp(T,x_0) \geq g(x_0) + \varepsilon$, for $\varepsilon > 0$ sufficiently small. Then we use as test function

$$\varphi(t,x) = \alpha(T-t) + \beta|x - x_0|^2 + g(x_0) + \frac{\varepsilon}{2}$$

with $\alpha > -C\beta$, with C a constant depending just on the data of the problem and the point x_0 (in the Euclidean case $C = -2(n-1)$) and $\beta > 1$ sufficiently large. Now we can find a sequence $(t_k, x_k) \to (T, x_0)$ and $q_k \to +\infty$ as $k \to +\infty$ such that $V_{q_k} - \varphi$ has a positive local maximum at some point (s_k, y_k), for any $k > 1$ (see [BCQ] for more details). To get the contradiction we will use the fact that V_q is a solution (so in particular a subsolution) of equation (30) in order to estimate $\alpha + C\beta \leq 0$, which contradicts the choice $\alpha > -C\beta$.

Unfortunately here, unlike in the Euclidean case, the test function, inserted in the equation for V_q, does not give a constant number since the Hamiltonian depends on the space-variable.
Nevertheless, we can observe that the functions V_q are bounded uniformly in p so, by the growth of $|x-x_0|$, the maximum points are such that $y_k \in \overline{B_R(x_0)} =: K$, with R independent of k.
First we remark that

$$\varphi_t(t,x) = -\alpha, \quad D\varphi(t,x) = 2\beta(x-x_0), \quad D^2\varphi(t,x) = 2\beta Id.$$

Remarking that at the point (s_k, y_k), we have

$$0 \geq \alpha - H(y_k, (q-1)\varphi^{-1}D\varphi(D\varphi)^T + D^2\varphi) \geq \alpha - 2\beta \mathrm{Tr}\big(\sigma(y_k)\sigma^T(y_x) + A(y_k, y_k - x_0)\big) \\ + 2\beta\lambda_{\min}\big(\sigma(y_k)\sigma^T(y_k) + (A(y_k, y_k - x_0)\big).$$

Recalling that there is a compact set K such that $y_k \in K$ for all k, by continuity, we get $0 \geq \alpha + 2C\beta$, with

$$C = -\max_{x\in K}\mathrm{Tr}(\sigma(x)\sigma^T(x)) - \max_{x\in K} A(x, x-x_0) + \min_{x\in K}\lambda_{\min}(\sigma(x)\sigma^T(x)) + \min_{x\in K}\lambda_{\min}(A(x, x-x_0)).$$

With such an estimate, we are able to obtain the same contradiction as in the Euclidean case, choosing $\alpha > -C\beta$. □

Corollary 5.1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be bounded and Hölder continuous, $T > 0$ and $\sigma(x)$ an $m \times n$-Hörmander matrix like in Theorem 5.1. If comparison principles for (22) hold, then the value function $V(t,x)$ defined by (21) is the unique continuous viscosity solution of the level set equation (22), satisfying $V(T,x) = g(x)$.

Proof. In the viscosity proof of 5.1 we have shown that $V^*(t,x) = V^\sharp(t,x)$ is a viscosity subsolution while $V_*(t,x) = V(t,x)$ is a viscosity supersolution of equation (22). Since, by Lemma 5.3, $V^\sharp(T,x) \leq g(x)$ while $g(x) = V(T,x)$, comparison principles imply $V^\sharp(t,x) \leq V(t,x)$. Moreover $V^\sharp(t,x) \geq V(t,x)$ by definition. Hence $V^\sharp(t,x) = V(t,x)$, which means $V(t,x)$ is upper semicontinuous. Since $V(t,x)$ is already lower semicontinuous as supremum of continuous functions, we conclude that $V(t,x)$ is continuous. □

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