A Counterexample for Hencky Plasticity in the Case of a Thin Plate under Vertical Load

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Abstract

We show that the variational problem for the bending of an elastic perfetcly plastic plate under vertical load does not always have a solution.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ the middle surface of a thin flat and plastic perfectly plastic plate in its undisturbed state and let $f : \Omega \mapsto \mathbb{R}$ be an outer force acting onto it in perpependicular direction. Let $v_0 : \Omega \mapsto \mathbb{R}$ describe the streching of the plate at the boundary. In the equilibrum state the vertical displacement (bending) $w : \Omega \mapsto \mathbb{R}$ of the surface and the corresponding moments inside the plate $M : \Omega \mapsto \mathbb{M}$ (with $\mathbb{M} = \mathbb{R}^{2\times 2}_{sym}$ we denote the space of symmetric 2×2-matrices, equipped with the scalar product $\tau : \chi = \tau_{ij}\chi_{ij}$) satisfy the equations

1) $\operatorname{div}\operatorname{div} M = f$

2) $\begin{array}{l} \mathcal{F}(M) \leq 0 \quad \nabla^2 w = B\,M + \Lambda \\ \Lambda : (m - M) \leq 0 \quad \forall\, \mathcal{F}(m) \leq 0 \end{array}$

3) $\begin{array}{l} w = v_0 \\ \nabla w = \nabla v_0 \end{array}$ on $\partial\Omega$

where $\Lambda : \Omega \mapsto \mathbb{M}$ can be considered as the plastic part of the curvature and $B : \mathbb{M} \mapsto \mathbb{M}$ is a linear, symmetric and positively defined mapping. The mapping

$$\mathcal{F} : \mathbb{M} \mapsto \mathbb{R} \text{ convex}$$

is the given *yield function* of the material. We consider the easiest case when

$$\mathcal{F}(\tau) = \sqrt{\tau_{ij}\tau_{ij}} - 1$$

which is known as Hencky's constituent law. We also set $B = \mathrm{Id}$.
By means of duality we can construct variational problems that allow

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to seek w and M indenpendently (see [2]). According to the terminology in [3] we call these problems $\mathcal{P}$ and $\mathcal{P}^*$ respectively. The problem $\mathcal{P}$ for w has the form

$$I(v) \to \min\{I(v) \,|\, v \in v_0 + W_0^{2,1}(\Omega)\} = \inf \mathcal{P}$$

$$I(v) := G(\nabla^2 v) - (f, v)$$
$$v_0 \in W^{2,1}(\Omega)$$

with $G : L^1(\Omega, \mathbb{M}) \mapsto \mathbb{R}$

$$\begin{aligned} G(n) &= \sup_{\substack{m:\Omega \mapsto \mathbb{M},\\ \mathcal{F}(m) \le 0 \text{ a.e.}}} \{(n, m) - \frac{1}{2}\int_\Omega m_{ij} m_{ij}\, dx\} \\ &= \int_\Omega g_0(|n|)\, dx \quad \text{ with } g_0(t) = \begin{cases} \frac{1}{2}t^2 & \text{for } t < 1 \\ t - \frac{1}{2} & \text{for } t \ge 1 \end{cases} \end{aligned}$$

(where $(\ ,\ )$ we denotes the duality product in $L^p(\Omega, X) \times L^{p'}(\Omega, X)$). Thus I is a functional with asymptotically linear growth. This feature gives rise to the special difficulities of the problem because it forces us to seek a solution in the nonreflexive space $W^{2,1}(\Omega)$ where the classic direct conclusion

coercivity[1] + weak lower semicontinuity of $I \Rightarrow$ existence of a minimizing element

fails by the lack of weak compactness for bounded sequences. This means that we cannot expect I to attain its minimum and in this note we give an example where in fact $\mathcal{P}$ has no solution.

1.1 Repetition of Duality Theory

Therefore we make use of duality theory ([3]) which provides a strong tool for characterising the weak solutions of $\mathcal{P}$ and also for analyzing its regularity properties ([5, 6]). The process of relaxing $\mathcal{P}$ consists of two steps. First we construct a Lagrangian $l : W^{2,1}(\Omega) \times L^\infty(\Omega, \mathbb{M}) \to \mathbb{R}$

$$l(v, m) \quad = \quad < m, \nabla^2 v > -(f, v) - \int g^*(m)\, dx$$

where

$$g_0^*(s) = \begin{cases} \frac{1}{2}s^2 & \text{for } s \le 1 \\ +\infty & \text{for } s > 1 \end{cases}$$

which gives a representation for $\mathcal{P}$ and $\mathcal{P}^*$ as inf-sup and sup-inf problems for l respectively.
Following the scheme in [4] the second step is to perform integration by parts in l which allows to extend l in v (under simultanious restriction in m). We obtain the extended lagrangian $L : (v_0 + W_0^{1,2}(\Omega)) \times (K \cap D) \to \mathbb{R}$

$$L(v, m) =< \operatorname{div}\operatorname{div} m - f, v - v_0 > - \int_\Omega g^*(m) dx + (m, \nabla^2 v_0) - (f, v_0)$$

with

[1] The coercivity of I can, for instance, be guaranteed by the *Save-Load-Condition*: $\exists M^1 \in L^\infty(\Omega, \mathbb{M}) :$ ($\operatorname{div}\operatorname{div} M^1 = f$ $\exists \delta_1 > 0 : |M^1| \le 1 - \delta_1$ a.e. in Ω)

$$K := \{m \in L^\infty(\Omega, \mathbb{M}) : |m(x)| \leq 1 \text{ a.e. in } \Omega\}$$
$$D := \{m \in L^\infty(\Omega, \mathbb{M}) : \ \operatorname{div}\operatorname{div} m \in \left(W_0^{1,2}(\Omega)\right)^*\}.$$

($<\ ,\ >$ denotes the duality product in $X \times X^*$)
The corresponding inf-sup problem $\mathcal{P}^{rel}$

$$I^{rel}(v) \to \min\{I^{rel}(v) \,|\, v \in v_0 + W_0^{1,2}(\Omega)\} = \inf \mathcal{P}^{rel}$$
$$I^{rel}(v) = \sup_{m \in K \cap D} L(v, m)$$

is the relaxed problem for $\mathcal{P}$ and we find that

(v, M) is a *saddle point* of L, i.e.

$$L(v, n) \leq L(v, M) \leq L(u, M) \tag{1}$$
$$\forall (u, n) \in (v_0 + W_0^{1,2}(\Omega)) \times (K \cap D)$$

$$\Leftrightarrow \begin{cases} v \text{ is a solution to } \mathcal{P}^{rel} \\ M \text{ is a solution to } \mathcal{P}^* \end{cases}$$

Finally we can state the following existence result which contains a necessary condition of weak solutions:

Theorem (Existence and Neccesary Condition of Weak Solutions)
Suppose that $f \in (W_0^{1,2}(\Omega))^*$, $-\infty < (f, v_0) < +\infty$ and let the save-load condition hold. Then every minimizing sequence of I in $v_0 + W_0^{2,1}(\Omega)$ contains a subsequence that converges weakly in $W^{1,2}(\Omega)$ and for any $1 \leq p < 2$ in $W^{1,p}(\Omega)$ strongly to a solution $w \in v_0 + W_0^{1,2}(\Omega)$ of $\mathcal{P}^{rel}$.
Moreover we have $\inf \mathcal{P} = \sup \mathcal{P}^* = \inf \mathcal{P}^{rel}$ and if w is any weak solution of $\mathcal{P}$ then (w, M) is a saddle point of the extended lagrangian L in $(v_0 + W_0^{1,2}(\Omega)) \times (K \cap D)$ where M is the solution of $\mathcal{P}^*$.

2 A Counterexample

We now give a concrete setting of the problem which does not posses a solution in $W^{2,1}(\Omega)$. We chose

$$\Omega = B(0, r_1) \setminus B(0, r_0) \text{ and } f \equiv 0 \text{ in } \Omega$$

where $0 < r_0 < r_1 < 1$ are to be fixed later. We define $w_0 \in C^\infty(\overline{\Omega})$

$$w_0(x) = m_0(r) = \tfrac{1}{4(\log r_1 + 1)}\left(r^2 \log r - r_1^2(2\log r_1 + 1)\log r\right) \quad r = |x|$$

As for any $u(x) = m(r)$ with $r = |x|$ trivially

$$\Delta u(x) = \tfrac{1}{r}\tfrac{d}{dr} r \tfrac{d}{dr} m(r)$$
$$(\nabla^2 u(x))_{ij} = \left(\tfrac{m''(r)}{r^2} - \tfrac{m'(r)}{r^3}\right) x_i x_j + \tfrac{m'(r)}{r}\delta_{ij}$$
$$|\nabla^2 u(x)|^2 = (m''(r))^2 + \left(\tfrac{m'(r)}{r}\right)^2$$

we see that

$$\Delta^2 w_0 = 0 \qquad \text{in } \Omega$$

$$\left.\begin{array}{r} |\nabla^2 w_0(x)| = 1 \\ \nabla w_0(x) = 0 \\ (\nabla^2 w_0(x))_{ij} = \tfrac{m_0''(r_1)}{r_1^2} x_i x_j \end{array}\right. \qquad \text{on } \partial B(0, r_1)$$

In order to fix r_0 and r_1 we calculate

$$|\nabla^2 w_0(x)|^2 =$$
$$\left(\frac{1}{4(\log r_1+1)}\right)^2\left(8\log^2 r+16\log r+4(2\log r_1+1)\frac{r_1^2}{r^2}+2(2\log r_1+1)^2\frac{r_1^4}{r^4})+10\right)$$

for $|x|=r$ and thus

$$\frac{d}{d|x|}|\nabla^2 w_0(x)|^2_{\big| |x|=r_1} = \left(\frac{1}{4(\log r_1+1)}\right)^2\frac{32}{r_1}\left(-\log r_1-\log^2 r_1\right)>0$$
$$\text{for } r_1 \text{ close to } 1$$

Consequently we can find $0<r_0<r_1<1$ such that

$$|\nabla^2 w_0(x)|<1 \text{ in } \Omega$$

Let now $v_0 \in W^{2,1}(\Omega)$ be the solution to the equation of linearily elastic deformation

$$\Delta^2 v_0 \equiv 0 \text{ in } \Omega$$
$$v_0 = w_0 \text{ on } \partial\Omega \quad \nabla v_0 = \nabla w_0 \text{ on } \partial B(0,r_0)$$
$$\nabla v_0 = m_0''(r_1)\nu \text{ on } \partial B(0,r_1)$$

where ν denotes the outer normal unit vector in $\partial B(0,r_1)$. Then with this choice of Ω, g_0, f, v_0 and $B=\mathrm{Id}:\mathbb{M}\mapsto\mathbb{M}$ we can state:

$$\mathcal{P} \text{ has no solution in } W^{2,1}(\Omega).$$

First let us show that w_0 and $M=\nabla^2 w_0$ are solutions of $\mathcal{P}^{rel}$ and $\mathcal{P}^*$ respectively. We remember that this is equivalent to $M\in K\cap D$ and

$$L(w_0,m)\le L(v,M) \quad \forall\,(v,m)\in(W_0^{1,2}(\Omega)+v_0)\times(K\cap D) \tag{2}$$

with

$$L(v,m) = <\operatorname{div}\operatorname{div} m-f, v-v_0> - \int g^*(m)dx+(m,\nabla^2 v_0)-M(v_0)$$

In our case $f\equiv 0$, thus $M(v)=0$ for all $v\in v_0+W_0^{2,1}(\Omega)$ and $f=0$ in $(W_0^{1,2}(\Omega))^*$. Further by definition we have for all $u\in W_0^{2,1}(\Omega)$

$$\begin{aligned}<\operatorname{div}\operatorname{div} M, u> &= \int M:\nabla^2 u\,dx=\int \operatorname{div}\operatorname{div} M u\,dx\\ &= \int \operatorname{div}\operatorname{div}\nabla^2 w_0 u\,dx = \int \Delta^2 w_0 u\,dx=0\end{aligned}$$

Thus trivially

$$\operatorname{div}\operatorname{div} M = 0 \in \left(W_0^{1,2}(\Omega)\right)^*$$

and we can write

$$<\operatorname{div}\operatorname{div} M, v-v_0> = <\operatorname{div}\operatorname{div} M, w_0-v_0> = 0$$

Thus (2) takes the form

$$<\operatorname{div}\operatorname{div} m, w_0-v_0> - \int(g^*(m)-g^*(M))+(m-M,\nabla^2 v_0)\le 0\,\forall\, m\in(K\cap D) \tag{3}$$

Using the special outlook of g^* and the elementary inequality

$$-\tfrac{1}{2}|\tau_1|^2+\tfrac{1}{2}|\tau_2|^2\le \tau_2:(\tau_2-\tau_1)$$

we see that (3) will follow from

$$<\operatorname{div}\operatorname{div} m, w_0-v_0> + \int M:(M-m)\,dx+(m-M,\nabla^2 v_0)\le 0\forall m\in(K\cap D) \tag{4}$$

In order to show this we first assumme $m\in C^\infty(\overline{\Omega},\mathbb{M})\cap K\cap D$. Again by definition of $\operatorname{div}\operatorname{div} m$ we have for all $u\in W_0^{2,1}(\Omega)$

$$
\begin{aligned}
< \operatorname{div}\operatorname{div} m, u > &= \int m : \nabla^2 u \, dx \\
&= -\int \operatorname{div} m \cdot \nabla u \, dx \\
&\leq ||\operatorname{div} m||_{L^2(\Omega,\mathbb{R}^2)} ||u||_{W^{1,2}(\Omega)}
\end{aligned}
$$

and thus by approximation

$$
< \operatorname{div}\operatorname{div} m, w_0 - v_0 >= -\int \operatorname{div} m \nabla(w_0 - v_0)\, dx
$$

$$
= -\int_{\partial B(0,r_1)} m : (\nabla(w_0 - v_0) \odot \nu) d\mathcal{H}^1(x) + \int m : (\nabla^2 w_0 - \nabla^2 v_0)\, dx
$$

where we make use of the notation

$$
\mathbb{R}^{m\times n} \ni \xi \odot \zeta = (\xi_i \zeta_j) \text{ if } (\xi, \zeta) \in \mathbb{R}^m \times \mathbb{R}^n
$$

Inserting this into (4) and writing $M = \nabla^2 w_0$ gives

$$
-\int_{\partial B(0,r_1)} m : (\nabla(w_0 - v_0) \odot \nu) d\mathcal{H}^1(x) + \int \nabla^2 w_0 : (\nabla^2 w_0 - \nabla^2 v_0)\, dx \leq 0
$$

which upon integrating by parts twice and using $\Delta^2 w_0 = 0$ leads to

$$
\int\limits_{\partial B(0,r_1)} m : (\nabla(w_0 - v_0) \odot \nu) d\mathcal{H}^1(x) \geq \int\limits_{\partial B(0,r_1)} \nabla^2 w_0 : (\nabla(w_0 - v_0) \odot \nu) d\mathcal{H}^1(x) \tag{5}
$$

We recall that $\nabla w_0 = 0$ on $\partial B(0, r_1)$ and thus

$$
(\nabla(w_0 - v_0) \odot \nu)_{ij} = -m''(r_1)(\nu \odot \nu)_{ij} = -\frac{m''(r_1)}{r_1^2} x_i x_j = -(\nabla^2 w_0(x))_{ij}
$$

As $|\nabla^2 w_0| = 1$ and $|m| \leq 1$ this means that (5) is true.
For general $m \in K \cap D$ the assertion follows from approximation in (4) with the standard smoothening mollifier. Clearly the sequence m_ρ of mollified functions converges to m strongly in any L^p, $1 \leq p < \infty$, and $\operatorname{div}\operatorname{div} m_\rho$ converges to m weakly in $(W_0^{1,2}(\Omega))^*$.

Assume now that $u_0 \in W^{2,1}(\Omega)$ is a solution to $\mathcal{P}$. Then u_0 is also a weak solution and by the uniqeness of the solution of $\mathcal{P}^*$ this implies that (u_0, M) is a saddle point of L, i.e. we have the variatonal inequality (2) with w_0 replaced by u_0. As $|M| = |\nabla^2 w_0| < 1$ in Ω we can test (2) with $m = M + \lambda \tau$ with any $\tau \in C_0^\infty(\Omega, \mathbb{M})$ and $\lambda \in \mathbb{R}$, $|\lambda| \leq C = C(\tau)$ and thus we see that

$$
\nabla^2 u_0 = M = \nabla^2 w_0
$$

This means that $u_0 = w_0 + \eta \cdot x + c$ with some constant $\eta \in \mathbb{R}^2$ and $c \in \mathbb{R}$. But as $u_0 = w_0$ on $\partial\Omega$ this implies

$$
u_0 = w_0 \text{ a.e. in } \Omega
$$

in contradiction to

$$
\nabla u_0 = \nabla v_0 \neq \nabla w_0 = 0 \text{ on } \partial B(0, r_1)
$$

□

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