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Problem sheet 10
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The Zeta function and the Riemann Hypothesis
Solution

Problem 37

Recall that $N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + O(\log T)$. Hence

$$\begin{aligned}
N(T+1) - N(T) &= \frac{T+1}{2\pi}\left(\log\frac{T+1}{2\pi} - 1\right) - \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + O(\log T) \\
&= \frac{T+1}{2\pi}\left(\log\frac{T+1}{2\pi}\right) - \frac{T}{2\pi}\left(\log\frac{T}{2\pi}\right) - \frac{1}{2\pi} + O(\log T) \\
&= \frac{T+1}{2\pi}\left(\log(T+1)\right) - \frac{T}{2\pi}\left(\log T\right) - \frac{\log(2\pi)}{2\pi} - \frac{1}{2\pi} + O(\log T)
\end{aligned}$$

Hence it is enough to show that $T(\log(T+1) - \log(T)) = O(\log(T))$. Consider $\lim_{T\to 0}\frac{1}{T\log(1/T)}(\log(1+T))$ instead. By the rule of *L'Hôpitale* we get

$$\lim_{T\to 0}\frac{\log(1+T)}{T\log(1/T)} = \lim_{T\to 0}\frac{\log(1+T)}{-T\log(T)} = \lim_{T\to 0}\frac{1/(T+1)}{-\log(T) - 1} = 0.$$

Similarly $\lim_{T\to\infty}\log(1+1/T)^T \to \log(e) = 1$ and hence $\lim_{T\to\infty} T\log(1+1/T)/\log(T) = 0$.

Problem 38

We will use problem 9, i.e. for $\operatorname{Re} s > 0$, $s \neq 1, \forall x \geqslant 1$:

$$\zeta(s) = \sum_{n\leqslant x}\frac{1}{n^s} + \frac{x^{1-s}}{x-1} + R(s,x), \quad |R(s,x)| \leqslant \left(2 + \frac{|T|}{\sigma}\right)\frac{1}{x^\sigma}.$$

Note that $\zeta(\sigma + iT)$ is bounded on $[2,\infty[\times[\pm 1, \pm 2]$ since $|\zeta(\sigma+iT)| \leqslant \sum_{n\geqslant 1} n^{-\sigma} < \infty$. As a holomorphic function it is also bounded on $[\delta, 2]\times[\pm 1, \pm 2]$. Hence we may assume $|T| > 2$. For $x = |T|$ and $|T| > 2, \sigma > \delta, 1 > \delta > 0$ we get

$$\begin{aligned}
|\zeta(\sigma+iT)| &\leqslant \sum_{n\leqslant |T|}\frac{1}{n^\sigma} + \frac{|T|^{1-\sigma}}{|T|-1} + R(s,|T|), \quad |R(s,|T|)| \leqslant \left(2 + \frac{|T|}{\sigma}\right)\frac{1}{|T|^\sigma}. \\
&\leqslant \int_2^{|T|+1}\frac{1}{(n-1)^\delta}\,\mathrm{d}\,n + 1 + |T|^{1-\delta} + 2 + \frac{1}{\delta}|T|^{1-\delta} \\
&\leqslant \frac{|T|^{1-\delta} - 1}{1-\delta} + 4|T|^{1-\delta} + \frac{1}{\delta}|T|^{1-\delta} \leqslant (4 + \delta^{-1} + (1-\delta)^{-1})\cdot|T|^{1-\delta}.
\end{aligned}$$