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Problem sheet 6
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The Zeta function and the Riemann Hypothesis
Solution

Problem 22

We use the following estimate: There is a $\varepsilon > 0$ such that $(1-\varepsilon)n\log(n) < p_n < (1+\varepsilon)n\log(n)$ for p_n the n-th prime number and n big enough.[1] Define $\varphi(x) = \frac{(1+\varepsilon)x}{(1-\varepsilon)\log(x)}$. Since $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is monotone increasing, so is φ^{-1}:

$$\frac{\partial}{\partial x}\varphi \circ \varphi^{-1}(x) = 1 \Rightarrow \frac{\partial}{\partial x}\varphi^{-1}(x) = \frac{1}{\frac{\partial \varphi}{\partial x}(\varphi^{-1}(x))}.$$

Hence $\varphi(x^{1-\varepsilon}) = \frac{1+\varepsilon}{1-\varepsilon}\frac{x^{1-\varepsilon}}{(1-\varepsilon)\log(x)} \leqslant \frac{(1-\varepsilon)x}{\log(x)} \Rightarrow \varphi^{-1}\left(\frac{(1-\varepsilon)x}{\log(x)}\right) \geqslant x^{1-\varepsilon}$ for large x. Hence for x big enough

$$\begin{aligned}
\sum_{p\geqslant x} f(p) &\leqslant \sum_{p_n \geqslant x} f((1-\varepsilon)n\log(n)) = \sum_{n\geqslant \pi(x)} f((1-\varepsilon)n\log(n)) \\
&\leqslant \int_{n\geqslant (1-\varepsilon)x/\log(x)} f((1-\varepsilon)n\log(n))\,\mathrm{d}\,n \\
&\leqslant \frac{(1+\varepsilon)}{(1-\varepsilon)}\int_{n\geqslant x^{1-\varepsilon}} f\left(n\cdot\frac{\log(n)-\log\log(n^{(1-\varepsilon)/(1+\varepsilon)})}{(1+\varepsilon)^{-1}\log(n)}\right)\left(\frac{1}{\log(n)}-\frac{1}{(\log(n))^2}\right)\mathrm{d}\,n \\
&\leqslant \frac{(1+\varepsilon)}{(1-\varepsilon)}\int_{n\geqslant x^{1-\varepsilon}} \frac{f\,(n)}{\log(n)}\,\mathrm{d}\,n
\end{aligned}$$

where we used that for large x: $1+\varepsilon > (1+\varepsilon)\left(1-\frac{\log\log(n^{(1-\varepsilon)/(1+\varepsilon)})}{\log(n)}\right) > 1$.[2]

Analogously for $\phi(x) = \frac{(1-\varepsilon)x}{(1+\varepsilon)\log(x)}$ we get $\phi(x^{1+\varepsilon}) = \frac{1-\varepsilon}{1+\varepsilon}\frac{x^{1+\varepsilon}}{(1+\varepsilon)\log(x)} \geqslant \frac{(1+\varepsilon)x}{\log(x)} \Rightarrow \phi^{-1}\left(\frac{(1+\varepsilon)x}{\log(x)}\right) \leqslant x^{1+\varepsilon}$ for large x and

$$\begin{aligned}
\sum_{p\geqslant x} f(p) &\geqslant \sum_{p_n \geqslant x} f((1+\varepsilon)n\log(n)) = \sum_{n\geqslant \pi(x)} f((1+\varepsilon)n\log(n)) \\
&\geqslant \int_{n\geqslant (1+\varepsilon)x/\log(x)} f((1+\varepsilon)n\log(n))\,\mathrm{d}\,n \\
&\geqslant \frac{(1-\varepsilon)}{(1+\varepsilon)}\int_{n\geqslant x^{1+\varepsilon}} f\left(n\cdot\frac{\log(n)-\log\log(n^{(1+\varepsilon)/(1-\varepsilon)})}{(1-\varepsilon)^{-1}\log(n)}\right)\left(\frac{1}{\log(n)}-\frac{1}{(\log(n))^2}\right)\mathrm{d}\,n \\
&\geqslant \frac{(1-\varepsilon)}{2(1+\varepsilon)}\int_{n\geqslant x^{1+\varepsilon}} \frac{f\,(n)}{\log(n)}\,\mathrm{d}\,n.
\end{aligned}$$

[1] See Problem 20.
[2] $\partial_x(\log\log(x)/\log(x)) = (1-\log\log(x))/(x\log(x)^2) < 0$ for large x and $\log\log(e^x)/\log(e^x) = x/\log(x) \to 0$.

Next consider

$$\begin{aligned}
\sum_{p\geqslant x} f(p)\log(p) &\leqslant \sum_{n\geqslant \pi(x)} f((1-\varepsilon)n\log(n))\log((1+\varepsilon)n\log(n)) \\
&\leqslant \int_{n\geqslant (1-\varepsilon)x/\log(x)} f((1-\varepsilon)n\log(n))\log((1+\varepsilon)n\log(n))\,\mathrm{d}\,n \\
&\leqslant \frac{(1+\varepsilon)}{(1-\varepsilon)}\int_{n\geqslant x^{1-\varepsilon}} f\left(n\cdot\frac{\log(n)-\log\log(n^{(1-\varepsilon)/(1+\varepsilon)})}{(1+\varepsilon)^{-1}\log(n)}\right)\left(\frac{\log\left(n^{\frac{(1+\varepsilon)^2)}{1-\varepsilon}}\right)}{\log(n)}\right)\mathrm{d}\,n \\
&\leqslant \frac{(1+\varepsilon)^3}{(1-\varepsilon)^2}\int_{n\geqslant x^{1-\varepsilon}} f\,(n)\,\mathrm{d}\,n
\end{aligned}$$

and

$$\begin{aligned}
\sum_{p\geqslant x} f(p)\log(p) &\geqslant \sum_{n\geqslant \pi(x)} f((1+\varepsilon)n\log(n))\log((1-\varepsilon)n\log(n)) \\
&\geqslant \int_{n\geqslant (1+\varepsilon)x/\log(x)} f((1+\varepsilon)n\log(n))\log((1-\varepsilon)n\log(n))\,\mathrm{d}\,n \\
&\geqslant \frac{(1-\varepsilon)}{(1+\varepsilon)}\int_{n\geqslant x^{1+\varepsilon}} f\left(n\cdot\frac{\log(n)-\log\log(n^{(1+\varepsilon)/(1-\varepsilon)})}{(1-\varepsilon)^{-1}\log(n)}\right)\left(\frac{\log\left(n^{\frac{(1-\varepsilon)^2}{1+\varepsilon}}\right)}{2\log(n)}\right)\mathrm{d}\,n \\
&\geqslant \frac{(1-\varepsilon)^3}{2(1+\varepsilon)^2}\int_{n\geqslant x^{1+\varepsilon}} f\,(n)\,\mathrm{d}\,n.
\end{aligned}$$

Problem 24

$\sum_{n\leqslant x}\lambda(n) = \nu_{\mathrm{ev}}(x)-\nu_{\mathrm{odd}}(x)$ where λ is the Liouville function $\lambda(n) = (-1)^{\sum_i k_i}$ for $n = \prod_i p_i^{k_i}$. But $\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}$: Hence if $|\sum_{n\leqslant x}\lambda(n)| = O(x^{\theta+\varepsilon})$ then by problem 15[3] $\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}$ converges for all $\mathrm{Re}(s)>\theta$ and the equality $\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}$ holds for $\mathrm{Re}(s)>\theta$. Thus $\zeta(s)$ cannot become zero there.[4]

[3] $\sum_{n=1}^{\infty}\lambda(n)$ diverges.

[4] Note that $\zeta(s)=0\Rightarrow\zeta(2s)=0\Rightarrow\zeta(4s)=0\Rightarrow\cdots$ leads to a contradiction (using the identity theorem for Dirichlet series).