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Summer Term 2016
Problem sheet 6
May 25th, 2016

Riemann Surfaces
Solution

Problem 21

a) The presheaf property of $\mathcal{O}$ transfers to $\mathcal{B}$, so does (Sh1) by the identity theorem. (Sh2) does not hold: Take an open set $U \overset{\varphi}{\simeq} B_1(0)$ and $U_k = \varphi^{-1}B_{1-1/k}(0)$. Then $\bigcup_{k\in\mathbb{Z}} U_k = U$ and $f(z) = \frac{1}{1-z}$ is bounded on each U_k (by k) but the unique extension f to $B_1(0)$ is not bounded.

b) $\mathcal{O}^*(U)/\exp(\mathcal{O}(U))$ is a presheaf with the usual restrictions, i.e. for $V \subset U$, $f\cdot\exp(\mathcal{O}(U)) \mapsto f|_V \cdot \exp(\mathcal{O}(V))$. The restrictions are well-defined as $f \cdot \exp(h) \mapsto f|_V \cdot \exp(h|_V) \in f|_V \cdot \exp(\mathcal{O}(V))$. Of course we have for $W \subset V \subset U$: $f|_W \cdot \exp(\mathcal{O}(W)) = (f|_V \cdot \exp(\mathcal{O}(V)))|_W$. (Sh1) does not hold: Take $U \overset{\varphi}{\simeq} B_1^*(0)$ and $U_1 = \varphi^{-1}(V_i), V_1 = \{z \in B_1^*(0) : \operatorname{Im}(z)\operatorname{Re}(z) > -1\}, V_2 = \{z \in B_1^*(0) : \operatorname{Im}(z)\operatorname{Re}(z) < 1\}$. Then U_1, U_2, U_{12} are simply-connected, but U is not simply connected.[1] For $f \in \mathcal{O}^*(U)$, $f_i = f|_{U_i} = \exp(h_i)$ for some holomorphic function h_i on U_i. Hence $[f_i] = 1$ in $\mathcal{O}^*(U_i)/\exp(\mathcal{O}(U_i)), i = 1, 2$. But $[f] \neq 1$. To give a particular example take $f = z$, then $[f] = 1$ would be satisfied if $z = \exp(h(z))$ for some holomorphic function h on the punctured unit disc. The logarithm however cannot be extended analytically to $B_1^*(0)$.[2]

Problem 22

a) $\tilde{\mathcal{F}}$ is a presheaf: Let $V \subset U$ be open and define $\tilde{\varrho}_V^U((\varphi_x)_{x\in U}) = (\varphi_x)_{x\in V}$ for $(\varphi_x)_{x\in U} \in \tilde{\mathcal{F}}(U)$. Then $\tilde{\varrho}_V^U : \tilde{\mathcal{F}}(U) \to \tilde{\mathcal{F}}(V)$ is well-defined: For every $x \in U$ there is a $x \in V_x \subset U$ and a $f^{(x)} \in \tilde{\mathcal{F}}(V_x)$ such that $\varphi_y = \varrho_y^{V_x}(f^{(x)})$. Now consider the image family $(\varphi_x)_{x\in V}$, then $x \in V_x \cap V \subset V$ and $\varrho_{V\cap V_x}^{V_x}(f^{(x)})$ satisfies $\varrho_y^{V_x\cap V}(\varrho_{V\cap V_x}^{V_x}(f^{(x)})) = \varrho_y^{V_x}(f^{(x)}) = \varphi_y$ for all $y \in V_x \cap V$.
By definition we atomatically get for $W \subset V \subset U$: $\tilde{\varrho}_W^U = \tilde{\varrho}_W^V \circ \tilde{\varrho}_V^U$ (restriction of sets).
$\tilde{\mathcal{F}}$ fulfils (Sh1): Take $(f_x)_{x\in U}, (g_x)_{x\in U} \in \tilde{\mathcal{F}}(U)$ with $(f_x)_{x\in U_i} = (g_x)_{x\in U_i}$ for all $i \in I$, $U = \bigcup_{i\in I} U_i$. Hence $f_x = g_x$ for every $x \in U$, i.e. $(f_x)_{x\in U} = (g_x)_{x\in U}$.
$\tilde{\mathcal{F}}$ fulfils (Sh2): Define $U_{ij} = U_i \cap U_j$. Assume $(f_x^i)_{x\in U_{ij}} = (f_x^j)_{x\in U_{ij}}$ for some $(f_x^i)_{x\in U_i} \in \tilde{\mathcal{F}}(U_i)$. We define $f = (f_x)_{x\in U}$ and $f_x = f_x^i$ if $x \in U_i$. This certainly gives a well-defined $f \in \bigcup_{x\in U} \mathcal{F}_x$. Moreover we know that every $x \in U$ is in some U_i. Then we find $V_i \subset U_i \subset U$ and a $g_i \in \mathcal{F}(V_i)$ with $\varrho_y^{V_i}(g_i) = f_y$ for all $y \in V_i$. Thus $f \in \tilde{\mathcal{F}}(U)$.

b) The map α is obviously well-defined, since for every $x \in U$ we have $f \in \mathcal{F}(U)$ and $\varrho_x^U(f) = (\alpha_U(f))_x$. We still need to show that $\alpha_V \circ \varrho_V^U = \tilde{\varrho}_V^U \circ \alpha_U$ for all $V \subset U$. By definition we already get for $f \in \mathcal{F}(U)$: $\alpha_V \circ \varrho_V^U(f) = (\varrho_x^V \circ \varrho_V^U(f))_{x\in V} = (\varrho_x^U(f))_{x\in V}$ and $\tilde{\varrho}_V^U \circ \alpha_U = \tilde{\varrho}_V^U \circ (\varrho_x^U(f))_{x\in U} = (\varrho_x^U(f))_{x\in V}$.

[1] Note that we do not need $\pi_1(U_{12}) = 0$ and that U_{12} is not connected.

[2] $1 = z' = h'(z)\exp(h(z)) = h'(z)z \Rightarrow h(\gamma(1)) - h(\gamma(0)) = \int_\gamma h'(z) = \int_\gamma z^{-1} = 2\pi i$ for a path $\gamma(t) = \exp(2\pi i t)$.

α induces an isomorphism of stalks: Fix x. Consider $g_x \in \tilde{\mathcal{F}}_x$. Take a neighbourhood V of x and $g \in \mathcal{F}(V)$ such that $\varrho_x^V(g) = g_x \in \tilde{\mathcal{F}}_x$. The function g exists by our assumption. We define $\beta_x(g_x) = [g]_x \in \mathcal{F}_x$. The map is certainly well-defined, since any other g' on V' with the same germ must coincide with g on some (possibly smaller) open set. But then $\alpha_x \circ \beta_x(g_x) = (\varrho_x^V(g))_x = g_x$ and $\beta_x \circ \alpha_x([f]_x) = \beta_x \circ \varrho_x^V(f) = [f]_x$ for all f with germ $[f]_x$.

Problem 23

a) If $\mathcal{F}$ satisfies (Sh1) and $\alpha_U(f) = \alpha_U(g)$, then for every $x \in U$ there is a neighbourhood V_x such that $f|_{V_x} = g|_{V_x}$. The V_x cover U and thus $f = g$ on U.
On the other hand, if α_U is injective and we have $f|_{U_i} = g|_{U_i}$ for a covering U_i of U, then $f_x = g_x$ for all $x \in U$, i.e. $\alpha_U(f) = \alpha_U(g) \Rightarrow f = g$.
As Alexander Trost pointed out, the following proof of the second statement works if we assume that $\mathcal{F}$ already satisfies (Sh1) resp. α_U is already injective.
If $\mathcal{F}$ satisfies (Sh2) and $q = (q_x)_{x \in U} \in \tilde{\mathcal{F}}$, then there exists locally on U_x a $f^{(x)} \in \mathcal{F}(U_x)$: $\varrho_y^{U_x}(f^{(x)}) = q_y$ for all $y \in U_x$. On the interesection $U_x \cap U_{x'}$, $f^{(x)}$ and $f^{(x')}$ coincide everywhere locally and (Sh1) guarantees that they satisfy $f^{(x)}|_{U_x \cap U_{x'}} = f^{(x')}|_{U_x \cap U_{x'}}$. Now the $f^{(x)}$ glue to a global function f with $\alpha_U(f) = q$. On the other hand if α_U is surjective and we have $f^i|_{U_{ij}} = f^j|_{U_{ij}}$, then we find a global section f with $f_x = f_x^i$ for all $x \in U_i$. Again using (Sh1) we get $f|_{U_i} = f^i$.

b) S is a sheaf, thus $S \simeq \tilde{S}$, so we only have to show, that $\gamma.(\tilde{\mathcal{F}}) = \tilde{\mathcal{S}}$. Of course if $(q_y)_{y \in U} \in \tilde{\mathcal{F}}(U)$ is locally on V given by f, then $\gamma_y(q_y) = (\gamma(f))_y$ and $\gamma.(\tilde{\mathcal{F}}) \subset \tilde{\mathcal{S}}$. On the other hand if $(r_y)_{y \in U} \in \tilde{\mathcal{S}}$ and locally on V around x: $r_y = g_y$ for a section g. Define $q_y = \gamma_y^{-1}(r_y)$. But there is a local section f on V' with $f_x = q_x$ and $\gamma(f)_x = r_x = g_x$ hence $\gamma(f) = g$ on $V'' \subset V \cap V'$. Thus $q_y = \gamma_y^{-1}(r_y) = \gamma_y^{-1}\gamma(f)_y = f_y$ for all $y \in V''$, i.e. $\gamma.(\tilde{\mathcal{F}}) = \tilde{\mathcal{S}}$.

c) We use part b). There is an obvious morphism $\mathcal{B} \to \mathcal{O}, f \mapsto f$. Since every holomorphic function is locally bounded, this map induces an isomorphism on stalks and hence by part b): $\tilde{\mathcal{B}} = \mathcal{O}$.
We have already seen, that $\mathcal{G} = \mathcal{O}^*(U)/\exp(\mathcal{O}(U))$ is locally trivial, thus $\tilde{\mathcal{G}} = 1$ the trivial sheaf.

Problem 24

There is nothing to be shown. p is continuous, thus the preimage of every open set is open. The restriction maps are just the restrictions on the preimages and all properties transfer.